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SOME CONGRUENCE PROPERTIES FOR ℓ-REGULAR PARTITIONS, WHERE $\ell \in \{4, 8, 13, 17, 19, 25, 40, 55\}$

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Abstract. A partition of n is ℓ-regular if none of its part is divisible by ℓ. Let $b_\ell(n)$ denote the number of ℓ-regular partitions of n. In this article, using theta functions due to ramanujan, we establish several new congruences modulo k for $b_\ell(n)$, where $k \in \{2, 11, 17, 19, 25\}$ and $\ell \in \{4, 8, 13, 17, 19, 25, 40, 55\}$.

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1. Introduction

A partition of a positive integer n is a finite non-increasing sequence of positive integers λ_1, λ_2,..., λ_k such that $\sum_{i=1}^{k}\lambda_i{=}n$. For a positive integer $\ell \geq 2$, a partition of n is called ℓ-regular if none of its part is divisible by ℓ. Let $b_\ell(n)$ denote the number of ℓ-regular partitions of n. The generating function for $b_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{f_\ell}{f_1}, \tag{1.1}$$

where for any integer k, $f_k = f(-q^k)$.
Throughout this paper, we assume that $|q| < 1$. For a positive integer k, we use the standard notation

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k).$$

Several mathematicians had extensively studied the arithmetic properties of $b_\ell(n)$. Cui and Gu [12] established several infinite families of congruences for $b_\ell(n)$, where $\ell \in \{2, 4, 5, 8, 13, 16\}$. Dai et al. [13] proved various results on the distribution of odd values of 2^a-regular partition functions. Xia [21] established several congruences for $b_\ell(n)$, where $\ell \in \{13, 17, 19\}$ by employing some theta functions due to Ramanujan. Carlson and Webb [10] established congruences modulo 5 for $b_\ell(n)$, where $\ell \in \{10, 15, 20\}$. For more details, see [2, 8, 14, 16, 18, 20].
Recently, Abinash et al. [1] found a new infinite families of congruence for $b_\ell(n)$, where $\ell \in \{17, 23, 65\}$ using theta function identities due to Ramanujan.

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In sequel, in the present paper, we establish some infinite families of congruences for $b_\ell(n)$, $\ell \in \{4, 8, 13, 17, 19, 25, 40, 55\}$ using Ramanujan functions. The following are our main results.

Theorem 1.1. *For any integer $n \geq 0$ and $k \geq 1$,*

$$b_{40}\left(5^{4k}n + \frac{13}{8}\left(5^{4k} - 1\right)\right) \equiv b_{40}(n) \pmod 2, \tag{1.2}$$

$$b_{40}\left(25n + 9\right) \equiv b_{40}\left(25n + 19\right) \equiv 0 \pmod 2. \tag{1.3}$$

Theorem 1.2. *For any integer $n \geq 0$ and $k \geq 1$,*

$$b_{55}\left(5^{3k}n + \frac{9}{4}\left(5^{3k} - 1\right)\right) \equiv 6^k b_{55}(n) \pmod{11}. \tag{1.4}$$

Theorem 1.3. *For any integer $n \geq 0$ and $k \geq 1$,*

$$b_{4}\left(7^{2k}n + \frac{1}{8}\left(7^{2k} - 1\right)\right) \equiv b_{4}(n) \pmod 2, \tag{1.5}$$

$$b_{4}\left(7^{k+1}n + \frac{1}{6}\left(7r(7^{k} - 1) + 36\right)\right) \equiv 0 \pmod 2, \tag{1.6}$$

where $r \in \{1, 2, 3, 4, 5, 6\}$.

