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ON THE EXTREMA OF THE FUNDAMENTAL EIGENVALUE OF A FAMILY OF SCHRÖDINGER OPERATORS ON A POLYGONAL DOMAIN IN $\mathbb{H}^2$

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Abstract. Let $\wp, \wp_0$ be two regular polygons of n sides in the hyperbolic plane $\mathbb{H}^2$ such that $\wp_0 \subset \wp$ and having the same center of mass. Let $\wp_0$ be circumscribed by a circle C contained in $\wp$. Fix $\wp$ and vary $\wp_0$ by rotating it in C about the center of mass and let $\wp_t$ ($t \in \mathbb{R}$) be the family of polygons obtained in this fashion. Let $\chi_{\wp_t}$ denote the indicator function of the subset $\wp_t$ of $\wp$. For any fixed non-zero constant $\alpha \in \mathbb{R}$, it is shown that the fundamental eigenvalue of Schrödinger Operators $-\Delta + \alpha\chi_{\wp_t}\mathrm{Id}$ attains its extremum when the axes of symmetry of $\wp_0$ coincide with those of $\wp$.

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1. Introduction And Statement Of Main Result

The hyperbolic plane $\mathbb{H}^2$ is the open set $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ of $\mathbb{C}$ with Riemannian metric

$$g((z,u),(z,v)) := \frac{\mathrm{Re}(u\overline{v})}{(\mathrm{Im}(z))^2} \ \ (z \in \mathbb{H}^2,\ u, v \in \mathbb{C}).$$

The inner metric d of the hyperbolic plane $(\mathbb{H}^2, g)$ is given by

$$\cosh d(z,w) = 1 + \frac{|z-w|^2}{2\mathrm{Im}(z)\mathrm{Im}(w)} \ \ \forall\, z, w \in \mathbb{H}^2.$$

And $(\mathbb{H}^2, d)$ is a complete metric space.
Consider the natural parametrisation $\phi : \mathbb{H}^2 \to \mathbb{R}^2$ defined by $\phi(x+iy) := (x,y)$ $\forall\, x+iy \in \mathbb{H}^2$. The corresponding two coordinate functions on $\mathbb{H}^2$ are $x(z) := \mathrm{Re}(z)$, $y(z) := \mathrm{Im}(z)$. For any $f \in \mathcal{C}^\infty(\mathbb{H}^2)$, the partial derivatives are defined by

$$\frac{\partial f}{\partial x}(z) := D_1(f \circ \phi^{-1})(\phi(z)), \frac{\partial f}{\partial y}(z) := D_2(f \circ \phi^{-1})(\phi(z)) \ \ (z \in \mathbb{H}^2).$$

For a smooth function $f : \mathbb{H}^2 \to \mathbb{R}$, df_z denotes its total derivative at the point $z \in \mathbb{H}^2$. Let D denote the Levi-Civita connection of $(\mathbb{H}^2, g)$. Then in the geometry of $(\mathbb{H}^2, g)$ we have :

(1) for a smooth function $f : \mathbb{H}^2 \to \mathbb{R}$, the gradient vector field ∇f is defined by the relation

$$g(\nabla f(z), (z,v)) = df(z)(v) \ (z \in \mathbb{H}^2,\ v \in \mathbb{C}),$$

(2) for any smooth tangent vector field X of $\mathbb{H}^2$, the divergence $\operatorname{div}(X)$ is defined as $\operatorname{trace}(DX)$,
(3) and the Laplace-Beltrami operator Δ is defined by

$$\Delta f = \operatorname{div}(\nabla f).$$

Let Δ^E , ∇^E denote the Laplacian operator and the gradient operator respectively of the open set $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ of $\mathbb{R}^2$ having the Euclidean metric. Then we have

$$\Delta = y^2\Delta^E \text{ and } \nabla = y^2\nabla^E. \tag{1.0.1}$$

For every $r > 0$, let $B_E((x, y), r)$ denote the Euclidean ball of radius r having its center at (x, y) in $\mathbb{R}^2$. Let $B_{\mathbb{H}^2}(z, r)$ denote the open ball of the hyperbolic plane $\mathbb{H}^2$ of radius r having its center at $z \in \mathbb{H}^2$. Then

$$B_{\mathbb{H}^2}(i, r) = B_E((0, \cosh r),\ \sinh r).$$

Fix $0 < r < R$ and a natural number $n \geq 3$. Let $\wp, \wp_0$ denote open regular polygons in $\mathbb{H}^2$ of n-sides circumscribed by the Euclidean circles $|z - i\cosh R| = \sinh R$ and $|z - i\cosh r| = \sinh r$ respectively such that one side of $\wp$ and one side $[b, c]$ of $\wp_0$ are orthogonal to the geodesic $\{iy \mid y > 0\}$ (see fig. 1).

Let ρ_t $(t \in \mathbb{R})$ denote the rotation maps of $\mathbb{H}^2$ about the point i, i.e.

$$\rho_t(z) = \frac{z\cos t + \sin t}{-z\sin t + \cos t} \quad (z \in \mathbb{H}^2). \tag{1.0.2}$$

Put

$$\wp_t := \rho_t(\wp_0) \quad (t \in \mathbb{R}). \tag{1.0.3}$$

Fix any non-zero $\alpha \in \mathbb{R}$. We consider the family of Schrödinger Operators

$$L_{t,\alpha} := \Delta - \alpha\chi_{\wp_t} Id,\ (t \in \mathbb{R})$$

defined on the Sobolev space $H^2(\wp, \mu) \cap H_0^1(\wp, \mu)$. The first eigenvalue $\lambda_{t,\alpha}$ of $-L_{t,\alpha}$ is positive when $\alpha \geq 0$ and it might be negative if α is negative.

If two vertices of $\wp_t$ and $\wp$ lie on the same half-axes of symmetry emanating from i, then we say that $\wp_t$ occupies *on position* in $\wp$. If a vertex of $\wp_t$ lie on the half-axes of symmetry emanating from i and passing through the mid-point of a side of $\wp$, then we say that $\wp_t$ occupies *off position* in $\wp_1$. In the general position of $\wp_t$, neither of the two conditions is satisfied. These concepts were first introduced in [2].

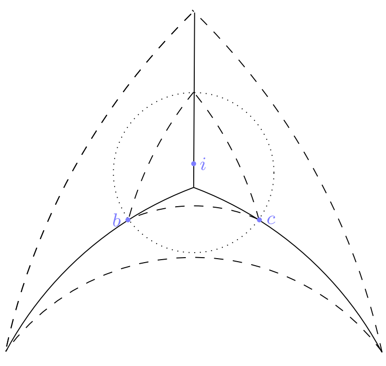

fig.1 : $n = 3$, $\wp_0 \subset \wp$ both in 'on position'

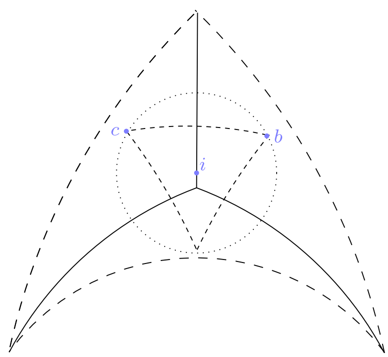

fig. 2 : n=3, $\wp_{\frac{2\pi}{3}}$ in 'off position'

Now we state the main result of this paper.

Theorem 1.1. *The first eigenvalue* $\lambda_{t,\alpha}$ *of* $-L_{t,\alpha}$ $(t \in \mathbb{R})$, *is optimized exactly when the axes of symmetry of* $\wp_t$ *coincide with those of* $\wp$.

For $\alpha > 0$, *the maximizing configuration for* $\lambda_{t,\alpha}$ *corresponds to the case when* $\wp_t$ *occupies 'on position' in* $\wp$ *and the minimizing configuration for* $\lambda_{t,\alpha}$ *corresponds to the case when* $\wp_t$ *occupies 'off position' in* $\wp$.

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For the case of Euclidean plane corresponding results were proved in [3] for $\alpha = 0$ and in [9] for $\alpha \neq 0$.

The proof relies on the Hadamard formula, and the moving plane method first introduced in [6].

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2. Preliminaries

We introduce some preliminary notations and state some known results needed in the sequel.

Notation: For any nonempty open set U of $\mathbb{R}^2$, $L^2(U)$ denotes the equivalence class of real valued Lebesgue measurable functions defined on U which are square integrable with respect to the Lebesgue measure of U. For $u, v \in L^2(U)$, define $\langle u, v\rangle_{L^2(U)} := \int_\Omega u(x)v(x)dxdy$. This is an inner product of $L^2(U)$ and the induced norm is denoted by $||.||_{L^2(U)}$. Also $(L^2(U), \langle\cdot,\cdot\rangle_{L^2(U)})$ is a Hilbert space.

