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# SCHUR CONVEXITIES OF RATIO OF ONE PARAMETER POWER EXPONENTIAL MEAN AND ITS INVARIANT

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Abstract. In this paper, the ratio of one parameter power exponential and its invariant mean defined as a functions $F_\alpha(a,b)$ and $F^*_\alpha(a,b)$. Further, by finding the partial derivatives the various convexity conditions such as Schur, Schur geometric, Schur harmonic and $m$-power Schur convexities (concavities) are verified.

## 1. Introduction

The Greek Mathematicians introduced the concept of Mathematical means based on proportions in the fourth century A.D in the Pythagorean School. In literature it is evident that Mathematical means have a lot of applications in geometry and music. Later on several authors contributed and developed good number of results which are applicable to various branches of science and technology. In recent years, Lokesha et al., obtained the solution for an open problem raised by Rooin involving means[3] and an investigations on the homogeneous functions as a result some inequalities involving means are established [6, 7]. Studies on Greek means, the approach of new means and their generalizations leads to several inequality results were found in [5, 10, 11]. The good number of results on convexity, Schur convexity and related properties of means were discussed by Nagaraja et al., ([1, 2, 4], [12]-[19]). The results on ratio of difference of means are discussed in [17].The concept and detailed study on invariant and complementary means found in [20]. In [8, 9, 21] Zhen Hang yang et al., proposed the power exponential mean and invariant power exponential mean are respectively given by;

$$Z(a,b) = (a^a b^b)^{\frac{1}{a+b}} \quad \text{and} \quad Z^i(a,b) = (a^b b^a)^{\frac{1}{a+b}}.$$

In [22, 23], authors discussed some interesting results on invariant power exponential mean and one parameter power exponential mean. This work motivate to develop this paper.

**Definition 1.1.** For all positive real values $a$ and $b$, the parameter $\alpha > 0$, then one parameter power exponential mean is defined as:

$$Z(a,b,\alpha) = \begin{cases} (a^{a^\alpha} b^{b^\alpha})^{\frac{1}{a^\alpha + b^\alpha}} & where \quad \alpha \neq 0 \\ \sqrt{ab} & where \quad \alpha = 0 \\ (a^a b^b)^{\frac{1}{a+b}} & where \quad \alpha = 1 \end{cases}$$

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**Definition 1.2.** For all positive real values $a$ and $b$, the parameter $\alpha > 0$, then one parameter invariant power exponential mean is defined as:

$$Z^i(a,b,\alpha) = \begin{cases} (a^{b^\alpha} b^{a^\alpha})^{\frac{1}{a^\alpha+b^\alpha}} & where \quad \alpha \neq 0 \\ \sqrt{ab} & where \quad \alpha = 0 \\ (a^b b^a)^{\frac{1}{a+b}} & where \quad \alpha = 1 \end{cases}$$

**Lemma 1.3.** [2] *Let $\Omega \subseteq R^n$ be symmetric with non empty interior geometrically convex set and let $\varphi : \Omega \to R_+$ be continuous on $\Omega$ and differentiable in $\Omega^0$. If $\varphi$ is symmetric on $\Omega$ and*

$$(a-b)\left(\frac{\partial\varphi}{\partial a} - \frac{\partial\varphi}{\partial b}\right) \geq 0 (\leq 0). \tag{1.1}$$

$$(lna - lnb)\left(a\frac{\partial\varphi}{\partial a} - b\frac{\partial\varphi}{\partial b}\right) \geq 0 (\leq 0). \tag{1.2}$$

$$(a-b)\left(a^2\frac{\partial\varphi}{\partial a} - b^2\frac{\partial\varphi}{\partial b}\right) \geq 0 (\leq 0). \tag{1.3}$$

$$(a^m - b^m)\left(a^{1-m}\frac{\partial\varphi}{\partial a} - b^{1-m}\frac{\partial\varphi}{\partial b}\right) \geq 0 (\leq 0) \tag{1.4}$$

*holds for any $a, b \in \Omega^0$, then $\varphi$ is a Schur convex (concave), Schur-geometrically convex(concave), Schur-harmonically convex(concave) and m-power Schur convex(concave) function respectively.*

## 2. Main Results

In this section, the ratio of one parameter power exponential mean and its invariant are defined and hence the different kinds of Schur convexity conditions are discussed.

