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NEW CLASSIFICATIONS OVER WREATH PRODUCTS OF GROUPS

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Abstract. It is known that a group provides the property P-by-Q if there exists a normal subgroup which satisfies the property P and the quotient group obtained by this normal subgroup satisfies the property Q. The main goal of this paper is to state and prove some distinguish results on special types of finite wreath products via P-by-Q property.

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1. Introduction and Preliminaries

Considering a group extension brings several benefits such as unification known results in a new structure or an extended version. Presenting classifications on some special type of characterizations over specific group (or other structure) extensions have taken so much interest due to extensions built on the known structures, and so have to satisfy known results same as the groups in which extended. Furthermore, an algebraic extension allows us not only to specify with a concise way but also to derive a new design as well as to present an economical way for proving the correctness of some properties. Even if we think of the concept of extensions as a new structure by combining known structures, this approach also provides similar benefits as above in another effective way. One of the most interested extension is obviously wreath products (cf. [8]) which is one step next of the split extensions. As indicated in [16], any two groups can be combined, sometimes in more than one nonequivalent way, into a third group known as a wreath product of two groups. We may refer the marvelous book [8] for the definition and some important constructions on wreath products of groups and semigroups.

However there are so many different extensions on groups in which studied on special algebraic properties for the same targets depicted in the above paragraph. For example, to classify (cyclic) subgroup separability, the *HNN* extension has been considered (cf. [1–4, 7, 9, 17]). Another example for the similar classification of subgroup separability is the split extensions. In [2], by studying on split extensions, the authors established very important generalization "not every metacyclic groups are subgroup separable" which shows a certain benefit of considering extensions.

Of course one may consider some algebraic properties other than separability to investigate certain benefits on (group) extensions. One of them is

This work contains some material from the first author's thesis.

P-by-Q property, where P and Q represent any algebraic properties (see, for instance, [13, 14]). In detail, a group G provides the property P-by-Q if there exists a normal subgroup N of G satisfies the property P and the quotient group G/N satisfies the property Q.

For example, in [10], the author proved that if G is a semidirect product of P-by-Q and H is another group such that the endomorphism semigroups of G and H are isomorphic, then H is also a semidirect product of P-by-Q. In fact a group G is called a semidirect product of P-by-Q if each Schmidt group (that is a non-nilpotent finite group in which each proper subgroup is nilpotent) G is solvable of order $p^s q^v$ (where p and q are different primes) with a unique Sylow p-subgroup P and a cyclic Sylow q-subgroup Q. This also implies another certain benefit of considering extensions, specifiably, split extensions. Further on this, by considering split extensions, in [5] Cevik proved that there exists a generalization of the result "if the finitely generated group G is free-by-$\mathbb{Z}$, then G is large whenever the free group F is infinitely generated, or whenever $\mathbb{Z} \times \mathbb{Z} \leqslant G$ and F has rank at least 2".

In the light of the above motivation material on extensions, the aim of this paper is to establish some quite important results on special types of finite wreath products via P-by-Q property.

The following two pre-results will be needed in the proofs of our main theorems.

Lemma 1.1 (cf. [11]). *For any group H and a finite group K such that for all $K_i \lhd K$, there exists a normal subgroup $H^{|K|} \rtimes_\theta K_i \lhd H \wr K$ which yields an isomorphism $H \wr K/(H^{|K|} \rtimes_\theta K_i) \cong K_j$, where $K/K_i \cong K_j$.*

Note that the order of the group K need not to be finite in wreath products in general. But the results will be built on the case K is finite throughout in this paper. We also note that Lemma 1.1 used originally in [11] exposing the proofs of the Nielsen-Schreier Theorem and the Kurosh Subgroup Theorem via wreath products.

One of the conclusion of Lemma 1.1 is the following.

Proposition 1.2. *For any group H and a finite group K, the wreath product $H \wr K$ provides the property P-by-Q if the direct product $H^{|K|}$ satisfies the property P and the group K satisfies the property Q.*

Proof. According to Lemma 1.1, it is known that $H^{|K|} \rtimes_\theta K_i \lhd H \wr K$ for all $K_i \lhd K$. Now if we take $K_i = \{1_K\}$, then $H^{|K|} \rtimes_\theta \{1_K\} \cong H^{|K|}$. This implies that there must exist a normal subgroup of $H \wr K$ which is isomorphic to $H^{|K|}$. Since algebraic properties will be preserved between isomorphic groups, if the group $H^{|K|}$ satisfies the property P, then $H^{|K|} \rtimes_\theta \{1_K\}$ will also satisfy it. Moreover, by the definition of semidirect products, we have $H \wr K/H^{|K|} \cong K$. Therefore, if K satisfies the property Q, then the wreath product $H \wr K$ provides P-by-Q, as required. □

2. Main Theorems

In this section, we will state and proof our main theories as separate cases in different subsections.