Let Ω be any nonempty open subset of the hyperbolic plane $\mathbb{H}^2$ such that $\overline{\Omega}$ is compact. Let $\mathcal{D}(\Omega)$ denote the collection of all real valued smooth functions defined on Ω having their compact support contained in Ω.

A function $f : \Omega \to \mathbb{R}$ is said to be a measurable function if the pre-image $f^{-1}((a, b))$ lies in the Borel σ-algebra of Ω for all open intervals (a, b) of $\mathbb{R}$. Let μ denote the Borel measure of the Hyperbolic plane $\mathbb{H}^2$ induced by the hyperbolic metric g. We denote $L^2(\Omega, \mu)$ the equivalence class of all real valued measurable functions u defined on Ω which are square integrable with respect to the measure μ, i.e. $\int_\Omega u^2 d\mu < \infty$.

Define $\langle u, v\rangle_{L^2(\Omega,\mu)} := \int_\Omega u(x)v(x)\, d\mu\ \ \forall\, u, v \in L^2(\Omega, \mu)$. This is an inner product of $L^2(\Omega, \mu)$ and the induced norm is denoted by $||.||_{L^2(\Omega,\mu)}$. Also $L^2(\Omega, \mu)$ is a Hilbert space.

Since $\overline{\Omega}$ is a compact subset of $\mathbb{H}^2$, $\exists$ real constants $0 < m < M$ such that $m \le y(z) \le M\ \forall\ z \in \Omega$. Then for any continuous real valued function f having compact support in Ω,

$$\frac{1}{M^2}\|f\|^2_{L^2(\Omega)} \le \int_\Omega f^2 \frac{1}{y^2} dxdy \le \frac{1}{m^2}\|f\|^2_{L^2(\Omega)}.$$

It follows that a function $u \in L^2(\Omega, \mu)$ if and only if u is a Lebesgue measurable function which is square integrable on Ω with respect to the Lebesgue measure of Ω. Thus $L^2(\Omega, \mu) = L^2(\Omega)$ as sets and their norms are equivalent.

For $\alpha = (\alpha_1, \alpha_2) \in (\mathbb{Z}^+)^2$, we put $|\alpha| = \alpha_1 + \alpha_2$ and denote $D_1^{\alpha_1} \circ D_2^{\alpha_2}$ by D^α. Here $D_1 = \frac{\partial}{\partial x}$ and $D_2 = \frac{\partial}{\partial y}$.

For a function $u \in L^2(\Omega, \mu)$, a function $v \in L^1_{\text{loc}}(\Omega)$ is said to be an '*α-th weak partial derivative*' of u if

$$\int_\Omega u D^\alpha \phi \, dx \, dy = (-1)^{|\alpha|} \int_\Omega v \phi \, dx \, dy \;\; \forall \;\; \phi \in \mathcal{D}(\Omega).$$

The α-th weak partial derivative of u, denoted by $\partial^\alpha u$, is unique when it exists.

We define for $k \in \mathbb{N}$ the Sobolev space

$H^k(\Omega, \mu) := \{u \in L^2(\Omega, \mu) \mid \alpha-\text{th weak partial derivative } \partial^\alpha u \in L^2(\Omega, \mu) \; \forall \; |\alpha| \leq k\}$

with the Sobolev norm given by

$$||u||_{H^k(\Omega,\mu)} = \left(\sum_{|\alpha| \leq k} ||\partial^\alpha u||^2_{L^2(\Omega,\mu)} \right)^{\frac{1}{2}}.$$

It is naturally a Hilbert Space.

The closure of $\mathcal{D}(\Omega)$ in $H^k(\Omega, \mu)$ is denoted by $H^k_0(\Omega, \mu)$.

We state few preliminary results needed in the sequel. We begin with a density result.

Notation: $\mathcal{D}(\overline{\Omega}) := \{\phi_{|_{\overline{\Omega}}} \mid \phi \in \mathcal{D}(\mathbb{H}^2)\}$.

Proposition 2.1. Let Ω be any open subset of $\mathbb{H}^2$ such that $\overline{\Omega} \subseteq \mathbb{H}^2$ and $\overline{\Omega}$ is compact. Suppose that Ω has Lipschitz boundary. Then $\mathcal{D}(\overline{\Omega})$ is dense in $H^k(\Omega, \mu)$ $\forall \; k \geq 0$.

Proof. Let $k \geq 0$ be any integer. Let $H^k(\Omega)$ denote the Sobolev space of Ω with respect to the Euclidean measure. Then as sets $H^k(\Omega, \mu) = H^k(\Omega)$ and their norms are equivalent. Also by Theorem 1.2.7 of [7], $\mathcal{D}(\overline{\Omega})$ is dense in $H^k(\Omega)$. Hence $\mathcal{D}(\overline{\Omega})$ is dense in $H^k(\Omega, \mu)$.

□

Theorem 2.2 (Elliptic Regularity Theorem). *Let Ω be a convex polygonal domain in $\mathbb{H}^2$ such that $\overline{\Omega}$ is compact and $\overline{\Omega} \subset \mathbb{H}^2$. Fix $f \in L^2(\Omega, \mu)$. Suppose $u \in H^1_0(\Omega, \mu)$ satisfies $\Delta u = f$ in the sense of distribution on Ω. Then $u \in H^2(\Omega, \mu) \cap H^1_0(\Omega, \mu)$.*

Proof. Let $u \in H^1_0(\Omega, \mu)$ such that $\Delta u = f$. Then $u \in H^1_0(\Omega)$ and $y^2 \Delta^E u = f$, hence $\Delta^E u = \frac{f}{y^2}$. Since $\overline{\Omega} \subset \mathbb{H}^2$, there exists a constant $M = M(\Omega) > 0$ such that $\frac{1}{\text{Im}(z)^2} \leq M \; \forall \; z \in \overline{\Omega}$. Then $\frac{|f|}{y^2} \leq M\,|f|$ and $M\,f \in L^2(\Omega)$. Therefore $\frac{f}{y^2} \in L^2(\Omega)$. Then by Theorem 2.4.3 of [7], $u \in H^2(\Omega)$. Thus $u \in H^2(\Omega, \mu) \cap H^1_0(\Omega, \mu)$.

□

The following two results follow easily from the results referred to therein.

Theorem 2.3. *(Rellich)* [5] *For any bounded open subset Ω of $(\mathbb{H}^2, g)$ the inclusion map*

$$j : H^1_0(\Omega, \mu) \rightarrow L^2(\Omega, \mu)$$

is a compact operator.

Proposition 2.4. (Poincare's inequality) [10] Let Ω be any bounded open set of $(\mathbb{H}^2, g)$. Then there exists a constant $C = C(\Omega) > 0$ such that

$$||u||_{L^2(\Omega,\mu)} \leq C ||\nabla u||_{L^2(\Omega,\mu)} \; \forall \; u \in H^1_0(\Omega, \mu). \tag{2.4.1}$$

Throughout this paper, $C = C(\Omega)$ denotes a constant satisfying the inequality (2.4.1).

3. On the eigenvalues of $-L_{t,\alpha}$

Notation: $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product of $\mathbb{R}^2$.

Note that $L_{t,\alpha} : \left(H^2(\Omega,\mu) \cap H_0^1(\Omega,\mu), ||.||_{H^2(\Omega,\mu)}\right) \to L^2(\Omega,\mu)$ is a bounded operator.

Proposition 3.1. Let $u, v \in \left(H^2(\Omega,\mu) \cap H_0^1(\Omega,\mu), ||.||_{H^2(\Omega,\mu)}\right)$ be arbitrary. Let $C = C(\Omega)$ be a constant in the Poincare's inequality (2.4.1). Then

(1) $\langle -L_{t,\alpha}(u), v\rangle_{L^2(\Omega,\mu)} = \langle u, -L_{t,\alpha}(v)\rangle_{L^2(\Omega,\mu)}$,

(2) $\langle -L_{t,\alpha}(u), u\rangle_{L^2(\Omega,\mu)} \geq \inf\left\{C^{-2}, C^{-2}+\alpha\right\} ||u||_{L^2(\Omega,\mu)}$.

(3) Suppose $-L_{t,\alpha}(u) = \lambda u$ for some $u \neq 0$ in $H^2(\Omega,\mu) \cap H_0^1(\Omega,\mu)$. Then $\lambda \in \mathbb{R}$ and $\lambda \geq \inf\left\{C^{-2}, C^{-2}+\alpha\right\}$.