**Definition 2.1.** For all positive real values $a$ and $b$, the parameter $\alpha > 0$, then ratio of one parameter power exponential mean and its invariant function is defined as:

$$F_\alpha(a,b) = \begin{cases} \left(\frac{a}{b}\right)^{\frac{a^\alpha - b^\alpha}{a^\alpha+b^\alpha}} & where \quad \alpha \neq 0 \\ 1 & where \quad \alpha = 0 \\ F(a,b) = \left(\frac{a}{b}\right)^{\frac{a-b}{a+b}} & where \quad \alpha = 1 \end{cases}$$

**Definition 2.2.** For all positive real values $a$ and $b$, the parameter $\alpha > 0$, then The ratio of one parameter invariant power exponential mean and power exponential mean function is defined as:

$$F^*_\alpha(a,b) = \begin{cases} \left(\frac{b}{a}\right)^{\frac{a^\alpha - b^\alpha}{a^\alpha+b^\alpha}} & where \quad \alpha \neq 0 \\ 1 & where \quad \alpha = 0 \\ F^*(a,b) = \left(\frac{b}{a}\right)^{\frac{a-b}{a+b}} & where \quad \alpha = 1 \end{cases}$$

**Proposition 2.3.** *The ratio of one parameter power exponential mean and its invariant function $F_\alpha(a,b)$ is homogeneous.*

**Proposition 2.4.** *The ratio of one parameter invariant power exponential mean and power exponential mean function $F^*_\alpha(a,b)$ is homogeneous.*

*Proof.* The proofs of proposition 1 and proposition 2 are obvious. □

*Remark* 2.5. The ratio of one parameter power exponential mean and its invariant function $F_\alpha(a,b)$ and the ratio of one parameter invariant power exponential mean and power exponential mean function $F^*_\alpha(a,b)$ are not symmetric.

**Theorem 2.6.** *For $0 \le a \le b$, and $\alpha \ge 0$, the function $F_\alpha(a,b) = \left(\frac{a}{b}\right)^{\frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}}$ is Schur convex.*

*Proof.* Let $F_\alpha = \left(\frac{a}{b}\right)^{\frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}}$, take log on both sides, then

$$logF_\alpha = \frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}[loga - logb]$$

By finding the partial derivatives with respect to $a$ and $b$ gives

$$\frac{1}{F_\alpha}\frac{\partial F_\alpha}{\partial a} = \frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}\left(\frac{1}{a}\right) + log\left(\frac{a}{b}\right)(\alpha a^{\alpha-1})\left[\frac{2b^\alpha}{(a^\alpha + b^\alpha)^2}\right] \tag{2.1}$$

and

$$\frac{1}{F_\alpha}\frac{\partial F_\alpha}{\partial b} = \frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}\left(\frac{-1}{b}\right) + log\left(\frac{a}{b}\right)(\alpha b^{\alpha-1})\left[\frac{-2a^\alpha}{(a^\alpha + b^\alpha)^2}\right] \tag{2.2}$$

Thus,

$$(a-b)\left(\frac{\partial F_\alpha}{\partial a} - \frac{\partial F_\alpha}{\partial b}\right) = (a-b)F_\alpha\left[\frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}\left(\frac{a+b}{ab}\right) + log\left(\frac{a}{b}\right)\frac{2\alpha a^\alpha b^\alpha\left(\frac{a+b}{ab}\right)}{(a^\alpha + b^\alpha)^2}\right] > 0$$

This proves that $F_\alpha(a,b)$ satisfies the Schur convexity condition.

Hence, $F_\alpha(a,b) = \left(\frac{a}{b}\right)^{\frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}}$ is Schur convex. □

**Theorem 2.7.** *For $0 \le a \le b$, and $\alpha \ge 0$, the function $F_\alpha(a,b) = \left(\frac{a}{b}\right)^{\frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}}$ is Schur geometrically convex.*

*Proof.* Let $F_\alpha = \left(\frac{a}{b}\right)^{\frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}}$, take log on both sides, then

$$logF_\alpha = \frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}[loga - logb]$$