By taking into account Lemma 1.1 and Proposition 1.2, unless stated otherwise, whole proofs of the results will be based on the following algorithm.

Algorithm 2.1. *In order for the group $H \wr K$ to provide the property P-by-Q, it will be sufficient for the normal subgroup $H^{|K|} \rtimes_\theta K_i$ satisfying the property P and for the group K/K_i satisfying the property Q. Specially, supposing $H^{|K|}$ satisfies the property P, if the group K satisfies the property Q, then the group $H \wr K$ provides P-by-Q.*

In the first three subsections, P will denote any algebraic property while the property Q will be considered as finite, cyclic and metacyclic. However in the final subsection, the property P will assume nilpotency while Q will be assumed the properties nilpotency and solvability. Some proofs of the results in these subsections will be omitted since these can be easily obtained by a simple replacement.

2.1. **Case I: P-by-*finite*.** The first classification on special wreath products is given in the following theorem.

Theorem 2.2. *Suppose that $m, n, k \in \mathbb{Z}^+$. Then*

(a) *$D_{2n} \wr \mathbb{Z}_m$ is metabelian-by-finite.*
(b) *$(\mathbb{Z}_m \rtimes_\theta \mathbb{Z}_n) \wr D_{2k}$ is polycyclic-by-finite.*
(c) *For an abelian group H and a finite group K, the wreath product $H \wr K$ is abelian-by-finite.*

Proof.

(a) We remind that a group G is called metabelian if there exists a normal subgroup N of G such that both N and G/N are abelian. In fact such these groups also provides abelian-by-abelian. We also recall that the (semi) direct product of two metabelian groups is also metabelian (see [12]).

Now consider the groups $D_{2n} = \langle x, y\,;\, x^n, y^2, (xy)^2 \rangle$ and $\mathbb{Z}_m = \langle z\,;\, z^m \rangle$. By [13], we clearly know that $D_{2n} \cong \mathbb{Z}_n \rtimes_\theta \mathbb{Z}_2$ such that $\mathbb{Z}_n = \langle x\,;\, x^n \rangle \lhd D_{2n}$ and $D_{2n}/\mathbb{Z}_n \cong \mathbb{Z}_2$. Thus $\mathbb{Z}_n \rtimes_\theta \mathbb{Z}_2$ and so D_{2n}^m ($m \in \mathbb{Z}^+$) are metabelian according to the above paragraph. Further, since $D_{2n} \wr \mathbb{Z}_m \cong D_{2n}^m \rtimes_\theta \mathbb{Z}_m$, it is seen that $D_{2n}^m \lhd D_{2n} \wr \mathbb{Z}_m$ and $D_{2n} \wr \mathbb{Z}_m / D_{2n}^m \cong \mathbb{Z}_m$. Hence, by Algorithm 2.1, $D_{2n} \wr \mathbb{Z}_m$ is metabelian-by-finite, as required.

(b) We first note that finite (semi) direct product of polycyclic groups is polycyclic (see, for instance, [6]). By the definition of wreath products, we know that $(\mathbb{Z}_m \rtimes_\theta \mathbb{Z}_n) \wr D_{2k} \cong (\mathbb{Z}_m \rtimes_\theta \mathbb{Z}_n)^{2k} \rtimes_\theta D_{2k}$ which implies, similarly in (a), $(\mathbb{Z}_m \rtimes_\theta \mathbb{Z}_n)^{2k} \lhd (\mathbb{Z}_m \rtimes_\theta \mathbb{Z}_n) \wr D_{2k}$ and $(\mathbb{Z}_m \rtimes_\theta \mathbb{Z}_n) \wr D_{2k} / \left[(\mathbb{Z}_m \rtimes_\theta \mathbb{Z}_n)^{2k} \right] \cong D_{2k}$. In here, since the normal subgroup $\mathbb{Z}_m$ of $\mathbb{Z}_m \rtimes_\theta \mathbb{Z}_n$ is cyclic and so the quotient group of them is also cyclic, we get the semidirect product $\mathbb{Z}_m \rtimes_\theta \mathbb{Z}_n$ is metacyclic. On the other hand, since each metacyclic group is polycyclic (cf. [6]), we reach that $(\mathbb{Z}_m \rtimes_\theta \mathbb{Z}_n)^{2k}$ is polycyclic. By considering the fact D_{2k} is finite and then using Algorithm 1, we obtain $(\mathbb{Z}_m \rtimes_\theta \mathbb{Z}_n) \wr D_{2k}$ is polycyclic-by-finite.