Proof. The Green's identity in the Euclidean plane is

$$\int_\Omega (-\Delta^E u)v \; dx\, dy = \int_\Omega \langle \nabla^E u, \nabla^E v\rangle \; dx\, dy = \int_\Omega u(-\Delta^E v) \; dx\, dy.$$

Let $dV := \frac{1}{y^2} dx \wedge \; dy$ denote the volume form of the hyperbolic plane $\mathbb{H}^2$. Then

$$\begin{aligned}
\langle \nabla^E u, \nabla^E v\rangle \; dx \wedge dy &= y^2 g\left(\nabla^E u, \nabla^E v\right) \; dx \wedge dy \\
&= y^2 g\left(\frac{1}{y^2}\nabla u, \frac{1}{y^2}\nabla v\right) \; dx \wedge \; dy \\
&= g\left(\nabla u, \nabla v\right) \; \frac{1}{y^2} dx \wedge \; dy \\
&= g\left(\nabla u, \nabla v\right) \; dV.
\end{aligned}$$

Next, $(-\Delta^E u)v \; dx \wedge \; dy = (-\Delta u)v \; \frac{1}{y^2} dx \wedge \; dy = (-\Delta u)v \; dV$ and similarly $(-u\Delta^E v) \; dx \wedge \; dy = u(-\Delta v) \; dV$. So we get

$$\begin{aligned}
\int_\Omega (-\Delta u)v \; d\mu &= \int_\Omega g\left(\nabla u, \nabla v\right) \; d\mu \\
&= \int_\Omega u(-\Delta v) \; d\mu \quad (u, v \in H^2(\Omega,\mu) \cap H_0^1(\Omega,\mu)). \quad (3.1.1)
\end{aligned}$$

Now given any $u, v \in H^2(\Omega,\mu) \cap H_0^1(\Omega,\mu)$ we have

$$\begin{aligned}
\langle -L_{t,\alpha}(u), v\rangle_{L^2(\Omega,\mu)} &= \int_\Omega (-\Delta u + \alpha\chi_{\wp_t} u)v \; d\mu \\
&= \int_\Omega (-\Delta u)v \; d\mu + \alpha \int_\Omega \chi_{\wp_t} uv \; d\mu \\
&= \int_\Omega u(-\Delta v) \; d\mu + \alpha \int_\Omega \chi_{\wp_t} uv \; d\mu \qquad (\text{ Using } (3.1.1)) \\
&= \int_\Omega (-\Delta v + \alpha\chi_{\wp_t} v)u \; d\mu \\
&= \langle u, -L_{t,\alpha}(v)\rangle_{L^2(\Omega,\mu)}.
\end{aligned}$$

Thus proof of (1) is complete.

Next,

$$\begin{aligned}
\langle -L_{t,\alpha}(u), u\rangle_{L^2(\Omega,\mu)} &= \int_\Omega (-\Delta u + \alpha\chi_{\wp_t} u)u \; d\mu \\
&= \int_\Omega (-\Delta u)u \; d\mu + \alpha \int_\Omega \chi_{\wp_t} u^2 \; d\mu \\
&= \int_\Omega g\left(\nabla u, \nabla u\right) \; d\mu + \alpha \int_\Omega \chi_{\wp_t} u^2 \; d\mu \quad (\text{ Using } (3.1.1)) \\
&\geq ||\nabla u||^2_{L^2(\Omega,\mu)} + \alpha \int_\Omega \chi_{\wp_t} u^2 \; d\mu \\
&\geq C^{-2}||u||^2_{L^2(\Omega,\mu)} + \alpha \int_\Omega \chi_{\wp_t} u^2 \; d\mu \quad (\text{ Using } (2.4.1))
\end{aligned}$$

Now inequality in (2) follows from following inequalities:

$$\langle -L_{t,\alpha}(u), u\rangle_{L^2(\Omega,\mu)} \geq \begin{cases} C^{-2}||u||^2_{L^2(\Omega,\mu)} & \text{if } \alpha > 0, \\ (C^{-2}+\alpha)||u||^2_{L^2(\Omega,\mu)} & \text{if } \alpha < 0. \end{cases}$$

By (1) it follows that $\lambda \in \mathbb{R}$. Then (3) follows from (2).

□

Theorem 3.2. *Suppose $\alpha > 0$. Then for any $f \in L^2(\Omega,\mu)$, there exist unique $u = u(f) \in H^1_0(\Omega,\mu)$ such that $-L_{t,\alpha}(u) = f$ in the sense of distributions on Ω.*

Proof. Define a symmetric bilinear form $\langle\cdot,\cdot\rangle_t$ on $H^1_0(\Omega,\mu)$ by

$$\langle u, v\rangle_t := \int_\Omega g\left(\nabla u, \nabla v\right) \; d\mu + \alpha \int_\Omega \chi_{\wp_t} uv \; d\mu. \tag{3.2.1}$$

Then $\langle\cdot,\cdot\rangle_t$ is an inner product on $H^1_0(\Omega,\mu)$ and

$$\begin{aligned}
||u||^2_t &\leq ||\nabla u||^2_{L^2(\Omega,\mu)} + \alpha||u||^2_{L^2(\Omega,\mu)} \\
&\leq (1+\alpha)\left(||\nabla u||^2_{L^2(\Omega,\mu)} + ||u||^2_{L^2(\Omega,\mu)}\right) \\
&= (1+\alpha)||u||^2_{H^1_0(\Omega,\mu)}.
\end{aligned}$$

Also by the Poincare Inequality (2.4.1),

$$||u||^2_{H^1_0(\Omega,\mu)} = ||\nabla u||^2_{L^2(\Omega,\mu)} + ||u||^2_{L^2(\Omega,\mu)} \leq (1+C^2)||\nabla u||^2_{L^2(\Omega,\mu)} \leq (1+C^2)||u||^2_t.$$

Thus the two norms $||.||_t$ and $||.||_{H^1_0(\Omega,\mu)}$ on the vector space $H^1_0(\Omega,\mu)$ are equivalent and hence $\left(H^1_0(\Omega,\mu), \langle\cdot,\cdot\rangle_t\right)$ is also a Hilbert space. Now given $f \in L^2(\Omega,\mu)$, define a map $\Lambda_f : \left(H^1_0(\Omega,\mu), \langle\cdot,\cdot\rangle_t\right) \to \mathbb{R}$ by

$$\Lambda_f(v) := \langle f, v\rangle_{L^2(\Omega,\mu)}, (v \in H^1_0(\Omega,\mu)).$$

Then Λ_f is a bounded linear functional on the Hilbert space $\left(H^1_0(\Omega,\mu), \langle\cdot,\cdot\rangle_t\right)$. By the Riesz representation theorem there exists unique $u = u_f \in H^1_0(\Omega,\mu)$ such that $\Lambda_f(v) := \langle u, v\rangle_t \; \forall \; v \in H^1_0(\Omega,\mu)$. So

$$\langle f, v\rangle_{L^2(\Omega,\mu)} = \int_\Omega g\left(\nabla u, \nabla v\right) \; d\mu + \alpha \int_\Omega \chi_{\wp_t} uv \; d\mu$$

and hence $-L_{t,\alpha}(u) = f$ in the sense of distributions on the domain Ω.

It remains to prove that u is unique. Suppose there exists $u_1, u_2 \in H_0^1(\Omega,\mu)$ such that $-L_{t,\alpha}(u_1) = f = -L_{t,\alpha}(u_2)$ in the sense of distributions on the domain Ω. Then $\langle u_1, v\rangle_t = -\langle f, v\rangle_{L^2(\Omega,\mu)} = \langle u_2, v\rangle_t \ \forall\ v \in H_0^1(\Omega,\mu)$. So $\langle u_1 - u_2, v\rangle_t = 0$ $\forall\ v \in H_0^1(\Omega,\mu)$ and hence $u_1 = u_2$.

□

Theorem 3.3. *Suppose* $\alpha > 0$. *Then*

$$-L_{t,\alpha} = -\Delta + \alpha\chi_{\wp_t} Id : \left(H^2(\Omega)\cap H_0^1(\Omega), ||.||_{H^2(\Omega)}\right) \to L^2(\Omega,\mu)$$

is an invertible bounded operator.