By finding the partial derivatives with respect to $a$ and $b$ gives

$$\frac{1}{F_\alpha}\frac{\partial F_\alpha}{\partial a} = \frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}\left(\frac{1}{a}\right) + log\left(\frac{a}{b}\right)(\alpha a^{\alpha-1})\left[\frac{2b^\alpha}{(a^\alpha + b^\alpha)^2}\right] \tag{2.3}$$

and

$$\frac{1}{F_\alpha}\frac{\partial F_\alpha}{\partial b} = \frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}\left(\frac{-1}{b}\right) + log\left(\frac{a}{b}\right)(\alpha b^{\alpha-1})\left[\frac{-2a^\alpha}{(a^\alpha + b^\alpha)^2}\right] \tag{2.4}$$

Thus,

$$(lna - lnb)\left(a\frac{\partial F_\alpha}{\partial a} - b\frac{\partial F_\alpha}{\partial b}\right) = log\left(\frac{a}{b}\right)F_\alpha\left[\frac{2(a^\alpha - b^\alpha)}{a^\alpha + b^\alpha} + log\left(\frac{a}{b}\right)\frac{2a^\alpha b^\alpha}{(a^\alpha + b^\alpha)^2}\right] > 0$$

This proves that $F_\alpha(a,b)$ satisfies the Schur geometrically convexity condition.
Hence, $F_\alpha(a,b) = \left(\frac{a}{b}\right)^{\frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}}$ is Schur geometrically convex. □

**Theorem 2.8.** *For $0 \le a \le b$, and $\alpha \ge 0$, the function $F_\alpha(a,b) = \left(\frac{a}{b}\right)^{\frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}}$ is Schur harmonically convex.*

*Proof.* Let $F_\alpha = \left(\frac{a}{b}\right)^{\frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}}$, take log on both sides, then

$$logF_\alpha = \frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}[loga - logb]$$

By finding the partial derivatives with respect to $a$ and $b$ gives

$$\frac{1}{F_\alpha}\frac{\partial F_\alpha}{\partial a} = \frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}\left(\frac{1}{a}\right) + log\left(\frac{a}{b}\right)(\alpha a^{\alpha-1})\left[\frac{2b^\alpha}{(a^\alpha + b^\alpha)^2}\right] \tag{2.5}$$

and

$$\frac{1}{F_\alpha}\frac{\partial F_\alpha}{\partial b} = \frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}\left(\frac{-1}{b}\right) + log\left(\frac{a}{b}\right)(\alpha b^{\alpha-1})\left[\frac{-2a^\alpha}{(a^\alpha + b^\alpha)^2}\right] \tag{2.6}$$

Thus,

$$(a-b)\left(a^2\frac{\partial F_\alpha}{\partial a} - b^2\frac{\partial F_\alpha}{\partial b}\right) = (a-b)F_\alpha\left[\frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}(a+b) + log\left(\frac{a}{b}\right)\frac{2a^\alpha b^\alpha (a+b)}{(a^\alpha + b^\alpha)^2}\right] > 0$$

This proves that $F_\alpha(a,b)$ satisfies the Schur harmonic convexity condition.
Hence, $F_\alpha(a,b) = \left(\frac{a}{b}\right)^{\frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}}$ is Schur harmonically convex. □

**Theorem 2.9.** *For $0 \le a \le b$, and $\alpha \ge 0$, the function $F_\alpha(a,b) = \left(\frac{a}{b}\right)^{\frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}}$ is $m$-power Schur convex, if $m > 0$ and $m$- power Schur concave, if $m < 0$.*

*Proof.* Let $F_\alpha = \left(\frac{a}{b}\right)^{\frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}}$, take log on both sides, then

$$logF_\alpha = \frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}[loga - logb]$$

By finding the partial derivatives with respect to $a$ and $b$ gives

$$\frac{1}{F_\alpha}\frac{\partial F_\alpha}{\partial a} = \frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}\left(\frac{1}{a}\right) + log\left(\frac{a}{b}\right)(\alpha a^{\alpha-1})\left[\frac{2b^\alpha}{(a^\alpha + b^\alpha)^2}\right] \tag{2.7}$$

and

$$\frac{1}{F_\alpha}\frac{\partial F_\alpha}{\partial b} = \frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}\left(\frac{-1}{b}\right) + log\left(\frac{a}{b}\right)(\alpha b^{\alpha-1})\left[\frac{-2a^\alpha}{(a^\alpha + b^\alpha)^2}\right] \tag{2.8}$$