(c) Let $|K| = t$ and H be an abelian group. Then t times direct product of H, shortly H^t, is also abelian. Since $H^t \lhd H \wr K$ and $H \wr K / K^t \cong K$, by Algorithm 2.1, we achieved the result. □

The group $\mathbb{Z}_m \wr \mathbb{Z}_n$ can be thought as an example for Theorem 2.2-(c). Moreover, by a famous theorem of [6], since every metacyclic groups are also metabelian, it is also true that $\mathbb{Z}_m \wr \mathbb{Z}_2$ and $D_{2n} \wr \mathbb{Z}_m$ are metacyclic-by-finite.

2.2. **Case II: P-by-*cyclic*.** The second type of classification on special wreath products is given in the next theorem.

Theorem 2.3. *Let $m, n, k \in \mathbb{Z}^+$ and p be any prime number. The following are true.*

(a) $D_{2n} \wr D_{2m}$ *is finite-by-cyclic.*
(b) $(\mathbb{Z}_m \times \mathbb{Z}_n) \wr \mathbb{Z}_k$ *is abelian-by-cyclic.*
(c) $(\mathbb{Z}_m \wr \mathbb{Z}_n) \wr \mathbb{Z}_k$ *is metabelian-by-cyclic.*
(d) $(\mathbb{Z}_{p^m} \wr \mathbb{Z}_{p^n}) \wr D_{2p^k}$ *is nilpotent-by-cyclic.*
(e) $(D_{2m} \wr D_{2n}) \wr D_{2k}$ *is solvable-by-cyclic.*
(f) $D_{2n} \wr \mathbb{Z}_2$ *is virtually metabelian-by-cyclic.*
(g) $D_{2^n} \wr D_{2m}$ *is virtually nilpotent-by-cyclic.*
(h) $\mathbb{Z}_m \wr \mathbb{Z}_n$ *is Chernikov-by-cyclic.*

Proof.

(a) By combining the facts $D_{2n} \wr D_{2m} \cong D_{2n}^{2m} \rtimes_\theta D_{2m}$ and $D_{2m} \cong \mathbb{Z}_m \rtimes_\theta \mathbb{Z}_2$, and also considering the first part of Lemma 1.1, we get a normal subgroup

$$D_{2n}^{2m} \rtimes_\theta \mathbb{Z}_m \lhd D_{2n}^{2m} \rtimes_\theta (\mathbb{Z}_m \rtimes_\theta \mathbb{Z}_2)$$

having order $m(2n)^{2m}$. On the other hand, by the second part of Lemma 1.1, we reach an isomorphism

$$D_{2n}^{2m} \rtimes_\theta (\mathbb{Z}_m \rtimes_\theta \mathbb{Z}_2) \,/\, \left[D_{2n}^{2m} \rtimes_\theta \mathbb{Z}_m\right] \cong \mathbb{Z}_2.$$

Thus, by Algorithm 2.1, the group $D_{2n}^{2m} \rtimes_\theta (\mathbb{Z}_m \rtimes_\theta \mathbb{Z}_2)$ is finite-by-cyclic, and so up to isomorphism $D_{2n} \wr D_{2m}$ is finite-by-cyclic, as required.

(b) - (c) Since the proofs can be easily obtained by Lemma 1.1, Proposition 1.2 and Algorithm 2.1, it will be omitted.