Proof. Clearly $-L_{t,\alpha} : \left(H^2(\Omega,\mu)\cap H_0^1(\Omega,\mu), ||.||_{H^2(\Omega,\mu)}\right) \to L^2(\Omega,\mu)$ is a bounded operator. By theorem 3.2 it is an injective map. To prove the operator $-L_{t,\alpha}$ is surjective, let $f \in L^2(\Omega,\mu)$ be arbitrary. By theorem 3.2 there exists $u \in H_0^1(\Omega,\mu)$ such that $-L_{t,\alpha}(u) = f$ in the sense of distributions in Ω. Then $-\Delta u = f - \alpha\chi_{\wp_t} u$ in the sense of distributions on Ω and $f - \alpha\chi_{\wp_t} u \in L^2(\Omega,\mu)$. Hence by theorem 2.2 $u \in H^2(\Omega,\mu)\cap H_0^1(\Omega,\mu)$. Therefore $-L_{t,\alpha} = f$ (in the operator sense) and $-L_{t,\alpha}$ is a surjective map. Since $-L_{t,\alpha} : \left(H^2(\Omega,\mu)\cap H_0^1(\Omega,\mu), ||.||_{H^2(\Omega,\mu)}\right) \to L^2(\Omega,\mu)$ is a bijective map and $H^2(\Omega,\mu), L^2(\Omega,\mu)$ are both Banach spaces. Now the open Mapping Theorem implies that $-L_{t,\alpha}$ is an open map and hence

$$(-L_{t,\alpha})^{-1} : L^2(\Omega,\mu) \to \left(H^2(\Omega,\mu)\cap H_0^1(\Omega,\mu), ||.||_{H^2(\Omega,\mu)}\right)$$

is a bounded operator.

□

Now we describe the spectrum of the operator $-L_{t,\alpha}$. First we consider the case where $\alpha > 0$. By theorem 3.3, $(-L_{t,\alpha})^{-1} : L^2(\Omega,\mu) \to \left(H^2(\Omega,\mu)\cap H_0^1(\Omega,\mu), ||.||_{H^2(\Omega,\mu)}\right)$ is a bounded operator and by theorem 2.3,

$$j : \left(H^2(\Omega,\mu)\cap H_0^1(\Omega,\mu), ||.||_{H^2(\Omega,\mu)}\right) \to L^2(\Omega,\mu)$$

is a compact operator. Hence the composite map

$$A_t := j\circ(-L_{t,\alpha})^{-1} : L^2(\Omega,\mu) \to L^2(\Omega,\mu)$$

is a compact operator. Since $H^2(\Omega,\mu)\cap H_0^1(\Omega,\mu)$ is dense in $L^2(\Omega,\mu)$, by the proposition 3.1, A_t is a non-negative symmetric operator with domain $L^2(\Omega,\mu)$. Since A_t is injective, its eigenvalues are strictly positive. By the theory of compact self adjoint operator [8], there exists a decreasing sequence $(\mu_n)_{n\in\mathbb{N}}$ converging to 0 consisting of all the eigenvalues of A_t counted with multiplicity and there exists a complete orthonormal basis of $L^2(\Omega,\mu)$ consisting of eigenfunctions of A_t. Let $u_n \in L^2(\Omega,\mu)$ be an eigenfunction of A_t associated with the eigenvalue μ_n. Then $\left(j\circ(-L_{t,\alpha})^{-1}\right)(u_n) = \mu_n u_n$ and hence $u_n \in H^2(\Omega,\mu)\cap H_0^1(\Omega,\mu)$. Therefore for each $n \in \mathbb{N}$, μ_n^{-1} is is an eigenvalue of $L_{t,\alpha}$. Also it is easy to see that if λ is an eigenvalue of the map $(-L_{t,\alpha} : H^2(\Omega,\mu)\cap H_0^1(\Omega,\mu) \to L^2(\Omega,\mu)$ then by theorem 3.3, $\lambda \neq 0$ and then λ^{-1} is an eigenvalue of the operator A_t. Thus $(\mu_n^{-1})_{n\in\mathbb{N}}$ is a sequence of all the eigenvalues of $-L_{t,\alpha}$ where each μ_n^{-1} is counted with multiplicity and $\mu_n^{-1} \to \infty$ as $n \to \infty$. By (3) of proposition 3.1, $\mu_n^{-1} \in [C^{-2}, \infty)\ \forall\ n \in \mathbb{N}$.

Next we consider the case $\alpha < 0$. Define $H_{t,\alpha} := L_{t,\alpha} + \alpha Id$. Then analogous results to theorem 3.2 and proposition 3.1 hold for operator $-H_{t,\alpha} := -L_{t,\alpha} + (-\alpha)Id$. Since $-\alpha > 0$, all eigenvalues of $-H_{t,\alpha}$ lie in $[C^{-2}, \infty)$. Clearly if η is an eigenvalue of $-H_{t,\alpha}$ with $-H_{t,\alpha}(u) = \eta u$ for some non-zero $u \in H_0^1(\Omega)$ if and only if $\eta + \alpha$

is an eigenvalue of $-L_{t,\alpha}$ with $-L_{t,\alpha}(u) = (\eta + \alpha)u$. Hence all the eigenvalues of $-L_{t,\alpha}$ lie in $[C^{-2} + \alpha, \infty)$ and both operators $-H_{t,\alpha}, -L_{t,\alpha}$ have the same collection of eigenfunctions, when $\alpha < 0$.

In the rest of this paper we denote the first eigenvalue of $-L_{t,\alpha}$ by $\lambda_{t,\alpha}$. Let

$$S := \{u \in H_0^1(\Omega, \mu) \ : \ ||u||_{L^2(\Omega,\mu)} = 1\}.$$

Define a function $J_{t,\alpha} : S \to \mathbb{R}$ by

$$J_{t,\alpha}(u) := \int_\Omega (||\nabla u||^2_{L^2(\Omega,\mu)} + \alpha \chi_{\wp_t} u^2)\, d\mu.$$

Then the function $J_{t,\alpha}$ is bounded below by the constant minimum$\{C^{-2}, C^{-2} + \alpha\}$. Let $(u_n)_{n\in\mathbb{N}}$ be a minimizing sequence in S for the functional $J_{t,\alpha}$. Since $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in $H_0^1(\Omega, \mu)$, it has a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ weakly converging to $H_0^1(\Omega, \mu)$ but strongly converging in $L^2(\Omega, \mu)$. As a consequence $u \in S$ and $J_{t,\alpha}$ attains its infimum at u. If $v \in H_0^1(\Omega, \mu)$, then so is $|v|$ ([4]), and $J_{t,\alpha}(v) = J_{t,\alpha}(|v|)$. Then by the variation principle $|u|$ is an eigenfunction of $-L_{t,\alpha}$ associated with the first eigenvalue $\inf_{v\in S} J_{t,\alpha}(v)$. Thus the infimum of the functional $J_{t,\alpha}$ is attained at a unique function $y_{t,\alpha} \in S$ characterized by

$$\int_\Omega y_{t,\alpha}^2\, d\mu = 1 \text{ and } y_{t,\alpha}(z) > 0 \ \forall\, z \in \Omega, \tag{3.3.1}$$

$$-L_{t,\alpha}(y_{t,\alpha}) = \lambda_{t,\alpha} y_{t,\alpha} \text{ on } \Omega \text{ in the sense of distributions,} \tag{3.3.2}$$

$$\lambda_{t,\alpha} = \inf_{v\in S} J_{t,\alpha}(v). \tag{3.3.3}$$

So $y_{t,\alpha}$ is the unique eigenfunction of $-L_{t,\alpha}$ associated to the first eigenvalue $\lambda_{t,\alpha}$ satisfying (3.3.1)

4. **Proof of Main Theorem 1.1**

Let D denote the Levi-Civita connection of the Riemannian manifold $(\mathbb{H}^2, g)$. Let $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$ be coordinate vectors on $\mathbb{H}^2$ with respect to coordinates $(x_1 := x, x_2 := y)$ of $\mathbb{H}^2$. Put $g_{i,j} := g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$ $(1 \le i, j \le 2)$. Let g^{ij} $(1 \le i, j \le 2)$ denote the (i, j)-th entry of the inverse of matrix $(g_{ij})_{2\times 2}$.