Theorem 1.4. *For any integer $n \geq 0$,*

$$b_{8}\left(49n + 21\right) \equiv b_{8}\left(49n + 28\right) \equiv b_{8}\left(49n + 42\right) \equiv 0 \pmod 2. \tag{1.7}$$

Theorem 1.5. *For any integer $n \geq 0$ and $k \geq 1$,*

$$b_{4}\left(13^{2k}n + \frac{1}{8}\left(13^{2k} - 1\right)\right) \equiv b_{4}(n) \pmod 2, \tag{1.8}$$

$$b_{4}\left(13^{k+1}n + \frac{13r}{12}\left(13^{k} - 1\right) + 8\right) \equiv 0 \pmod 2, \tag{1.9}$$

where $r \in \{0, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

Theorem 1.6. *For any integer $n \geq 0$ and $k, l \geq 0$,*

$$\sum_{n=0}^{\infty} b_{13}\left(4 \cdot 7^{2k} \cdot 13^{2l}n + \frac{7^{2k} \cdot 13^{2l} - 1}{2}\right)q^n \equiv f_1^3 \pmod 2, \tag{1.10}$$

$$b_{13}\left(4 \cdot 7^{2k+2}n + 4 \cdot 7^{2k+1}r + \frac{7^{2k+2} - 1}{2}\right) \equiv 0 \pmod 2, \tag{1.11}$$

$$b_{13}\left(4 \cdot 7^{2k} \cdot 13^{2l+2}n + \frac{(8s + 5)7^{2k} \cdot 13^{2l+1} - 1}{2}\right) \equiv 0 \pmod 2, \tag{1.12}$$

where $r \in \{1, 2, 3, 4, 5, 6\}$ and $s \in \{0, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

Theorem 1.7. *For any integer $n \geq 0$ and $k, l \geq 0$,*

$$\sum_{n=0}^{\infty} b_{13}\left(4 \cdot 13^{2k} \cdot 7^{2l} n + \frac{13^{2k} \cdot 7^{2l} - 1}{2}\right) q^n \equiv f_1^3 \pmod{2}, \tag{1.13}$$

$$b_{13}\left(4 \cdot 13^{2k+2} n + \frac{(8r+5)13^{2k+1} - 1}{2}\right) \equiv 0 \pmod{2}, \tag{1.14}$$

$$b_{13}\left(4 \cdot 13^{2k} \cdot 7^{2l+2} n + 4 \cdot 13^{2k} \cdot 7^{2l+1} s + \frac{13^{2k} \cdot 7^{2l+2} - 1}{2}\right) \equiv 0 \pmod{2}, \tag{1.15}$$

where $r \in \{0, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and $s \in \{1, 2, 3, 4, 5, 6\}$.

Theorem 1.8. *For any integer $n \geq 0$ and $k, l \geq 0$,*

$$\sum_{n=0}^{\infty} b_{13}\left(4. \cdot 7^{2k} \cdot 13^{2l+1} n + \frac{7^{2k} \cdot 13^{2l+1} - 1}{2}\right) q^n \equiv f_1^3 \pmod{2}, \tag{1.16}$$

$$b_{13}\left(52n + 4r + 2\right) \equiv 0 \pmod{2}, \tag{1.17}$$

$$b_{13}\left(52 \cdot 7^{2k+2} n + 52 \cdot 7^{2k+1} s + \frac{13 \cdot 7^{2k+2} - 1}{2}\right) \equiv 0 \pmod{2}, \tag{1.18}$$

$$b_{13}\left(4 \cdot 7^{2k} \cdot 13^{2l+3} n + \frac{(8t+5)7^{2k} \cdot 13^{2l+2} - 1}{2}\right) \equiv 0 \pmod{2}, \tag{1.19}$$

where $r \in \{0, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, $s \in \{1, 2, 3, 4, 5, 6\}$ and $t \in \{0, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

Theorem 1.9. *For any integer $n \geq 0$ and $k, l \geq 0$,*

$$\sum_{n=0}^{\infty} b_{13}\left(4. \cdot 13^{2k+1} \cdot 7^{2l} n + \frac{13^{2k+1} \cdot 7^{2l} - 1}{2}\right) q^n \equiv f_1^3 \pmod{2}, \tag{1.20}$$

$$b_{13}\left(4 \cdot 13^{2k+3} n + \frac{(8r+5)13^{2k+2} - 1}{2}\right) \equiv 0 \pmod{2}, \tag{1.21}$$

$$\begin{aligned} b_{13}\Big(4 \cdot 13^{2k+1} \cdot 7^{2l+2} n &+ 4 \cdot 13^{2k+1} \cdot 7^{2l+1} s \\ &+ \frac{13^{2k+1} \cdot 7^{2l+2} - 1}{2}\Big) \equiv 0 \pmod{2}, \end{aligned} \tag{1.22}$$

where $r \in \{0, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and $s \in \{1, 2, 3, 4, 5, 6\}$.

