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ON SOME NEW MIXED MODULAR EQUATIONS OF COMPOSITE DEGREES 45 AND 63

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Abstract. On pages 242-250 of his second notebook, Ramanujan records modular equations of composite degrees. All these have been proved by B. C. Berndt using either the method of parametrization or the theory of modular forms. In this paper, we establish new P - Q modular equations of composite degrees 45 and 63 using Ramanujan's eta-function identities. We also establish some new Ramanujan's type mixed modular equations of composite degrees 45 and 63 using P - Q modular equations so obtained.

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1. Introduction

Ramanujan's general theta-function [6] $f(a,b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab|<1, \tag{1}$$

$$= (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2}$$

Three special cases of $f(a,b)$ are as follows:

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{3}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{4}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} q^{n(3n-1)/2} = (q;q)_\infty, \tag{5}$$

where

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1.$$

The complete elliptic integral of the first kind $K(k)$ is defined by

$$K(k) := \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2}k^{2n} = \frac{\pi}{2}\ {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right), \tag{6}$$

where $0<k<1$ and ${}_2F_1$ is the ordinary or Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n\,n!}z^n, \quad 0\le |z|<1,$$

where

$$(a)_0 = 1, \quad (a)_n = a(a+1)\cdots(a+n-1), \quad \text{for } \ n \text{ is a positive integer}$$

and a, b and c are complex numbers such that $c \neq 0,-1,-2,\ldots$. The number k is called the modulus of K, and $k' := \sqrt{1-k^2}$ is called the complementary modulus. Let K, K', L and L' denote the complete elliptic integrals of the first kind associated with the moduli k, k', l and l', respectively. Suppose that the equality

$$n\frac{K'}{K} = \frac{L'}{L} \tag{7}$$

holds for some positive integer n. Then, modular equation of degree n is a relation between the moduli k and l which is induced by (7). Following Ramanujan, set $\alpha = k^2$ and $\beta = l^2$. Then we say β is of degree n over α. The multiplier m is defined by

$$m = \frac{K}{L}. \tag{8}$$

Let K, K', L_1, L_1', L_2, L_2', L_3 and L_3' denote complete elliptic integrals of the first kind corresponding, in pairs, to the moduli $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$ and $\sqrt{\delta}$, and their complementary moduli respectively. Let n_1, n_2 and n_3 be positive integers such that $n_3 = n_1 n_2$. Suppose that the equalities

$$n_1\frac{K'}{K} = \frac{L_1'}{L_1}, \quad n_2\frac{K'}{K} = \frac{L_2'}{L_2} \ \text{ and } \ n_3\frac{K'}{K} = \frac{L_3'}{L_3} \tag{9}$$

holds. Then a "mixed" modular equation is a relation between the moduli $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$ and $\sqrt{\delta}$ that is induced by (9). We say that β, γ and δ are of degrees n_1, n_2 and n_3, respectively over α. The multipliers m and m' are associated with α, β, γ and δ.

M. S. Mahadeva Naika, S. Chandankumar and B. Hemanthkumar [5] established some new mixed modular equations relating A_1 with A_r, where

$$A_r = \frac{f^2(-q^{3r})}{q^{\frac{r}{6}} f(-q^r) f(-q^{9r})}$$

and $r \in \{2, 3, 5, 7, 11, 13\}$. They also established several explicit evaluations of cubic singular moduli.

In Chapter 19 and 20 of his second notebook, Ramanujan recorded modular equations of composite degrees. Infact, some of his modular equations involve product or quotients of the multipliers m and m' on one side and a combination of different powers of $\left(\frac{\alpha\beta}{\gamma\delta}\right)$ and $\left(\frac{(1-\alpha)(1-\beta)}{(1-\gamma)(1-\delta)}\right)$ on the other side. All these have been proved in [1] either by employing the method of parametrization or the theory of modular forms. Berndt often remarked that his methods astray from the path followed by Ramanujan.

In Chapter 25 of his Second Notebook, Ramanujan states twenty three beautiful P - Q eta-function identities or P - Q modular equations. These are identities involving quotients of eta-functions, which are designated as P or Q by Ramanujan. Elementary proofs of eighteen of these twenty three P - Q identities by employing the theory of theta-functions in the Spirit of Ramanujan and remaining five using theory of modular forms can be found in [2].

In section 2, we collect the identities which are useful in proving our main results. In section 3, we establish several new P - Q mixed modular equations of degrees 45 and 63. In section 4, we establish certain new mixed modular equations of composite degrees 45 and 63.