Definition 4.1. The divergence div(X) of a smooth tangent vector field X on $(\mathbb{H}^2, g)$ is defined as

$$\text{div}(X) := trace(DX) = \sum_{1\le i,j\le 2} g^{ij} g\left(D_{\frac{\partial}{\partial x_i}} X, \frac{\partial}{\partial x_j}\right).$$

Recall the notations $\wp_0, \wp_t$ introduced in (1.0.3). We construct a vector field V on $(\mathbb{H}^2, g)$ such that div$(V)(z) = 0 \ \forall\, z \in B_{\mathbb{H}^2}(i, r)$ and the 1-parameter group of isometries $\{\psi_t\}_{t\in\mathbb{R}}$ of $(\mathbb{H}^2, g)$ associated with V satisfy

$$\psi_t(\wp_0) = \wp_t \text{ and } \psi_t(\wp) = \wp \ \forall\, t \in \mathbb{R}.$$

Recall the rotation maps ρ_t of $(\mathbb{H}^2, g)$ such that $\rho_t(i) = i$ and the total derivative $(d\rho_t)_i$ is the rotation by angle t on the tangent space $T_z\mathbb{H}^2$ naturally identified with $\mathbb{C}$. See (1.0.2). The in-radius of $\wp$ is defined as $\sup\{s \in \mathbb{R} \ : \ \partial B_{\mathbb{H}^2}(i, s) \subset \wp\}$ and is denoted by inrad$(\wp)$. The polygon $\wp_0$ is circumscribed by the boundary circle $C = \partial B_{\mathbb{H}^2}(i, r)$. Let $r_1 := ((r + \text{inrad}(\wp)))/2$.

Let $\eta = \eta(t) : \mathbb{R} \to \mathbb{R}$ be a smooth function such that

$$\eta(t) \geq 0 \, \forall \, t \in \mathbb{R}, \; \eta(t) = 1 \, \forall \, |t| \leq r \; \& \; \eta(t) = 0 \, \forall \, |t| \geq r_1.$$

Define $\varphi : \mathbb{H}^2 \to \mathbb{R}$ by $\varphi(z) = \eta(d_{\mathbb{H}^2}(z, i)) \, \forall \, z \in \mathbb{H}^2$. Then

$$\varphi(z) \geq 0, \varphi(z) = 1 \, \forall \, z \in B_{\mathbb{H}^2}(i, r) \; \& \; \varphi(z) = 0 \, \forall \, z \in \mathbb{H}^2 \setminus B_{\mathbb{H}^2}(i, r_1).$$

Consider the vector field V of $(\mathbb{H}^2, g)$ defined by

$$V(z) := \varphi(z) \left(\frac{d}{dt} \rho_t(z) \bigg|_{t=0} \right) \; (z \in \mathbb{H}^2). \tag{4.1.1}$$

Then clearly $\psi_t(z) = \rho_t(z) \; \forall \; (t, z) \in \mathbb{R} \times B_{\mathbb{H}^2}(i, r)$, and $V(z) = \varphi(z)(1 + z^2)$ (see [3]).

Now we verify that $\operatorname{div}(V(z)) = 0 \, \forall \, z \in B_{\mathbb{H}^2}(i, r)$.

For $z \in B_{\mathbb{H}^2}(i, r)$, $V(z) = 1 + z^2 = \left(1 + x^2 - y^2, 2xy\right)$. Consider the coordinate vectors $e_1 = (1, 0), e_2 = (0, 1)$. Then $||e_1||_g = ||e_2||_g = \frac{1}{y}$ and $g(e_1, e_2) = 0$. The Christoffel symbols Γ^l_{ij} of $(\mathbb{H}^2, g)$ are calculated using

$$\Gamma^l_{ij} = \frac{1}{2} \sum_{k=1}^{2} (\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) g^{kl}.$$

Then $\Gamma^1_{11} = 0 = \Gamma^1_{22} = \Gamma^2_{21} = \Gamma^2_{12}$ and $\Gamma^2_{11} = \frac{1}{y}, \Gamma^2_{22} = -\frac{1}{y} = \Gamma^1_{12} = \Gamma^1_{21}$. So

$$\nabla_{e_1} e_1 = \frac{1}{y} e_2, \nabla_{e_1} e_2 = \nabla_{e_2} e_1 = \Gamma^1_{21} e_1 = -\frac{1}{y} e_1; \; \& \; \nabla_{e_2} e_2 = -\frac{1}{y} e_2.$$

Now

$$\begin{aligned}
& \nabla_{e_1} V \\
= \; & \nabla_{e_1} \left((1 + x^2 - y^2) \, e_1 + 2xy \, e_2 \right) \\
= \; & \frac{\partial}{\partial x} \left((1 + x^2 - y^2) \right) e_1 + \left((1 + x^2 - y^2) \right) \nabla_{e_1} e_1 + \frac{\partial}{\partial x} (2xy) \, e_2 + (2xy) \, \nabla_{e_1} e_2 \\
= \; & 2x e_1 + \left((1 + x^2 - y^2) \right) \left(\frac{1}{y} \right) e_2 + 2y e_2 + (2xy) \left(-\frac{1}{y} \right) e_1 \\
= \; & \left(\frac{1 + x^2 + y^2}{y} \right) e_2.
\end{aligned} \tag{4.1.2}$$

And

$$\begin{aligned}
& \nabla_{e_2} V \\
= \; & \nabla_{e_2} \left((1 + x^2 - y^2) e_1 + 2xy \, e_2 \right) \\
= \; & \frac{\partial}{\partial y} \left((1 + x^2 - y^2) \right) e_1 + \left((1 + x^2 - y^2) \right) \nabla_{e_2} e_1 + \frac{\partial}{\partial y} (2xy) \, e_2 + (2xy) \, \nabla_{e_2} e_2 \\
= \; & -2y e_1 + \left((1 + x^2 - y^2) \right) \left(-\frac{1}{y} \right) e_1 + 2x e_2 + (2xy) \left(-\frac{1}{y} \right) e_2 \\
= \; & -\left(\frac{1 + x^2 + y^2}{y} \right) e_1.
\end{aligned} \tag{4.1.3}$$

By (4.1.2) and (4.1.3),

$\operatorname{div}(V(z)) = g(\nabla_{e_1} V(z), e_1) + g(\nabla_{e_2} V(z), e_2) = \left(\frac{1 + x^2 + y^2}{y} \right) (g(e_2, e_1) - g(e_1, e_2)) = 0.$

Hence

$$\operatorname{div}(V(z)) = 0 \, \forall \, z \in B_{\mathbb{H}^2}(i, r). \tag{4.1.4}$$

Also

$$\rho_t(\wp_0) = \wp_t, \quad \rho_t(\wp) = \wp \; \forall \; t \in \mathbb{R}. \tag{4.1.5}$$

Let $j_t(z) := det(d\rho_t)_z, (z \in \mathbb{H}^2)$. Then

$$\left.\begin{array}{lll} j_0(z) & = 1 \; \forall \; z \in \wp, & \\ j_t(z) & = 1 \; \forall \; (t,z) \in \mathbb{R} \times B_{\mathbb{H}^2(i,r)}, \text{ and} & \\ j'_t(z)_{|t=0} & := \left.\frac{dj_t(z)}{dt}\right|_{t=0} = \operatorname{div}(V(z)) \; \forall \; z \in \wp. & \end{array}\right\} \tag{4.1.6}$$

Lemma 4.2. For $u \in \mathcal{C}^1(\overline{\wp})$,

$$\lim_{t\to 0} \frac{1}{t}\left(\int_{\wp_t} u \; d\mu - \int_{\wp_0} u \; d\mu\right) = \int_{\partial\wp_0} u \, g(V, \mathrm{n}) \; ds, \tag{4.2.1}$$

where $ds = \frac{dz}{(Im(z))^2}$ is the hyperbolic line element and n denotes the outward unit normal field of the domain $\wp_0$ on its boundary $\partial\wp_0$ away from its vertices.

Proof. By the change of variable formula for integration,

$$\lim_{t\to 0} \frac{1}{t}\left(\int_{\wp_t} u \; d\mu - \int_{\wp_0} u \; d\mu\right) = \lim_{t\to 0} \int_{\wp_0} \frac{(u \circ \rho_t)j_t - u}{t} \; d\mu. \tag{4.2.2}$$

Now on the open ball $B_{\mathbb{H}^2}(i,r)$, $\frac{\partial}{\partial t}((u\circ\rho_t)j_t) = g\left((\nabla u) \circ \rho_t, \frac{d\rho_t}{dt}\right) j_t + (u\circ\rho_t)j'_t$. Since $u \in \mathcal{C}^1(\overline{\wp})$, $(u \circ \psi_t)j_t \in \mathcal{C}^1(\mathbb{R} \times \wp)$. So for any $\delta > 0$, $\exists$ a constant $M > 0$ such that

$$\left|\frac{\partial}{\partial t}((u \circ \rho_t)j_t)(z)\right| \leq M \; \forall \; (t,z) \in [0,\delta] \times B_{\mathbb{H}^2}(i,r).$$

By (4.1.6), as functions on $B_{\mathbb{H}^2}(i,r)$

$$\begin{aligned} & g\left((\nabla u) \circ \rho_t, \frac{d\rho_t}{dt}\right) j_t + (u \circ \rho_t)j'_t \\ = \; & g\left((\nabla u) \circ \rho_t, \frac{d\rho_t}{dt}\right) \longrightarrow g(\nabla u, V) \text{ pointwise as } t \longrightarrow 0. \end{aligned}$$

So, on $B_{\mathbb{H}^2}(i,r)$, $\frac{\partial}{\partial t}|_{t=0}((u \circ \rho_t)j_t) = \lim_{t\to 0} \frac{(u\circ\rho_t)j_t - u}{t} = g(\nabla u, V)$.