Theorem 1.10. *For any integer $n \geq 0$ and $k \geq 0$,*

$$b_{17}\left(13^{9k} n + \frac{2}{3}\left(13^{9k} - 1\right)\right) \equiv (15)^k b_{17}(n) \pmod{17}, \tag{1.23}$$

$$b_{17}\left(13^{9k+9} n + \frac{1}{3}\left(5(r+1) \cdot 13^{9k+8} - 2\right)\right) \equiv 0 \pmod{17}, \tag{1.24}$$

where $r \in \{0, 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12\}$.

Theorem 1.11. *For any integer $n \geq 0$ and $k \geq 0$,*

$$b_{19}\left(13^{19k}n + \frac{3}{4}\left(13^{19k} - 1\right)\right) \equiv (15)^k b_{19}(n) \pmod{19}, \tag{1.25}$$

$$b_{19}\left(13^{19k+19}n + \frac{1}{4}\left(7(r+1)\cdot 13^{19k+18} - 3\right)\right) \equiv 0 \pmod{19}, \tag{1.26}$$

where $r \in \{0,1,2,3,4,5,6,7,8,10,11,12\}$.

Theorem 1.12. *For any integer $n \geq 0$ and $k \geq 0$,*

$$b_{25}\left(5\cdot 13^{30k}n + 5\cdot 13^{30k} - 1\right) \equiv (18)^k b_{25}(5n+4) \pmod{25}, \tag{1.27}$$

$$b_{25}\left(5\cdot 13^{30k+30}n + 5(r+1)\cdot 13^{30k+29} - 1\right) \equiv 0 \pmod{25}, \tag{1.28}$$

where $r \in \{0,1,2,3,4,5,6,7,8,9,10,11\}$.

2. Preliminaries

In this section, we recall some results to prove our main results.
The following identities holds.
[17, p. 212]

$$f_1 = f_{25}\left(\frac{1}{R(q^5)} - q - q^2R(q^5)\right), \tag{2.1}$$

[7, eq. (7.4.14), p. 165]

$$\begin{aligned}\frac{1}{f_1} = \frac{f_{25}^5}{f_5^6}\Bigg(&\frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 - 3q^5R(q^5) \\ &+ 2q^6R(q^5)^2 - q^7R(q^5)^3 + q^8R(q^5)^4\Bigg).\end{aligned} \tag{2.2}$$

The following identity hold [19].

$$\frac{f_1^6}{f_5^6} = \frac{1}{R(q)^5} - 11q - q^2R(q)^5, \tag{2.3}$$

where

$$R(q) = \frac{(q,q^4;q^5)_\infty}{(q^2,q^3;q^5)_\infty}.$$

From [4, p. 303, Entry 17(v)], we have

$$f_1 = f_{49}\left(\frac{B(q^7)}{C(q^7)} - q\frac{A(q^7)}{B(q^7)} - q^2 + q^5\frac{C(q^7)}{A(q^7)}\right), \tag{2.4}$$

where $A = A(q) =: \frac{f(-q^3,-q^4)}{f(-q^2)}$, $B = B(q) =: \frac{f(-q^2,-q^5)}{f(-q^2)}$,
and $C = C(q) =: \frac{f(-q,-q^6)}{f(-q^2)}$.

From [5, p. 174, Entry 31] and [6, eq. 3.11, and eq. 3.15]

$$\frac{B^5}{AC^4}-\frac{A^5}{B^4C}-\frac{C^5q^3}{A^4B} = 3q, \tag{2.5}$$

$$\frac{AB^2}{C^3}+\frac{A^2Cq}{B^3}+\frac{BC^2q^2}{A^3} = \frac{f_1^4}{f_7^4}+8q, \tag{2.6}$$

$$-\frac{A^3}{BC^2}-\frac{B^3q}{A^2C}-\frac{C^3q^2}{AB^2} = \frac{f_1^4}{f_7^4}+5q, \tag{2.7}$$

$$\frac{B^7}{C^7}-\frac{A^7q}{B^7}+\frac{C^7q^5}{A^7} = \frac{f_1^8}{f_7^8}+14q\frac{f_1^4}{f_7^4}+57q^2. \tag{2.8}$$