(d) It is clear that $(\mathbb{Z}_{p^m} \wr \mathbb{Z}_{p^n}) \wr D_{2p^k}$ is isomorphic to $(\mathbb{Z}_{p^m} \wr \mathbb{Z}_{p^n})^{2p^k} \rtimes D_{2p^k}$. With the fact $D_{2p^k} \cong \mathbb{Z}_{p^k} \rtimes_\theta \mathbb{Z}_2$, we can write

$$(\mathbb{Z}_{p^m} \wr \mathbb{Z}_{p^n}) \wr D_{2p^k} \cong (\mathbb{Z}_{p^m} \wr \mathbb{Z}_{p^n})^{2p^k} \rtimes_\theta \left(\mathbb{Z}_{p^k} \rtimes \mathbb{Z}_2\right).$$

By considering the normal subgroup $\mathbb{Z}_{p^k} \lhd \mathbb{Z}_{p^k} \rtimes_\theta \mathbb{Z}_2$ and using Lemma 1.1, we obtain a normal subgroup $(\mathbb{Z}_{p^m} \wr \mathbb{Z}_{p^n})^{2p^k} \rtimes \mathbb{Z}_{p^k}$ with the order $p^{2p^k(mn+m)+k}$. Therefore this normal subgroup is actually a p-group provides the nilpotency property in the theorem. In addition, the quotient group obtained from this normal (p)-subgroup is isomorphic to $\mathbb{Z}_2$. Thus, by Algorithm 2.1, $(\mathbb{Z}_{p^m} \wr \mathbb{Z}_{p^n})^{2p^k} \rtimes_\theta \left(\mathbb{Z}_{p^k} \rtimes \mathbb{Z}_2\right)$ is nilpotent-by-cyclic, and hence up to isomorphism $(\mathbb{Z}_{p^m} \wr \mathbb{Z}_{p^n}) \wr D_{2p^k}$ is too.

(e) This is a kind of iterated version of (a). In here, we just need to observe that if both $H \lhd G$ and G/H are solvable, then G is solvable due to a result in [1]. Also, by considering the Burnside's theorem, we achieve the result with the method given in Algorithm 2.1.

(f) We first recall that if a group G provides P-by-finite, where P corresponds any algebraic property, then it also provides virtually P property (see [10]). Observe that $D_{2n} \wr \mathbb{Z}_2 \cong D_{2n}^2 \rtimes_\theta \mathbb{Z}_2$, and hence there exist $D_{2n}^2 \lhd D_{2n} \wr \mathbb{Z}_2$ with the quotient group isomorphic to $\mathbb{Z}_2$. According to Theorem 2.2-(a), D_{2n}^2 is metabelian-by-finite. Due to the result in [10], it also provides virtually metabelian. Thus, by Algorithm 2.1, the group $D_{2n} \wr \mathbb{Z}_2$ is virtually metabelian-by-cyclic.

(g) Since $D_{2^n} \wr D_{2m}$ can be written as the semidirect product $D_{2^n}^{2m} \rtimes_\theta D_{2m}$, by using the isomorphisms $D_{2^n} \cong \mathbb{Z}_{2^{n-1}} \rtimes_\theta \mathbb{Z}_2$ and $D_{2m} \cong \mathbb{Z}_m \rtimes_\theta \mathbb{Z}_2$, we get

$$D_{2^n} \wr D_{2m} \cong (\mathbb{Z}_{2^{n-1}} \rtimes_\theta \mathbb{Z}_2)^{2m} \rtimes_\theta (\mathbb{Z}_m \rtimes_\theta \mathbb{Z}_2)$$

which yields the normal subgroup series

$$(\mathbb{Z}_{2^{n-1}} \rtimes_\theta \mathbb{Z}_2)^{2m} \lhd (\mathbb{Z}_{2^{n-1}} \rtimes_\theta \mathbb{Z}_2)^{2m} \rtimes_\theta \mathbb{Z}_m \lhd D_{2^n} \wr D_{2m}$$

by Lemma 1.1. Since the order of $(\mathbb{Z}_{2^{n-1}} \rtimes_\theta \mathbb{Z}_2)^{2m}$ is $(2^n)^{2m}$, we get that the normal subgroup $D_{2^n}^{2m}$ is a 2-group and so nilpotent. Hence, by using Algorithm 2.1 with the fact $\mathbb{Z}_m$ is finite, we obtain $(\mathbb{Z}_{2^{n-1}} \rtimes_\theta \mathbb{Z}_2)^{2m} \rtimes_\theta \mathbb{Z}_m$ is nilpotent-by-cyclic. Therefore, by the result of [10] mentioned in (f), this group is virtually nilpotent. As a result of all these above material, we obtain $D_{2^n} \wr D_{2m}$ is virtually nilpotent-by-cyclic, as required.

(h) Let us start the proof by reminding some fundamental facts on Chernikov groups. Details can be found in [15].

The following are equivalent for a group G:

- ∘ G is Chernikov.
- ∘ G is virtually abelian provides minimality condition.
- ∘ G is virtually solvable provides minimality condition.