Thus,

$$(a^m-b^m)\left(a^{1-m}\frac{\partial F_\alpha}{\partial a} - b^{1-m}\frac{\partial F_\alpha}{\partial b}\right) = (a^m-b^m)F_\alpha\left(\frac{a^m + b^m}{a^m b^m}\right)\left[\frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha} + \frac{2\alpha a^\alpha b^\alpha \log(\frac{a}{b})}{(a^\alpha + b^\alpha)^2}\right]$$

Hence, $F_\alpha(a,b) = \left(\frac{a}{b}\right)^{\frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}}$ is $m$-power Schur convex, if $m > 0$ and $m$- power Schur concave, if $m < 0$. □

**Theorem 2.10.** *For* $0 \leq a \leq b$, *and* $\alpha \geq 0$, *the function* $F_\alpha^*(a,b) = \left(\frac{b}{a}\right)^{\frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}}$ *is Schur concave.*

*Proof.* Let $F_\alpha^* = \left(\frac{b}{a}\right)^{\frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}}$, take log on both sides, then

$$logF_\alpha^* = \frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}[logb - loga]$$

By finding the partial derivatives with respect to $a$ and $b$ gives

$$\frac{1}{F_\alpha^*}\frac{\partial F_\alpha^*}{\partial a} = \frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}\left(\frac{-1}{a}\right) + log\left(\frac{b}{a}\right)(\alpha a^{\alpha-1})\left[\frac{2b^\alpha}{(a^\alpha + b^\alpha)^2}\right] \tag{2.9}$$

and

$$\frac{1}{F_\alpha^*}\frac{\partial F_\alpha^*}{\partial b} = \frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}\left(\frac{1}{b}\right) + log\left(\frac{b}{a}\right)(\alpha b^{\alpha-1})\left[\frac{-2a^\alpha}{(a^\alpha + b^\alpha)^2}\right] \tag{2.10}$$

Thus,

$$(a-b)\left(\frac{\partial F_\alpha^*}{\partial a} - \frac{\partial F_\alpha^*}{\partial b}\right) = (a-b)F_\alpha^*\left[\frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}\left(\frac{-a-b}{ab}\right) + log\left(\frac{b}{a}\right)\frac{2\alpha a^\alpha b^\alpha\left(\frac{a+b}{ab}\right)}{(a^\alpha + b^\alpha)^2}\right] < 0$$

This proves that $F_\alpha^*(a,b)$ satisfies the Schur concavity condition.
Hence, $F_\alpha^*(a,b) = \left(\frac{b}{a}\right)^{\frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}}$ is Schur concave. □

**Theorem 2.11.** *For* $0 \leq a \leq b$, *and* $\alpha \geq 0$, *the function* $F_\alpha^*(a,b) = \left(\frac{b}{a}\right)^{\frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}}$ *is Schur geometrically concave.*

*Proof.* Let $F_\alpha^* = \left(\frac{b}{a}\right)^{\frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}}$, take log on both sides, then

$$logF_\alpha^* = \frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}[logb - loga]$$

By finding the partial derivatives with respect to $a$ and $b$ gives

$$\frac{1}{F_\alpha^*}\frac{\partial F_\alpha^*}{\partial a} = \frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}\left(\frac{-1}{a}\right) + log\left(\frac{b}{a}\right)(\alpha a^{\alpha-1})\left[\frac{2b^\alpha}{(a^\alpha + b^\alpha)^2}\right] \tag{2.11}$$

and

$$\frac{1}{F_\alpha^*}\frac{\partial F_\alpha^*}{\partial b} = \frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}\left(\frac{1}{b}\right) + log\left(\frac{b}{a}\right)(\alpha b^{\alpha-1})\left[\frac{-2a^\alpha}{(a^\alpha + b^\alpha)^2}\right] \tag{2.12}$$

Thus,

$$(lna-lnb)\left(a\frac{\partial F_\alpha^*}{\partial a} - b\frac{\partial F_\alpha^*}{\partial b}\right) = log\left(\frac{a}{b}\right)F^*\left[\frac{-2(a^\alpha - b^\alpha)}{a^\alpha + b^\alpha} + log\left(\frac{b}{a}\right)\frac{2\alpha a^\alpha b^\alpha}{(a^\alpha + b^\alpha)^2}\right] < 0$$