2. Preliminary Results

In this section, we collect several identities which are useful in proving our main results.

Lemma 2.1. [1, Ch. 17, Entry 12 (i), (iii) and (iv), p. 124] *We have*

$$f(q) = \sqrt{z}\, 2^{-1/6}\{x(1-x)e^y\}^{1/24}, \tag{10}$$

$$f(-q^2) = \sqrt{z}\, 2^{-1/3}\{x(1-x)e^y\}^{1/12}, \tag{11}$$

$$f(-q^4) = \sqrt{z}\, 2^{-2/3}(1-x)^{1/24}\{xe^y\}^{1/6}. \tag{12}$$

Lemma 2.2. [1, Ch. 20, Entry 3 (xii), (xiii), pp. 352–353][6] *Let α, β and γ be of the first, third and ninth degrees respectively. Let m denote the multiplier connecting α and β and m' be the* *multiplier relating γ and δ. Then the following modular equations hold.*

$$\left(\frac{\beta^2}{\alpha\gamma}\right)^{1/4}+\left(\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}\right)^{1/4}-\left(\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/4}=-3\frac{m}{m'} \tag{13}$$

and

$$\left(\frac{\alpha\gamma}{\beta^2}\right)^{1/4}+\left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}\right)^{1/4}-\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}\right)^{1/4}=\frac{m'}{m}. \tag{14}$$

Lemma 2.3. [3, 4, 5] *If $M = A_1A_2$ and $N = \dfrac{A_1}{A_2}$, then*

$$M-\frac{3}{M}=N^3+\frac{1}{N^3}. \tag{15}$$

Lemma 2.4. [5] *If $M = A_1A_5$ and $N = \dfrac{A_1}{A_5}$, then*

$$N^3+\frac{1}{N^3}=M^2-\frac{9}{M^2}-5\left(M-\frac{3}{M}\right). \tag{16}$$

Lemma 2.5. [5] *If $M = A_1A_7$ and $N = \dfrac{A_1}{A_7}$, then*

$$N^4+\frac{1}{N^4}+7\left(N^3+\frac{1}{N^3}\right)+28\left(N^2+\frac{1}{N^2}\right)+56\left(N+\frac{1}{N}\right)+77=\left(M^3-\frac{27}{M^3}\right). \tag{17}$$

Lemma 2.6. *If $u = A_1^6$ and $v = A_2^6$, then*

$$u^3+12u^2v+27uv+12uv^2-u^2v^2+v^3=0. \tag{18}$$

Proof. The equation (15) can be written as

$$A_1^4A_2^4-3A_1^2A_2^2-(u+v)=0. \tag{19}$$

Solving for $A_1^2A_2^2$, we get

$$A_1^2A_2^2=\frac{3+t}{2}, \tag{20}$$

where $t^2 = 9 + 4(u + v)$.

Cubing both sides of the above identity, we obtain (18). □

3. New $P-Q$ mixed modular equations

In this section, we establish some new $P-Q$ mixed modular equations of degrees 45 and 63 using Ramanujan's eta-function identities.

Theorem 3.1. *If $P = A_1A_5$ and $Q = A_2A_{10}$, then*

$$P^3 + Q^3 - P^2Q^2 + 2P^2Q - 3PQ + 2PQ^2 = 0. \tag{21}$$

Proof. Solving the equation (16) for N^3, we get

$$N^3 = \frac{a+m}{2}, \tag{22}$$

where $a = P^2 - \frac{9}{P^2} - 5\left(P - \frac{3}{P}\right)$ and $m^2 = a^2 - 4$.
Substituting $N^3 = \left(\frac{A_1^2}{P}\right)^3$ in the above equation, we get

$$A_1^6 = P^3\left(\frac{a+m}{2}\right). \tag{23}$$

Replacing q by q^2 in the above equation, we find that

$$A_2^6 = Q^3\left(\frac{b+n}{2}\right), \tag{24}$$

where $b = Q^2 - \frac{9}{Q^2} - 5\left(Q - \frac{3}{Q}\right)$ and $n^2 = b^2 - 4$.
Using (23) and (24) in the equation (18), we deduce that