Thus by Dominated Convergence theorem

$$\lim_{t\to 0} \int_{\wp_0} \frac{(u \circ \rho_t)j_t - u}{t} \; d\mu = \int_{\wp_0} g(\nabla u, V) \; d\mu. \tag{4.2.3}$$

By 4.1.4, on $B_{\mathbb{H}^2}(i,r)$

$$\operatorname{div}(uV) = g(\nabla u, V) + u \operatorname{div}(V) = g(\nabla u, V). \tag{4.2.4}$$

By (4.2.4) and Divergence theorem of Gauss for Riemannian manifolds applied in the second equality below),

$$\begin{aligned} \int_{\wp_0} g(\nabla u, V) \; d\mu &= \int_{\wp_0} \operatorname{div}(uV) d\mu \\ &= \int_{\partial\wp_0} g(uV, \mathrm{n}) \; ds \\ &= \int_{\partial\wp_0} u \; g(V, \mathrm{n}) \; ds. \end{aligned} \tag{4.2.5}$$

By (4.2.2), (4.2.3) and (4.2.5) we get (4.2.1).

□

Let $\mathcal{C}_b(\wp)$ denote the space of real valued, bounded continuous functions defined on $\wp$ with the sup-norm denoted by $||.||_\infty$.

Theorem 4.3. *Fix* $\alpha \in \mathbb{R}$.

(1) *There exists a constant* $M > 0$ *such that* $||y_{t,\alpha}||_\infty \leq M < \infty\ \forall\, t \in \mathbb{R}$.
(2) *There exists a constant* $C > 0$ *such that* $||\chi_{\wp_{t+h}} - \chi_{\wp_t}||_{L^1(\wp,\mu)} \leq C|h|\ \forall\, t \in \mathbb{R}$.
(3) *There exists a constant* $C' > 0$ *such that* $|\lambda_{t,\alpha} - \lambda_{0,\alpha}| \leq C'|t|\ \forall\, t \in \mathbb{R}$.
(4) $y_{t,\alpha}$ *converges to* $y_{0,\alpha}$ *strongly in* $L^2(\wp)$ *as* $t \to 0$.
(5) *The curve* $t \mapsto y_{t,\alpha}$ *is continuous map from* $\mathbb{R}$ *to* $\mathcal{C}_b(\wp)$.

Proof. **(1):** Since $\wp$ is convex polygonal domain, by Sobolev embedding theorem (theorem 5.4 of [1]), there exists an embedding $j : H^2(\wp, \mu) \subseteq \mathcal{C}_b(\wp)$, i.e. $\exists$ a constant $C_1 > 0$ such that

$$\|j(u)\|_\infty \leq C_1 \|u\|_{H^2(\wp,\mu)}\ \forall\, u \in H^2(\wp, \mu). \tag{4.3.1}$$

We denote $j(u)$ again by u.

By theorem 2.2.3 of [7], $\exists$ a constant $C_2 > 0$ such that

$$\|u\|_{H^2(\wp,\mu)} \leq C_2 \|\Delta u\|_{L^2(\wp,\mu)}. \tag{4.3.2}$$

Therefore

$$\begin{aligned} ||y_{t,\alpha}||_\infty &\leq C_1 C_2 ||\Delta y_{t,\alpha}||_{L^2(\wp,\mu)} \\ &= C_1 C_2 (|\lambda_{t,\alpha}| + |\alpha|)\, ||y_{t,\alpha}||_{L^2(\wp,\mu)} \\ &= C_1 C_2 (|\lambda_{t,\alpha}| + |\alpha|)\ \forall\, t. \end{aligned}$$

So it suffices to prove that

$$|\lambda_{t,\alpha} - \lambda_{0,\alpha}| \leq |\alpha|\ \forall\, t \in \mathbb{R}. \tag{4.3.3}$$

By the Rayleigh Principle we get,

$$\lambda_{0,\alpha} \leq J_{0,\alpha}(y_{t,\alpha}) = \int_\wp (||\nabla y_{t,\alpha}||^2 + \alpha \chi_{\wp_0} y_{t,\alpha}^2)\, d\mu. \tag{4.3.4}$$

and

$$\lambda_{t,\alpha} = J_{t,\alpha}(y_{t,\alpha}) = \int_\wp (||\nabla y_{t,\alpha}||^2 + \alpha \chi_{\wp_t} y_{t,\alpha}^2)\, d\mu. \tag{4.3.5}$$

Hence $\forall\, t \in \mathbb{R}$,

$$\begin{aligned} \lambda_{0,\alpha} - \lambda_{t,\alpha} &\leq \alpha \int_\wp (\chi_{\wp_0} - \chi_{\wp_t})\, y_{t,\alpha}^2 d\mu \qquad (4.3.6) \\ &\leq |\alpha| \int_\wp |\chi_{\wp_0} - \chi_{\wp_t}| y_{t,\alpha}^2 d\mu \\ &\leq |\alpha|. \end{aligned}$$

Similarly

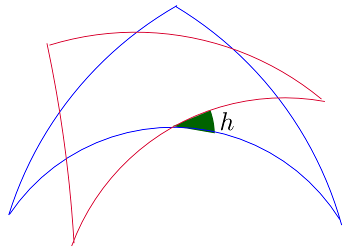
$$\lambda_{t,\alpha} - \lambda_{0,\alpha} \leq |\alpha|.$$

Thus estimate (4.3.3) is proved. So proof of **(1)** is now complete.

(2): Consider the difference T_h of sets $\wp_h, \wp_0$; i.e.

$$T_h := (\wp_h \cup \wp_0) \setminus (\wp_h \cap \wp_0)\ (h \in \mathbb{R}).$$

Then T_h is union of $2n$ small congruent triangles say $T_1, T_2, \cdots, T_{2n}$ with one angle as h and all sides of length not more than side length of $\wp_0$ equal to say a.



Now

$$|\chi_{\wp_h} - \chi_{\wp_0}|(x) = \begin{cases} 1 & x \in T_h, \\ 0 & \text{otherwise.} \end{cases}$$

and each triangle T_j $(1 \leq j \leq 2n)$ is contained in a sector of vertex angle h with two sides of lengths $\leq a$. Hence

$$\begin{aligned} ||\chi_{\wp_h} - \chi_{\wp_0}||_{L^1(\wp)} &= \sum_{j=1}^{2n} Area(T_j) \\ &\leq 2n \int_{r=0}^{a} \int_{\theta=0}^{h} \sinh(r)\, dr d\theta \\ &\leq 2n(\cosh a - 1)|h|. \end{aligned}$$

Proof of (2) is now complete.

(3): By the estimate (4.3.6),

$$|\lambda_{t,\alpha} - \lambda_{0,\alpha}| \leq |\alpha| \int_{\wp} |\chi_{\wp_t} - \chi_{\wp_0}|(y_{t,\alpha}^2 + y_{0,\alpha}^2)\, d\mu.$$

Then using (1) and (2) of theorem 4.3 we get

$$|\lambda_{t,\alpha} - \lambda_{0,\alpha}| \leq 2M|\alpha| \|\chi_{\wp_t} - \chi_{\wp_0}\|_{L^1(\wp,\mu)} \leq 2M|\alpha||t|.$$

Thus (3) is proved with $C' := 2M|\alpha|$.

(4): Consider any sequence $y_{t_n,\alpha}$ with $t_n \to 0$ as $n \to \infty$. By (1) above, $(y_{t_n,\alpha})_{n\in\mathbb{N}}$ is bounded in $H_0^1(\wp)$ and hence it converges weakly to some $v \in H_0^1(\wp)$. Since the embedding of $H_0^1(\wp)$ of $L^2(\wp)$ is a compact operator, $(y_{t_n,\alpha})_{n\in\mathbb{N}}$ has subsequence again denoted as $(y_{t_n,\alpha})_{n\in\mathbb{N}}$, which converges strongly in $L^2(\wp)$. Applying uniqueness of weak limits, we conclude that $(y_{t_n,\alpha})_{n\in\mathbb{N}}$ converges to v strongly in $L^2(\wp)$ and weakly in $H_0^1(\wp)$. Using the properties (3.3.1), (3.3.2) and (3.3.3) of the eigenfunction $y_{0,\alpha}$, it can be shown that $v = y_{0,\alpha}$. Proof of (4) is now complete.

