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INEQUALITIES AMONG MEANS OF TWO POSITIVE ARGUMENTS IN INDEX (CONJUGATE INDEX) SETS

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Abstract. In this paper, means of two arguments in index and conjugate index sets are introduced using relevant necessary conditions and some properties have also been stated. An attempt is made to prove some well known Ky Fan type inequalities involving them.

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1. Introduction

A systematic study of the means and its inequalities has been carried out in classical book Inequalities by Hardy et al. [1]. Though this book brought some orderliness in presenting the work in the field of inequalities, it made no attempt at completeness; rather it contained fundamental contributions of each of three front rank mathematicians in the field. Such collections of results were continued in the book Analytic Inequalities by Mitrinovi´c [2].

Some of remarkable inequalities proved by Ky Fan came to light through the publication of these results in 1961, in the acclaimed book Inequalities by Beckenbach and Bellman [4]. A large collection of inequalities can be found in the book A Dictionary of Inequalities by Bullen [3] and also some convexity results on various important means and their applications to mean inequalities were found in [5–13].

It is of prime importance to consider an interval to define index and conjugate index sets. Such a consideration can be methodically deduced starting form the complete set of reals. Let $\mathbb{R}$ be the set of index numbers which is nothing but the set of real numbers

$$(-\infty,\ 0] \cup [0, \infty) = (-\infty, \infty) = \mathbb{R}.$$

For $a, b \in \mathbb{R}$, as the geometric mean $G(a, b) = \sqrt{ab}$ is not defined for negative arguments, the current discussion is restricted to the set of positive real numbers $\mathbb{R}^+$ instead of $\mathbb{R}$.

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Let $a, b \in \mathbb{R}^+$ be the index set. As can be found in [3], the corresponding conjugate index set numbers are defined as:

$$a^c = \frac{a}{a-1} \quad \text{and} \quad b^c = \frac{b}{b-1}.$$

It is clearly evident that a^c, b^c are not defined for $a = 1$ and $b = 1$. Therefore the interval for elements of conjugate index set is $a^c, b^c \in \mathbb{R}^+ - (0, 1]$. Further for $a^c, b^c \in (0, 1)$, a^c and b^c are negative and therefor $G(a^c, b^c) = \sqrt{a^c b^c}$ is not defined. Hence the present study is carried in the set $a, b \in (1, \infty)$.

2. Definitions and Propositions

It is essential to recall some of the definitions and propositions required to the develop this work.

Definition 2.1. *[3] For any $a, b \in \mathbb{R}^+ - (0, 1]$, the respective conjugates denoted by a^c, b^c are*

$$a^c = \frac{a}{a-1}; \; b^c = \frac{b}{b-1}.$$

Definition 2.2. *[3] For any $a, b \in \mathbb{R}^+ - (0, 1]$, the standard means A, G, H are given by*

$$A(a, b) = \frac{a+b}{2}; \; G(a, b) = \sqrt{ab}; \; H(a, b) = \frac{2ab}{a+b}.$$

The corresponding means of conjugates of a^c, b^c are given by

$$A^c(a^c, b^c) = \frac{a^c \; + \; b^c}{2}; \; G^c(a^c, b^c) = \sqrt{a^c b^c}; \; H^c(a^c, b^c) = \frac{2a^c b^c}{a^c + b^c}.$$

Proposition 2.1. *Let $a, b \in \boldsymbol{R}^+ - (0, 1]$, $a^c = \frac{a}{a-1}$, $b^c = \frac{b}{b-1}$ be conjugates of a and b respectively then*

$$(a^c)^c = a, (b^c)^c = b \tag{1}$$

$$\frac{1}{a} + \frac{1}{a^c} = 1, \frac{1}{b} + \frac{1}{b^c} = 1 \tag{2}$$

and

$$if\, a, b \in (1, \infty) \quad then \quad a^c, b^c \in (1, \infty). \tag{3}$$

Proposition 2.2. *Let $a, b \in (1, 2]$, $a^c = \frac{a}{a-1}$, $b^c = \frac{b}{b-1}$ be conjugates of a and b respectively. Then*

$$(i) \quad a > a^c \quad if \quad a > 2 \tag{4}$$

$$(ii) \quad a < a^c \quad if \quad 1 < a < 2 \tag{5}$$

$$(iii) \quad a = a^c \quad if \quad a = 2. \tag{6}$$

Proof. Consider

$$a - a^c = a - \frac{a}{a-1} = a \left[\frac{a-2}{a-1} \right] \tag{7}$$

Case[1]: If $a = 2$, $a - a^c = 0$ which proves (6).
Case[2]: If $a > 2$ then $a - 1 > 0$; $a - 2 > 0$; $a > 0$ together gives $(a - a^c) >$

0, this proves (4) that is $a > a^c$ for $a > 2$.
Case[3]: If $1 < a < 2$ then $a - 2 < 0$, $a - 1 > 0$ together leads to $(a - a^c) < 0$ (or) $a < a^c$, this proves (5). □

From the above analysis further results can be discussed for $a, b \in (1,2) \cup (2,\infty)$. Accordingly the results are discussed in two cases:

(1) For $a, b \in (1,2)$
(2) For $a, b \in (2,\infty)$.

The plot of $f(a) = a^c$ for various values of a is shown in Figure 1.

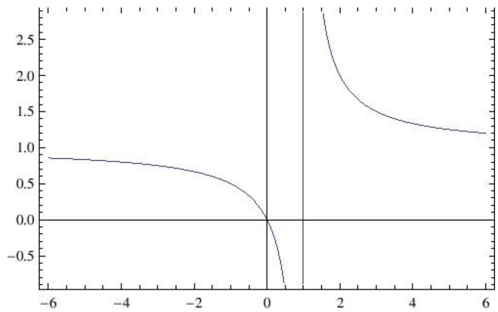

Figure 1. Graph of a^c for different values of a

3. Main Results

In this section some inequalities involving the means G, A, G^c and A^c have been established.