The following 13-dissection identity holds [4, p. 373, Entry 8(i), eq. (8.1)].

$$\begin{aligned} f_1 = f_{169}\bigg(&\frac{L(q^{13})}{J(q^{13})}-q\frac{N(q^{13})}{K(q^{13})}-q^2\frac{J(q^{13})}{I(q^{13})}+q^5\frac{M(q^{13})}{L(q^{13})} \\ &+q^7-q^{12}\frac{K(q^{13})}{M(q^{13})}+q^{22}\frac{I(q^{13})}{N(q^{13})}\bigg), \end{aligned} \tag{2.9}$$

where

$$I(q) =: f(-q,-q^{12}), J(q) =: f(-q^2,-q^{11}), K(q) =: f(-q^3,-q^{10}),$$
$$L(q) =: f(-q^4,-q^9), M(q) =: f(-q^5,-q^8), N(q) =: f(-q^6,-q^7).$$

From Hirschhorn [15, pp. 323-325] we note that, if

$$\eta := \frac{f_1}{q^7 f_{169}} \text{ and } S := \frac{f_{13}^2}{q^{13} f_{169}^2}, \tag{2.10}$$

then

$$\begin{aligned} H_{13}(\eta) &= 1, \\ H_{13}(\eta^2) &= -2S-1, \\ H_{13}(\eta^3) &= 13, \\ H_{13}(\eta^4) &= 2S^2-13, \\ H_{13}(\eta^5) &= -20S^2-10\times 13S-13^2, \\ H_{13}(\eta^6) &= 10S3-13^2, \\ H_{13}(\eta^7) &= 98S^3+28\times 13S^2-13^3, \\ H_{13}(\eta^8) &= -70S^4-13^3, \\ H_{13}(\eta^9) &= -162S^4+108\times 13S^3+72\times 13^2S^2+18\times 13^3S+13^4, \\ H_{13}(\eta^{10}) &= 238S^5=5-13^4, \\ H_{13}(\eta^{11}) &= -902S^5-1672\times 13S^4-792\times 13^2S^3-198\times 13^3S^2 \\ &\quad -22\times 13^4S-13^5, \\ H_{13}(\eta^{12}) &= -418S^6-13^5. \end{aligned} \tag{2.11}$$

Also, from [15, Eq.(6.65)], We have
(2.12)

$$\begin{aligned}\eta^{13} &= 13\eta^{12} - (13S+91)\eta^{11} + (169S+507)\eta^{10} - (78S^2+1183S+2535)\eta^9\\ &+ (962S^2+6253S+10985)\eta^8 - (260S^3+6422S^2+28561S+41743)\eta^7\\ &+ (2886S^3+31096S^2+112047S+142805)\eta^6\\ &- (494S^4+17238S^3+123032S^2+371293S+428415)\eta^5\\ &+ (4498S^4+71318S^3+404248S^2+1056757S+1113879)\eta^4\\ &- (468S^5+21294S^4+224094S^3+1085318S^2+2599051S+2599051)\eta^3\\ &+ (2652S^5+58474S^4+487734S^3+2113514S^2+4826809S\\ &+ 4826809)\eta^2\\ &- (143S^6+6084S^5+83486S^4+571220S^3+2227758S^2+48268095\\ &+ 4826809)\eta + S^7.\end{aligned}$$

By using (2.10), (2.11) and (2.12), Chern et al. [11, Eq.(4.4)] computed

$$\begin{aligned} H_{13}(\eta^{16}) &= 530S^8 - 13^7,\\ H_{13}(\eta^{18}) &= -27710S^9 - 13^8. \end{aligned} \tag{2.13}$$