(5):
$$\begin{aligned} \|y_{t,\alpha} - y_{0,\alpha}\|_\infty &\leq C_1 \|y_{t,\alpha} - y_{0,\alpha}\|_{H^2(\wp)} \quad \text{(by (4.3.1))} \\ &\leq C_1 C_2 \|\Delta(y_{t,\alpha} - y_{0,\alpha})\|_{L^2(\wp)} \quad \text{(by (4.3.2))} \\ &\leq C_1 C_2 (\|\alpha(\chi_{\wp_t} - \chi_{\wp_0}) y_{t,\alpha} + \alpha \chi_{\wp_0}(y_{t,\alpha} - y_{0,\alpha})\|_{L^2(\wp)} \\ &\quad + \|(\lambda_{0,\alpha} - \lambda_{t,\alpha}) y_{t,\alpha} - \lambda_{0,\alpha}(y_{t,\alpha} - y_{0,\alpha})\|_{L^2(\wp)}) \\ &\to 0 \text{ as } t \to 0 \quad \text{(by (1), (2), (3) \& (4) of theorem 4.3).} \end{aligned}$$

Proof of (5) is now complete.

□

Theorem 4.4 (Hadamard Formula). *Let* n_t *denote the outward unit normal field of the polygonal domain* $\wp_t$ *on its boundary* $\partial\wp_t$ *away from its vertices. Then* $t \mapsto \lambda_{t,\alpha}$ *is a differentiable function and*

$$\frac{d\lambda_{t,\alpha}}{dt} = \alpha \int_{\partial\wp_t} y_{t,\alpha}^2\, g(V, \mathrm{n}_t)\, ds. \tag{4.4.1}$$

Proof. Let $\phi, \psi \in H_0^1(\wp, \mu)$ be arbitrary. Then

$$\int_\wp g(\nabla y_{t+h,\alpha}, \nabla\phi)\, d\mu + \alpha \int_\wp (\chi_{\wp_{t+h}})(y_{t+h,\alpha})\phi\, d\mu = \lambda_{t+h,\alpha} \int_\wp (y_{t+h,\alpha})\phi\, d\mu, \tag{4.4.2}$$

and

$$\int_\wp g(\nabla y_{t,\alpha}, \nabla\psi)\, d\mu + \alpha \int_\wp (\chi_{\wp_t})(y_{t,\alpha})\psi\, d\mu = \lambda_{t,\alpha} \int_\wp (y_{t,\alpha})\psi\, d\mu. \tag{4.4.3}$$

Substitute $\phi = y_{t,\alpha}$ in (4.4.2), $\psi = y_{t+h,\alpha}$ in (4.4.3) and take the difference of two resulting equations. We get

$$\alpha \int_\wp (\chi_{\wp_{t+h}} - \chi_{\wp_t})(y_{t+h,\alpha})(y_{t,\alpha})\, d\mu = (\lambda_{t+h,\alpha} - \lambda_{t,\alpha}) \int_\wp (y_{t+h,\alpha})(y_{t,\alpha})\, d\mu, \tag{4.4.4}$$

Using substitution $\xi_{t,\alpha} := y_{t+h,\alpha} - y_{t,\alpha}$ in (4.4.4), for each $h \neq 0$,

$$\begin{aligned} &\frac{(\lambda_{t+h,\alpha} - \lambda_{t,\alpha})}{h} \int_\wp (y_{t+h,\alpha})(y_{t,\alpha})\, d\mu \\ = \;& \frac{\alpha}{h} \int_\wp (\chi_{\wp_{t+h}} - \chi_{\wp_t})(y_{t,\alpha}^2)\, d\mu + \alpha \int_\wp \frac{(\chi_{\wp_{t+h}} - \chi_{\wp_t})}{h} (\xi_{t,\alpha})(y_{t,\alpha})\, d\mu. \end{aligned} \tag{4.4.5}$$

By (3) of proposition (4.3) and (3.3.1),

$$\int_\wp (y_{t+h,\alpha})(y_{t,\alpha})\, d\mu \longrightarrow \int_\wp y_{t,\alpha}^2\, d\mu = 1 \text{ as } h \longrightarrow 0. \tag{4.4.6}$$

Next by the Lemma 4.2,

$$\begin{aligned} \lim_{h\to 0} \frac{1}{h} \int_\wp (\chi_{\wp_{t+h}} - \chi_{\wp_t})(y_{t,\alpha}^2)\, d\mu &= \lim_{h\to 0} \frac{1}{h} \left(\int_{\wp_{t+h}} y_{t,\alpha}^2\, d\mu - \int_{\wp_t} y_{t,\alpha}^2\, d\mu \right) \\ &= \int_{\partial\wp_t} y_{t,\alpha}^2\, g(V, \mathrm{n}_t)\, ds. \end{aligned} \tag{4.4.7}$$

Then

$$\begin{aligned} &\left| \int_\wp \frac{(\chi_{\wp_{t+h}} - \chi_{\wp_t})}{h} (\xi_{t,\alpha})(y_{t,\alpha})\, d\mu \right| \\ \leq \;& \int_\wp \left| \frac{\chi_{\wp_{t+h}} - \chi_{\wp_t}}{h} \right| |\xi_{t,\alpha}||y_{t,\alpha}|\, d\mu \\ \leq \;& M \int_\wp \left| \frac{\chi_{\wp_{t+h}} - \chi_{\wp_t}}{h} \right| |y_{t+h,\alpha} - y_{t,\alpha}|\, d\mu \qquad \text{by (1) of proposition 4.3} \\ \leq \;& MC \int_\wp |y_{t+h,\alpha} - y_{t,\alpha}|\, d\mu \qquad \text{by (2) of proposition 4.3} \\ = \;& MC ||y_{t+h,\alpha} - y_{t,\alpha}||_{L^1(\wp,\mu)}. \end{aligned}$$

So by (3) of proposition 4.3,

$$\left|\int_{\wp} \frac{(\chi_{\wp_{t+h}} - \chi_{\wp_t})}{h}(\xi_{t,\alpha})(y_{t,\alpha})\, d\mu\right| \leq MC||y_{t+h,\alpha} - y_{t,\alpha}||_{L^1(\wp,\mu)} \longrightarrow 0 \text{ as } h \longrightarrow 0. \tag{4.4.8}$$

Now (4.4.5),(4.4.6),(4.4.7) and (4.4.8) implies (4.4.1).

□

Proof of main theorem:

Recall that $\wp, \wp_0$ are open two regular polygons in $\mathbb{H}^2$ of n-sides circumscribed by the Euclidean circles $|z - i\cosh R| = \sinh R$ and $|z - i\cosh r| = \sinh r$ respectively such that one side of $\wp$ and one side $[b, c]$ of $\wp_0$ are orthogonal to the geodesic $\{iy \mid y > 0\}$ (see fig. 1). Further, recall that the first eigenvalue of the Schrödinger operator $-L_{t,\alpha}$ and the first eigenfunction $y_{t,\alpha}$ lies in $H^2(\wp,\mu) \cap H^1_0(\wp,\mu) \cap \mathcal{C}_b(\wp)$ and has the following properties:

$$\left.\begin{array}{rcll} -L_{t,\alpha}(y_{t,\alpha}) & = & \lambda_{t,\alpha} y_{t,\alpha} & \text{on } \wp, \\ y_{t,\alpha} & = & 0 & \text{on } \partial\wp, \\ y_{t,\alpha} & > & 0 & \text{on } \wp, \\ \int_{\wp} y^2_{t,\alpha}\, d\mu & = & 1 & \end{array}\right\} \tag{4.4.9}$$

The maps ρ_t are isometries of $(\mathbb{H}^2, g)$ with $\rho_{t+(2\pi/n)}(\wp_0) = \rho_t(\wp_0)$ and $\rho_{t+(2\pi/n)}(\wp) = \rho_t(\wp)\ \forall\, t \in \mathbb{R}$. Hence $\lambda_{t,\alpha}, y_{t,\alpha}$ are invariant under the rotation map $\rho_{2\pi/n}$.