In above, the meaning of provides minimality condition of the group is that every descending subgroup series is finite. Additionally, in [15], it is proved that the direct product of any two Chernikov groups is also Chernikov.

With a similar idea as in the above proofs, since $\mathbb{Z}_m \wr \mathbb{Z}_n$ can also be equivalent to the semidirect product $\mathbb{Z}_m^n \rtimes_\theta \mathbb{Z}_n$, we clearly have a normal subgroup $\mathbb{Z}_m^n$ and the quotient group related to it is isomorphic to $\mathbb{Z}_n$. Assume that $\mathbb{Z}_m = \langle x\,;\, x^m \rangle$ and the order m is decomposed as $p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m}$, where each p_i is a different prime. Thus we have the descending subgroup series

$$\mathbb{Z}_m \supset \langle x^{p_1} \rangle \supset \left\langle x^{p_1^2} \right\rangle \supset \ldots \supset \left\langle x^{p_1^{k_1}} \right\rangle \supset 1$$
$$\mathbb{Z}_m \supset \langle x^{p_2} \rangle \supset \left\langle x^{p_2^2} \right\rangle \supset \ldots \supset \left\langle x^{p_2^{k_2}} \right\rangle \supset 1$$
$$\vdots$$
$$\mathbb{Z}_m \supset \langle x^{p_m} \rangle \supset \left\langle x^{p_m^m} \right\rangle \supset \ldots \supset \left\langle x^{p_m^{k_m}} \right\rangle \supset 1$$

which are all finite. Thus $\mathbb{Z}_m$ provides the minimality condition. Besides, for a non prime m, there would be a case $m = m_1 m_2$ with $(m_1, m_2) = 1$, where $m_1, m_2 \in \mathbb{Z}^+$ having $\mathbb{Z}_m \cong \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}$. This implies that there exists a normal abelian subgroup $\mathbb{Z}_{m_1}$ and the quotient group related to it is isomorphic to $\mathbb{Z}_{m_2}$. So $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}$ is abelian-by-finite, and by the result indicated in (f), virtually abelian. Hence $\mathbb{Z}_m$, and so $\mathbb{Z}_m^n$ is Chernikov. After all, by Algorithm 2.1, $\mathbb{Z}_m \wr \mathbb{Z}_n$ is Chernikov-by-cyclic.

Hence the result. □

As a consequence of Theorems 2.2 and 2.3, we have the following result which the proof of it will be omitted.

Corollary 2.4. *The group $D_{2n} \wr \mathbb{Z}_m$ is both finite-by-cyclic, and (virtually) metabelian-by-cyclic as well as $(\mathbb{Z}_m \rtimes_\theta \mathbb{Z}_n) \wr D_{2k}$ is finite-by-cyclic.*

2.3. **Case III:** P**-by-*metacyclic*.**

Theorem 2.5. *Suppose that $a,b,c,d \in \mathbb{Z}^+$ and p is any prime. Therefore $\left(\mathbb{Z}_{p^a} \wr \mathbb{Z}_{p^b}\right) \wr (\mathbb{Z}_c \times \mathbb{Z}_d)$ is nilpotent-by-metacyclic.*

Proof. For simplicity, let us denote the group $\left(\mathbb{Z}_{p^a} \wr \mathbb{Z}_{p^b}\right) \wr (\mathbb{Z}_c \times \mathbb{Z}_d)$ by just G. Even though G is an iterated wreath product, according to the definition, it also can be written as the form $\left(\mathbb{Z}_{p^a} \wr \mathbb{Z}_{p^b}\right)^{cd} \rtimes_\theta (\mathbb{Z}_c \times \mathbb{Z}_d)$.

Therefore, by the definition of semidirect products, we have the following normal subgroup and so the quotient group related to G:

$$(1) \qquad \left(\mathbb{Z}_{p^a} \wr \mathbb{Z}_{p^b}\right)^{cd} \lhd G \quad \text{and} \quad G/\left(\mathbb{Z}_{p^a} \wr \mathbb{Z}_{p^b}\right)^{cd} \cong \mathbb{Z}_c \times \mathbb{Z}_d.$$

Since the order of $\left(\mathbb{Z}_{p^a} \wr \mathbb{Z}_{p^b}\right)^{cd}$ is $p^{cd(ap^b+b)}$, it is a p-group, and so nilpotent.