Also, Baruah and Das [3, Lemma 2.10] computed

$$H_{13}(\eta^{24}) = -577738S^8 - 13^{12}. \tag{2.14}$$

From binomial theorem for any prime p, we have

$$f_1^p \equiv f_p \pmod{p}. \tag{2.15}$$

3. Proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Setting $\ell = 40$ in (1.1), we obtain

$$\sum_{n=0}^{\infty} b_{40}(n)q^n = \frac{f_{40}}{f_1}. \tag{3.1}$$

Replacing $1/f_1$ by using (2.2) and extracting the terms involving q^{5n+4} on both sides then dividing the resulting identity by q^4 and changing $q \to q^{1/5}$, we obtain

$$\sum_{n=0}^{\infty} b_{40}(5n+4)q^n = 5f_8\frac{f_5^5}{f_1^6}.$$

Employing binomial theorem (2.15), we have

$$\sum_{n=0}^{\infty} b_{40}(5n+4)q^n \equiv f_1^2 f_5^5 \pmod{2}.$$

Replacing f_1 by using (2.1), we obtain

$$\sum_{n=0}^{\infty} b_{40}(5n+4)q^n \equiv f_5^5 f_{25}^2\left(\frac{1}{R(q^5)^2} + q^2 + q^4R(q^5)^2\right) \pmod{2}. \tag{3.2}$$

Now extracting the terms involving q^{5n+2} on both sides then dividing the resulting identity by q^2 and changing $q \to q^{1/5}$, we obtain

$$\sum_{n=0}^{\infty} b_{40}(5^2n+14)q^n \equiv f_1^5 f_5^2 \pmod 2.$$

Again by using (2.1) and extracting the terms involving q^{5n} on both sides then changing $q \to q^{1/5}$, we obtain

$$\sum_{n=0}^{\infty} b_{40}(5^3n+14)q^n \equiv f_1^2 f_5^5 \left(\frac{1}{R(q)^5} + q + q^2 R(q)^5 \right) \pmod 2. \tag{3.3}$$

Employing (2.3) in (3.3), we obtain

$$\sum_{n=0}^{\infty} b_{40}(5^3n+14)q^n \equiv \frac{f_1^8}{f_5} \equiv \frac{f_8}{f_5} \pmod 2. \tag{3.4}$$

Changing $q \to q^8$ in (2.1) then applying the resulting identity in (3.4), extracting the terms involving q^{5n+3} on both sides then dividing the resulting identity by q^3 and changing $q \to q^{1/5}$, we obtain

$$\sum_{n=0}^{\infty} b_{40}(5^4n+389)q^n \equiv q\frac{f_{40}}{f_1} \pmod 2.$$

From (1.1), it is evident that

$$\sum_{n=0}^{\infty} b_{40}(5^4n+389)q^n \equiv \sum_{n=0}^{\infty} b_{40}(n)q^{n+1} \pmod 2.$$

If we extract the coefficients of the form q^{n+1}, then we obtain that for all $n \geq 0$,

$$b_{40}\left(5^4n + \frac{13}{8}\left(5^4-1\right)\right) \equiv b_{40}(n) \pmod 2. \tag{3.5}$$

Now by mathematical induction, identity (1.2) follows from (3.5).
Extracting the terms involving q^{5n+1} and q^{5n+3} in (3.2) completes the proof of identity (1.3). □

Proof of Theorem 1.2. We omit the proof of Theorem 1.2, since its follows the same line as Theorem 1.1. □

4. Proofs of Theorems 1.3 and 1.4.

Proof of Theorem 1.3. Setting $\ell = 4$ in (1.1), we obtain

$$\sum_{n=0}^{\infty} b_4(n)q^n = \frac{f_4}{f_1} \equiv f_1^3 \pmod 2. \tag{4.1}$$

Replacing f_1 by using (2.4), we obtain

$$\sum_{n=0}^{\infty} b_4(n)q^n \equiv f_{49}^3\left(\frac{B(q^7)}{C(q^7)} - q\frac{A(q^7)}{B(q^7)} - q^2 + q^5\frac{C(q^7)}{A(q^7)}\right)^3 \pmod 2.$$

Now extracting the terms involving q^{7n+6} then dividing the resulting identity by q^6 and changing $q \to q^{1/7}$, we obtain

$$\sum_{n=0}^{\infty} b_4(7n+6)q^n \equiv f_7^3 \pmod 2.$$

From (4.1), we obtain

$$\sum_{n=0}^{\infty} b_4(7n+6)q^n \equiv \sum_{n=0}^{\infty} b_4(n)q^{7n} \pmod 2. \tag{4.2}$$