Theorem 3.1. *For* $1 < (a,b) < 2$*, then* $G \leq A \leq G^c \leq A^c$.

Proof. For $1 < (a,b) < 2$, then $a < a^c$, $b < b^c$; also $G \leq G^c$, $A \leq A^c$ and $G^c \leq A^c$. It is known that

$$a^c = f(a) = \frac{a}{a-1} \text{ is the function of } a$$
$$b^c = f(b) = \frac{b}{b-1} \text{ is the function of } b.$$

Further it is observed for $1 \leq (a,b) \leq 2$, $2 \leq (a^c, b^c) \leq \infty$, that is as (a,b) increases from 1 to 2 the value of (a^c, b^c) decreases from ∞ to 2; this proves that a^c and b^c are decreasing functions. According to the property of means

$$min(a,b) \leq A(a,b) \leq max(a,b)$$

and

$$min(a^c, b^c) \leq G^c(a^c, b^c) \leq max(a^c, b^c).$$

But the maximum value of $(a,b) \leq$ minimum value of (a^c, b^c). This proves that $A \leq G^c$. □

The plot of G, A, G^c and A^c are as shown below:

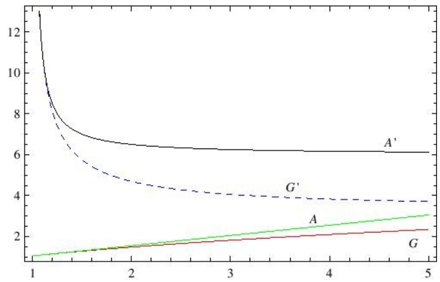

FIGURE 2. The plot of G, A, G^c and A^c

Theorem 3.2. *If $1 < a < b < 2$, then $G - G^c < A - A^c$.*

Proof. If $1 < a < b < 2$, then $a < a^c$, $b < b^c$. Consider,

$$(G - G^c) - (A - A^c) = \left(\sqrt{ab} - \frac{\sqrt{ab}}{\sqrt{AB}}\right) - \left[\frac{a+b}{2} - \frac{1}{2}\left(\frac{a}{A} + \frac{b}{B}\right)\right]$$

where $A = a - 1$ and $B = b - 1$, on simplification gives,

$$(G - G^c) - (A - A^c) = \left(\sqrt{AB} - 1\right)\frac{\sqrt{ab}}{\sqrt{AB}} - \frac{1}{2}\left(\frac{(AB)(a+b) - aB - bA}{AB}\right)$$

$$= \frac{1}{2AB}\left[2\left(AB - \sqrt{AB}\right)\sqrt{ab} - aAB - bAB + aB + bA\right]$$

$$= \frac{\triangle}{2AB}.$$

Here $\triangle$ on simplification leads to,

$$\triangle = -AB[\sqrt{a} - \sqrt{b}]^2 - [\sqrt{aB} - \sqrt{bA}]^2 < 0$$

This proves that $(G - G^c) - (A - A^c) < 0$. □

Theorem 3.3. *If $1 < a < b < 2$, then $\frac{A}{A^c} < \frac{G}{G^c}$ and equality holds when $a = b = 2$.*

Proof. Consider

$$AG^c - GA^c = \frac{(a+b)}{2}\sqrt{\frac{ab}{(a-1)(b-1)}} - \frac{\sqrt{ab}}{2}\left(\frac{a}{a-1} + \frac{b}{b-1}\right)$$

$$= \frac{\sqrt{ab}}{2}\left(\frac{a+b}{\sqrt{AB}} - \frac{a}{A} - \frac{b}{B}\right)$$

where $A = a - 1$ and $B = b - 1$. Since $a < b$ implies that $a - 1 < b - 1$ or $A < B$. Therefore

$$AG^c - GA^c = \frac{\sqrt{ab}}{2}\left(\frac{(a+b)\sqrt{AB} - aB - bA}{AB}\right)$$

$$= \frac{\sqrt{ab}}{2AB}\left[(\sqrt{A} - \sqrt{B})(a\sqrt{B} - b\sqrt{A})\right].$$

According to the property of mean for $1 \le (a, b) \le 2$ the harmonic mean $H(a, b)$ satisfies $1 \le H(a, b) \le\ 2$, on simplification gives

$$\frac{a+b}{2} \le ab \le a + b.$$

Case(i) For $a < b$; $(a - b)(ab - a - b) > 0$ on simplifications, we have $a^2(b - 1) > b^2(a - 1)$ and $(\sqrt{A} - \sqrt{B}) < 0$ on combining leads to:

$$\left[(\sqrt{A} - \sqrt{B})(a\sqrt{B} - b\sqrt{A})\right] < 0.$$

Case(ii) For $a > b$; $(a - b)(ab - a - b) < 0$ on simplifications, we have $a^2(b - 1) < b^2(a - 1)$ and $(\sqrt{A} - \sqrt{B}) > 0$ on combining leads to:

$$\left[(\sqrt{A} - \sqrt{B})(a\sqrt{B} - b\sqrt{A})\right] < 0.$$

From the above two cases proves that $AG^c - GA^c < 0$. □

4. Conclusion

Interesting inequalities have been proved among the important means of elements from index and conjugate index sets. More important result is the existence of KyFan type inequality between these means. A rigorous analysis may further lead to more important results.

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