On the other hand, since the quotient group in (1) is isomorphic to the direct product $\mathbb{Z}_c \times \mathbb{Z}_d$, and since we have the cyclic groups $\mathbb{Z}_c \lhd \mathbb{Z}_c \times \mathbb{Z}_d$ and $\mathbb{Z}_c \times \mathbb{Z}_d/\mathbb{Z}_c \cong \mathbb{Z}_d$, it implies that $\mathbb{Z}_c \times \mathbb{Z}_d$ is metacyclic, and so up to isomorphism the quotient group in (1) is metacyclic. After all, by considering Algorithm 2.1, we obtain G is nilpotent-by-metacyclic.

Hence the result. □

We note that in [18] the author studied the nilpotency of $\mathbb{Z}_{p^a} \wr \mathbb{Z}_{p^b}$ with a different approximation. However study in here the extended version of this result via P-by-Q property.

A simple application of Theorem 2.5 is the following.

Corollary 2.6. *Suppose that $a,b,c,d \in \mathbb{Z}^+$. Then the iterated wreath product $(\mathbb{Z}_a \wr \mathbb{Z}_b) \wr (\mathbb{Z}_c \times \mathbb{Z}_d)$ is solvable-by-metacyclic.*

Proof. By replacing each p^a (p is a prime) with only $a \in \mathbb{Z}^+$, and then following the same processes as in the proof of Theorem 2.5 with keeping in mind the connection between nilpotency and solvability, the required result can be easily obtained. □

2.4. **Case IV: nilpotent-by-**Q**.** Unlike other subsections, in this final part, we will take the property P as just nilpotent and investigate different variations of the property Q.

Thus the final main result of this paper is the following.

Theorem 2.7. *Suppose that $a,b,c,d \in \mathbb{Z}^+$ and p,q are prime numbers. Then*

(a) *the iterated wreath product $\left(\mathbb{Z}_{p^a} \wr \mathbb{Z}_{p^b}\right) \wr \left(\mathbb{Z}_{q^c} \wr \mathbb{Z}_{q^d}\right)$ is nilpotent-by-nilpotent.*

(b) *$\mathbb{Z}_{p^a} \wr D_{2p^b}$ is nilpotent-by-nilpotent.*

Proof.

(a) In the proof, we will follow completely the same as in the proofs of previous main results (Theorems 2.2-2.5). For simplicity, let us denote the whole group $\left(\mathbb{Z}_{p^a} \wr \mathbb{Z}_{p^b}\right) \wr \left(\mathbb{Z}_{q^c} \wr \mathbb{Z}_{q^d}\right)$ by only G, and also denote $\left(\mathbb{Z}_{p^a} \wr \mathbb{Z}_{p^b}\right)$ and $\left(\mathbb{Z}_{q^c} \wr \mathbb{Z}_{q^d}\right)$ by H and K, respectively.

Since $G \cong H^{|K|} \rtimes_\theta K$ and so since we have $H^{|K|} \lhd G$ with $G/H^{|K|} \cong K$, we clearly observe that the order of K is $q^{cq^d+q^d}$ which yields that K is

p-group and so nilpotent. On the other hand, a simple calculation shows that the order of $H^{|K|}$ is

$$p^{\left(q^{cq^d+q^d(ap^b+b)}\right)},$$

thus $H^{|K|}$ is a p-group and so nilpotent. By taking into account Algorithm 2.1, all these above processes sufficient to show that G is nilpotent-by-nilpotent.

(b) This follows directly from the same way as in (a). □

A consequence of Theorems 2.3 and 2.7 can be presented as follows.

Corollary 2.8. *Suppose that $a, b, c, d \in \mathbb{Z}^+$. Then the iterated wreath product $(D_{2^a} \wr D_{2^b}) \wr (D_c \wr D_d)$ is nilpotent-by-solvable.*

Proof. By replacing each p^a (p is a prime) with only 2 in the proof of Theorem 2.3-(a) and apply the same processes with keeping in mind every nilpotent groups are solvable, the proof of this corollary can be done easily. □

3. Conclusions and Future Problems

By adapting the known results to the wreath product via P-by-Q property, we made classifications over this so distinguish group extension. To do that, P and Q properties were chosen in accordance with the structure of the wreath product, and more specific investigations in classical group theory were made.

After the outcomes obtained in this paper, one may naturally try to generalize the results (Theorems 2.2-2.7) into the any groups H and K. This is of course the most important future problem on this subject. Another problem would be the choice of suitable properties for P and Q other than studied in here. One may also would like to choose another groups extension such as Zappa-Szep product, Knit product etc. to apply same or similar results.

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