Extracting the terms involving q^{7n} then by changing $q \to q^{1/7}$, we obtain for all $n \geq 0$,

$$b_4(7^2 n+6) \equiv b_4(n) \pmod 2. \tag{4.3}$$

Now by mathematical induction, (1.5) follows from (4.3).
If we extract the coefficients in terms of the form q^{7n+r} where $r \in \{1,2,3,4,5,6\}$ from (4.2), we obtain for all $n \geq 0$,

$$b_4(7\,(7n+r)+6) \equiv 0 \pmod 2. \tag{4.4}$$

Now from (1.5) and (4.4), we complete the proof of identity (1.6). □

Proof of Theorem 1.4. Setting $\ell = 8$ in (1.1), we obtain

$$\sum_{n=0}^{\infty} b_8(n)q^n = \frac{f_8}{f_1} \equiv f_1^7 \pmod 2. \tag{4.5}$$

Using (2.4) in (4.5), we obtain

$$\sum_{n=0}^{\infty} b_8(n)q^n \equiv f_{49}^7\left(\frac{B(q^7)}{C(q^7)} - q\frac{A(q^7)}{B(q^7)} - q^2 + q^5\frac{C(q^7)}{A(q^7)}\right)^7 \pmod 2.$$

Extracting the terms involving q^{7n} then changing $q \to q^{1/7}$, we obtain

$$\sum_{n=0}^{\infty} b_8(7n)q^n \equiv f_7^7\left(\frac{B^7}{C^7} + q\left(\frac{A^7}{B^7} + \frac{AB^2}{C^3}\right) + q^2\left(1 + \frac{A^2C}{B^3}\right)\right.$$
$$\left. + q^3\left(\frac{BC^2}{A^3}\right) + q^5\left(\frac{C^7}{A^7}\right)\right) \pmod 2,$$

$$\sum_{n=0}^{\infty} b_8(7n)q^n \equiv f_7^7\left(\left(\frac{B^7}{C^7} + q\frac{A^7}{B^7} + q^5\frac{C^7}{A^7}\right)\right.$$
$$\left. + q\left(\frac{AB^2}{C^3} + q\frac{A^2C}{B^3} + q^2\frac{BC^2}{A^3}\right) + q^2\right) \pmod 2.$$

Substituting (2.6) and (2.8) in above identity and then rearranging the resulting identity, we obtain

$$\sum_{n=0}^{\infty} b_8(7n)q^n \equiv \frac{f_1^8}{f_7} + qf_1^4f_7^3 \pmod 2. \tag{4.6}$$

Using (2.4) in (4.6) and extracting the terms involving q^{7n+3}, q^{7n+4} and q^{7n+6}, we completes the proof of identity (1.7). □

5. Proof of Theorem 1.5.

Proof of Theorem 1.5. Setting $\ell = 4$ in (1.1), we obtain

$$\sum_{n=0}^{\infty} b_4(n)q^n = \frac{f_4}{f_1} \equiv f_1^3 \pmod 2. \tag{5.1}$$

Applying (2.9) in (5.1), we obtain

$$\sum_{n=0}^{\infty} b_4(n)q^n \equiv f_{169}^3 \left(\frac{L(q^{13})}{J(q^{13})} - q\frac{N(q^{13})}{K(q^{13})} - q^2\frac{J(q^{13})}{I(q^{13})} + q^5\frac{M(q^{13})}{L(q^{13})} + q^7 - q^{12}\frac{K(q^{13})}{M(q^{13})} + q^{22}\frac{I(q^{13})}{N(q^{13})} \right)^3 \pmod 2.$$

Extracting the terms involving q^{13n+8} then dividing the resulting identity by q^8 and changing $q \to q^{1/13}$, we obtain

$$\sum_{n=0}^{\infty} b_4(13n+8)q^n \equiv qf_{13}^3 \pmod 2.$$

In view of (5.1), we have

$$\sum_{n=0}^{\infty} b_4(13n+8)q^n \equiv \sum_{n=0}^{\infty} b_4(n)q^{13n+1} \pmod 2. \tag{5.2}$$