This proves that $F_\alpha^*(a,b)$ satisfies the Schur geometrically concavity condition.
Hence, $F_\alpha^*(a,b) = \left(\frac{b}{a}\right)^{\frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}}$ is Schur geometrically concave. □

**Theorem 2.12.** *For $0 \leq a \leq b$, and $\alpha \geq 0$, the function $F_\alpha^*(a,b) = \left(\frac{b}{a}\right)^{\frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}}$ is Schur harmonically concave.*

*Proof.* Let $F_\alpha^* = \left(\frac{b}{a}\right)^{\frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}}$, take log on both sides, then

$$logF_\alpha^* = \frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}[logb - loga]$$

By finding the partial derivatives with respect to $a$ and $b$ gives

$$\frac{1}{F_\alpha^*}\frac{\partial F_\alpha^*}{\partial a} = \frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}\left(\frac{-1}{a}\right) + log\left(\frac{b}{a}\right)(\alpha a^{\alpha-1})\left[\frac{2b^\alpha}{(a^\alpha + b^\alpha)^2}\right] \tag{2.13}$$

$$\frac{1}{F_\alpha^*}\frac{\partial F_\alpha^*}{\partial b} = \frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}\left(\frac{1}{b}\right) + log\left(\frac{b}{a}\right)(\alpha b^{\alpha-1})\left[\frac{-2a^\alpha}{(a^\alpha + b^\alpha)^2}\right] \tag{2.14}$$

Thus,

$$(a-b)\left(a^2\frac{\partial F_\alpha^*}{\partial a} - b^2\frac{\partial F_\alpha^*}{\partial b}\right) = (a-b)F_\alpha^*\left[\frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}(-a-b) + log\left(\frac{b}{a}\right)\frac{2\alpha a^\alpha b^\alpha(a+b)}{(a^\alpha + b^\alpha)^2}\right] < 0$$

This proves that $F_\alpha^*(a,b)$ satisfies the Schur harmonic concavity condition.
Hence, $F_\alpha^*(a,b) = \left(\frac{b}{a}\right)^{\frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}}$ is Schur harmonically concave. □

**Theorem 2.13.** *For $0 \leq a \leq b$, and $\alpha \geq 0$, the function $F_\alpha^*(a,b) = \left(\frac{b}{a}\right)^{\frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}}$ is $m$-power Schur concave, if $m > 0$ and $m$- power Schur convex, if $m < 0$.*

*Proof.* Let $F_\alpha^* = \left(\frac{b}{a}\right)^{\frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}}$, take log on both sides, then

$$logF_\alpha^* = \frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}[logb - loga]$$

By finding the partial derivatives with respect to $a$ and $b$ gives

$$\frac{1}{F_\alpha^*}\frac{\partial F_\alpha^*}{\partial a} = \frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}\left(\frac{-1}{a}\right) + log\left(\frac{b}{a}\right)(\alpha a^{\alpha-1})\left[\frac{2b^\alpha}{(a^\alpha + b^\alpha)^2}\right] \tag{2.15}$$

and

$$\frac{1}{F_\alpha^*}\frac{\partial F_\alpha^*}{\partial b} = \frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}\left(\frac{1}{b}\right) + log\left(\frac{b}{a}\right)(\alpha b^{\alpha-1})\left[\frac{-2a^\alpha}{(a^\alpha + b^\alpha)^2}\right] \tag{2.16}$$

Thus,

$$(a^m-b^m)\left(a^{1-m}\frac{\partial F_\alpha^*}{\partial a} - b^{1-m}\frac{\partial F_\alpha^*}{\partial b}\right) = (a^m-b^m)F_\alpha^*\left(\frac{a^m + b^m}{a^m b^m}\right)\left[\frac{2\alpha a^\alpha b^\alpha \log(\frac{b}{a})}{(a^\alpha + b^\alpha)^2} - \frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}\right]$$

Hence, $F_\alpha^*(a,b) = \left(\frac{b}{a}\right)^{\frac{a^\alpha - b^\alpha}{a^\alpha + b^\alpha}}$ is $m$-power Schur concave, if $m > 0$ and $m$- power Schur convex, if $m < 0$. □

## 3. CONCLUSION

This paper provides the verification of various convexity conditions such as Schur, Schur geometric, Schur harmonic and $m$-power Schur convexities(concavities) of ratio of one parameter power exponential mean and its invariant.

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