Recall that the side $[b, c]$ of polygon $\wp_0$ is orthogonal to y-axis. Put $b_t := \rho_t(b)$ and $c_t := \rho_t(c)$. Then $[b_t, c_t]$ is a side of polygon $\wp_t$ (see fig.3).

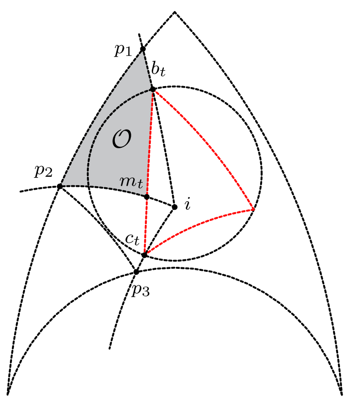


Figure 3 : $\wp_t$ ‘in general position’

Let n_t denote the outward unit field of $\wp_t$ on its boundary $\partial\wp_t$ away from its vertices. Since the rotation map $\rho_{2\pi/n}$ is an orientation preserving isometry of the polygon $\wp_t$, $\mathrm{n}_t(z) = \mathrm{n}_t(\rho_{2\pi/n}(z))\ \forall\, z \in \partial\wp_t$. Also by the construction of vector field V, the total derivative $(d\rho_t)_z(V(z)) = V(\rho_t(z))\ \forall\, t \in \mathbb{R}, \forall\, z \in \partial\wp_t$. Hence the function $g(\mathrm{n}_t, V)$ is invariant under rotation map $\rho_{2\pi/n}$. Therefore by the Hadamard formula,

$$\lambda'_{t,\alpha} = \alpha \int_{\partial\wp_t} y^2_{t,\alpha} g(\mathrm{n}_t, V)\, ds = n\alpha \int_{(b_t,c_t)} y^2_{t,\alpha} g(\mathrm{n}_t, V)\, ds$$

where $(b_t, c_t) := [b_t, c_t] \setminus \{b_t, c_t\}$ is an open side of $\partial\wp_t$ and n is the number of sides of the polygon $\wp_t$.

Let m be the mid-point of the side $[b, c]$ of $\wp_0$ and $\sigma : \mathbb{H}^2 \to \mathbb{H}^2$ be the reflection map about the y-axis (see figure1). Then $m_t = \rho_t(m)$ is the mid-point of side $[b_t, c_t]$ of $\wp_t$. Since the reflection map σ is a symmetry of $\wp_0$, $\sigma_t := \rho_t \circ \sigma \circ \rho_{-t}$ is an isometry in a neighborhood of $\wp_0$. Put $z^* = \sigma_t(z)\ \forall\ z \in \wp$. Thus

$$g(\mathrm{n}_t(z^*), V(z^*)) = g(d\sigma_t(\mathrm{n}_t(z)), -d\sigma_t(V(z))) = -g(\mathrm{n}_t(z), V(z))\ \forall\ z \in (b_t, m_t).$$

Thus

$$\lambda'_{t,\alpha} = n\alpha \int_{z\in(b_t,m_t)} \left(y^2_{t,\alpha}(z) - y^2_{t,\alpha}(z^*)\right) g(\mathrm{n}_t, V)\ ds. \tag{4.4.10}$$

When $t = 0$ or $t = 2\pi/n$, $\sigma_{|_\wp}$ is symmetry of both $\wp$, $\wp_0 = \wp_{2\pi/n}$ and hence $y_{t,\alpha}(\sigma(z)) = y_{t,\alpha}(z)\ \forall\ z \in \wp$. Then by (4.4.10), $\lambda'_{t,\alpha} = 0$ for $t = 0$ or $t = 2\pi/n$.

From now on we assume $0 < t < \pi/n$. It is proved in [3] that $g(\mathrm{n}_t(z), V(z)) > 0$ for all $z \in (b_t, m_t)$.

claim : $y_{t,\alpha}(z^*) > y_{t,\alpha}(z)$ for all $z \in (b_t, m_t)$.

Then by (4.4.10) it follows that $\lambda'_{t,\alpha} < 0$ if $\alpha > 0$ and $\lambda'_{t,\alpha} > 0$ if $\alpha < 0$. Hence the theorem.

Now we prove the claim. For $1 \le j \le 3$, let $\gamma_j : [0, \infty) \to \mathbb{H}^2$ be the rays which emanate from i and intersect $\partial\wp_t$ at b_t, m_t, c_t respectively. Further let the ray γ_j intersect $\partial\wp$ at the points p_j for each $j = 1, 2, 3$ (see figure 3). Let $\mathcal{O}$ denote the closed triangular region having i, p_1, p_2 as its vertices. Then $\sigma_t \circ \gamma_1 = \gamma_3$. Further, $\rho_{2\pi/n}$ is a symmetry of $\wp$ and $\rho_{2\pi/n}(z) = z^*$ for every point z of the ray γ_1. It follows that $\sigma_t(p_1) = p_3$ and $\sigma_t(\mathcal{O})$ is the triangle $[i, p_2, p_3]$. Thus $\sigma_t(\mathcal{O}) \subseteq \wp$.

Define a function $\omega : \mathcal{O} \to \mathbb{R}$ by

$$\omega(z) = y_{t,\alpha}(z) - y_{t,\alpha}(z^*)\ (z \in \mathcal{O}).$$

Since the rotation map $\rho_{2\pi/n}$ is a symmetry of both $\wp$ and $\wp_t$, $y_{t,\alpha}(z) = y_{t,\alpha}(z^*)$ $\forall\ z \in [i, p_1]$.Therefore $\omega = 0$ on $[i, p_1]$. By (4.4.9), $\omega < 0$ on open line segment (p_1, p_2). Also $z = z^*\ \forall\ z \in [i, p_2]$ and hence $\omega = 0$ on $[i, p_2]$.

Then $\omega^+(z) := \max\{\omega(z), 0\}$, $(z \in \mathcal{O})$ and $\omega^+ \in H^1_0(\mathcal{O})$.

Let $\mathcal{O}^+ := \{z \in \mathcal{O} \mid \omega(z) > 0\}$. Since $-L_{t,\alpha}\omega = \lambda_{t,\alpha}\omega$,

$$\begin{aligned} 0 &= \int_{\mathcal{O}} -L_{t,\alpha}(\omega)\omega^+\ d\mu - \lambda_{t,\alpha}\int_{\mathcal{O}} \omega\omega^+\ d\mu \\ &= \int_{\mathcal{O}^+} -L_{t,\alpha}(\omega^+)\omega^+\ d\mu - \lambda_{t,\alpha}\int_{\mathcal{O}^+} (\omega^+)^2\ d\mu \\ &= \int_{\mathcal{O}} -L_{t,\alpha}(\omega^+)\omega^+\ d\mu - \lambda_{t,\alpha}\int_{\mathcal{O}} (\omega^+)^2\ d\mu. \end{aligned}$$

Therefore,

$$\lambda_{t,\alpha}\int_{\mathcal{O}} (\omega^+)^2\ d\mu = \int_{\mathcal{O}} ||\nabla\omega^+||^2\ d\mu + \int_{\mathcal{O}} \chi_{\wp_t}(\omega^+)^2\ d\mu. \tag{4.4.11}$$

Let β be the first eigenvalue of $-\Delta + \alpha\chi_{\wp_t\cap\mathcal{O}}Id$ in $H^1_0(\mathcal{O})$. Then if $\omega^+ \neq 0$, then by the Raleigh Principle and (4.4.11),

$$\beta \le \frac{\int_{\mathcal{O}} ||\nabla\omega^+||^2\ d\mu + \int_{\mathcal{O}} \chi_{\wp_t}(\omega^+)^2\ d\mu}{\int_{\mathcal{O}}(\omega^+)^2\ d\mu} = \lambda_{t,\alpha}. \tag{4.4.12}$$

Since $\omega < 0$ on (p_1, p_2), $\mathcal{O}^+$ is a proper open subset of $\wp$. Now $\omega^+ \in H_0^1(\mathcal{O}) \subseteq H_0^1(\wp)$ and the eigenspace of $-L_{t,\alpha}$ corresponding to the first eigenvalue $\lambda_{t,\alpha}$ is one dimensional real vector subspace of $H_0^1(\wp)$. Hence by the Raleigh principle and (4.4.12) we have $\omega^+ = cy_{t,\alpha}$ for some non-zero real number c. This is not possible since $\omega^+(z) = 0\ \forall z \in \wp \setminus \mathcal{O}^+$ and $y_{t,\alpha}(z) > 0\ \forall z \in \wp$. Therefore $\omega^+ = 0$, i.e. $\omega < 0$ on ω. Hence the claim. The proof of the theorem is now complete.

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