Extracting the terms involving q^{13n+1}, for all $n \geq 0$,

$$b_4(13^2n+21) \equiv b_4(n) \pmod 2. \tag{5.3}$$

Now by mathematical induction, (1.8) follows from (5.3).
Extracting the terms involving q^{13n+r} where $r \in \{0, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, we obtain

$$b_4(13^2n+13r+8) \equiv 0 \pmod 2. \tag{5.4}$$

Hence, the identity (1.9) follows from (1.8) and (5.4). □

6. Proofs of Theorems 1.6 - 1.9.

Proof of Theorem 1.6. Setting $\ell = 13$ in (1.1), we obtain

$$\sum_{n=0}^{\infty} b_{13}(n)q^n = \frac{f_{13}}{f_1}.$$

From [9, Theorem 2] Calkin et al. showed that

$$\sum_{n=0}^{\infty} b_{13}(2n)q^n \equiv f_2^3 + q^3 f_{26}^3 \pmod 2. \tag{6.1}$$

Extracting the terms involving q^{2n} and changing $q \to q^{1/2}$, we obtain

$$\sum_{n=0}^{\infty} b_{13}(4n)q^n \equiv f_1^3 \pmod 2. \tag{6.2}$$

From (6.2), it is evident that (1.10) true for case $k = l = 0$.
Suppose that the congruence (1.10) holds for some integer $k \geq 0$ and $l = 0$.
Employing (2.4) in (1.10) with $l = 0$, we have

$$\sum_{n=0}^{\infty} b_{13}\left(4\cdot 7^{2k}n + \frac{7^{2k}-1}{2}\right)q^n \equiv$$
$$f_{49}^3\left(\frac{B(q^7)}{C(q^7)} - q\frac{A(q^7)}{B(q^7)} - q^2 + q^5\frac{C(q^7)}{A(q^7)}\right)^3 \pmod 2.$$

Now extracting the terms involving q^{7n+6} then dividing the resulting identity by q^6 and changing $q \to q^{1/7}$, we obtain

$$\sum_{n=0}^{\infty} b_{13}\left(4\cdot 7^{2k+1}n + \frac{7^{2k+2}-1}{2}\right)q^n \equiv f_7^3 \pmod 2. \tag{6.3}$$

Extracting the terms involving q^{7n} in (6.3) and changing $q \to q^{1/7}$, implies that (1.10) is true for $k+1$. By mathematical induction (1.10) is true for any integer $k \geq 0$ and $l = 0$.
Now, suppose that the congruence (1.10) holds for some integer k and $l \geq 0$.
Employing the identity (2.9) in (1.10), we have

$$\sum_{n=0}^{\infty} b_{13}\left(4\cdot 7^{2k}\cdot 13^{2l}n + 7^{2k}\cdot\frac{13^{2l}-1}{2} + \frac{7^{2k}-1}{2}\right)q^n$$
$$\equiv f_{169}^3\left(\frac{L(q^{13})}{J(q^{13})} - q\frac{N(q^{13})}{K(q^{13})} - q^2\frac{J(q^{13})}{I(q^{13})} + q^5\frac{M(q^{13})}{L(q^{13})}\right.$$
$$\left. + q^7 - q^{12}\frac{K(q^{13})}{M(q^{13})} + q^{22}\frac{I(q^{13})}{N(q^{13})}\right)^3 \pmod 2.$$

Extracting the terms involving q^{13n+8} from above identity then dividing the resulting identity by q^8 and changing $q \to q^{1/13}$, we obtain
(6.4)
$$\sum_{n=0}^{\infty} b_{13}\left(4\cdot 7^{2k}\cdot 13^{2l+1}n + 7^{2k}\cdot\frac{5\cdot 13^{2l+1}-1}{2} + \frac{7^{2k}-1}{2}\right)q^4 \equiv qf_{13}^3 \pmod 2.$$

Extracting the terms involving q^{13n+1} in (6.4) then dividing the resulting identity by q and changing $q \to q^{1/13}$, implies that (1.10) is true for $l+1$. By mathematical induction (1.10) is true for any integer k and $l \geq 0$. This completes the proof of identity (1.10).
Hence, the identity (1.11) follows from (6.3) by extracting the terms involving q^{7n+r} where $r \in \{1, 2, 3, 4, 5, 6\}$.
Also, the identity (1.12) follows from (6.4) by extracting the terms involving q^{13n+s} where $s \in \{0, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. □

Proof of Theorem 1.7. We omit the proof of Theorem 1.7, since its follows the same line as Theorem 1.6. □

Proofs of Theorems 1.8 ***and*** 1.9. From the identity (6.1) extracting the terms involving q^{2n+1} then dividing the resulting identity by q and changing

$q \to q^{1/2}$, we obtain

$$\sum_{n=0}^{\infty} b_{13}(4n+2)q^n \equiv q f_{13}^3 \pmod{2}.$$

Now, extracting the terms involving q^{13n+1} from above identity then dividing the resulting identity by q and changing $q \to q^{1/13}$, we obtain

$$\sum_{n=0}^{\infty} b_{13}(52n+6)q^n \equiv f_1^3 \pmod{2}.$$

We omit the proofs of Theorems 1.8 and 1.9, since its follows the same line as Theorem 1.6. □

7. Proofs of Theorems 1.10 - 1.12.

Proof of Theorem 1.10. Setting $\ell = 17$ in (1.1), we obtain

$$\sum_{n=0}^{\infty} b_{17}(n)q^n = \frac{f_{17}}{f_1} \equiv f_1^{16} \pmod{17}. \tag{7.1}$$

Substituting (2.9) in (7.1) , we have

$$\sum_{n=0}^{\infty} b_{17}(n)q^n \equiv q^{112} f_{169}^{16} \eta^{16} \pmod{17}.$$

Applying operator H_{13} in above identity, we obtain

$$\sum_{n=0}^{\infty} b_{17}(n)q^n \equiv q^{112} f_{169}^{16} H_{13}(\eta^{16}) \pmod{17}. \tag{7.2}$$

In view of identities (2.11) and (7.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty} b_{17}(n)q^n &\equiv q^{112} f_{169}^{16}(3S^8+13)\\ &\equiv 3q^8 f_{13}^{16} + 13q^{112} f_{169}^{16} \pmod{17}.\end{aligned}$$

Extracting the terms involving q^{13n+8} in above identity then dividing the resulting identity by q^8 and changing $q \to q^{1/13}$, we obtain

$$\sum_{n=0}^{\infty} b_{17}(13n+8)q^n \equiv 3\sum_{n=0}^{\infty} b_{17}(n)q^n + 13\sum_{n=0}^{\infty} b_{17}(n)q^{13n+8} \pmod{17}.$$

Extracting the terms involving q^{13n+8} in above identity then dividing the resulting identity by q^8 and changing $q \to q^{1/13}$, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty} b_{17}\left(13^2 n + \frac{2}{3}\left(13^2-1\right)\right)q^n \equiv\\ &3\sum_{n=0}^{\infty} b_{17}(13n+8)q^n + \sum_{n=0}^{\infty} b_{17}(n)q^n \pmod{17}.\end{aligned}$$

Repeating the same process as above, we obtain

$$\sum_{n=0}^{\infty} b_{17}\left(13^8 n + \frac{2}{3}\left(13^8-1\right)\right)q^n \equiv 15\sum_{n=0}^{\infty} b_{17}(n)q^{13n+8} \pmod{17}. \tag{7.3}$$

Extracting the terms involving q^{13n+8} in (7.3) then dividing the resulting identity by q^8 and changing $q \to q^{1/13}$, we obtain

$$b_{17}\left(13^9 n + \frac{2}{3}\left(13^9 - 1\right)\right) \equiv 15 b_{17}(n) \pmod{17}. \tag{7.4}$$

Now, by mathematical induction, (1.23) follows from (7.4).
Also, the identity (1.24) follows from (7.3) and (1.23) by extracting the terms involving q^{13n+r} where $r \in \{0, 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12\}$. □

***Proofs of Theorems* 1.11 *and* 1.12.** We omit the proofs of Theorems 1.11 and 1.12, since its follows the same line as Theorem 1.10. □

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