$$\begin{aligned}
&- P^6Q^6a^2b^2 - P^6Q^6a^2bn - P^6Q^6ab^2m - P^6Q^6abmn + 2P^6Q^6a^2 + 2P^6Q^6b^2 + 2P^6Q^6bn \\
&+ 2P^9a^3 + 2P^9a^2m - 4P^6Q^6 + 12P^6Q^3a^2b + 12P^6Q^3abm + 12P^6Q^3amn + 12P^3Q^6ab^2 \\
&+ 12P^3Q^6b^2m + 12P^3Q^6bmn + 2Q^9b^3 + 2Q^9b^2n - 6P^9a - 2P^9m - 24P^6Q^3b - 24P^6Q^3n \\
&+ 12P^3Q^6abn - 24P^3Q^6a - 24P^3Q^6m - 6Q^9b - 2Q^9n + 27P^3Q^3ab + 27P^3Q^3an \\
&+ 27P^3Q^3bm + 27P^3Q^3mn + 2P^6Q^6am + 12P^6Q^3a^2n = 0.
\end{aligned} \tag{25}$$

Collecting the terms having m on one side of the above equation and rest on the other side, squaring both sides and then substituting for m^2, n^2, a and b, we find that

$$M(P,Q)N(P,Q,n) = 0, \tag{26}$$

where

$$M(P,Q) = P^3 - P^2Q^2 + 2P^2Q + 2PQ^2 - 3PQ + Q^3$$

and

$$N(P,Q,n) = P^8Q^{23} + P^9Q^{21} - 23P^8Q^{22} + P^8Q^{21}n - 8P^7Q^{23} + P^6Q^{24} + P^{10}Q^{19} - 21P^9Q^{20}$$
$$+ P^9Q^{19}n + 198P^8Q^{21} - 18P^8Q^{20}n + 187P^7Q^{22} - 8P^7Q^{21}n - 12P^6Q^{23} + P^6Q^{22}n - 6P^5Q^{24}$$
$$+ P^4Q^{25} + P^{11}Q^{17} - 20P^{10}Q^{18} + P^{10}Q^{17}n + 158P^9Q^{19} - 16P^9Q^{18}n - 636P^8Q^{20} + 108P^8Q^{19}n$$
$$- 1694P^7Q^{21} + 147P^7Q^{20}n - 59P^6Q^{22} - 7P^6Q^{21}n + 179P^5Q^{23} - 6P^5Q^{22}n - 22P^4Q^{24} + P^4Q^{23}n$$
$$- 4P^3Q^{25} + P^2Q^{26} + P^{12}Q^{15} - 18P^{11}Q^{16} + P^{11}Q^{15}n + 140P^{10}Q^{17} - 15P^{10}Q^{16}n - 350P^9Q^{18}$$
$$+ 78P^9Q^{17}n - 1032P^8Q^{19} - 111P^8Q^{18}n + 6595P^7Q^{20} - 959P^7Q^{19}n + 1726P^6Q^{21} - 94P^6Q^{20}n$$
$$- 2190P^5Q^{22} + 149P^5Q^{21}n + 99P^4Q^{23} - 17P^4Q^{22}n + 125P^3Q^{24} - 4P^3Q^{23}n - 28P^2Q^{25} + P^2Q^{24}n$$
$$- 2PQ^{26} + Q^{27} - 16P^{12}Q^{14} + P^{12}Q^{13}n + 106P^{11}Q^{15} - 13P^{11}Q^{14}n - 258P^{10}Q^{16} + 65P^{10}Q^{15}n$$
$$- 1650P^9Q^{17} + 25P^9Q^{16}n + 12525P^8Q^{18} - 1306P^8Q^{17}n - 1340P^7Q^{19} + 14P^4Q^{21}n + 1920P^7Q^{18}n$$
$$+ 13443P^5Q^{21} - 10873P^6Q^{20} + 1241P^6Q^{19}n - 1445P^5Q^{20}n + 1382P^4Q^{22} - 1626P^3Q^{23} + 105P^3Q^{22}n$$
$$+ 303P^2Q^{24} - 23P^2Q^{23}n + 63PQ^{25} - 2PQ^{24}n - 30Q^{26} + Q^{25}n - P^{13}Q^{12} + 89P^{12}Q^{13} - 11P^{12}Q^{12}n$$
$$- 81P^{11}Q^{14} + 41P^{11}Q^{13}n - 1476P^{10}Q^{15} + 52P^{10}Q^{14}n + 10195P^9Q^{16} - 1274P^9Q^{15}n - 19733P^8Q^{17}$$
$$+ 4187P^8Q^{16}n - 77937P^7Q^{18} + 5967P^7Q^{17}n + 21132P^6Q^{19} - 4552P^6Q^{18}n - 35246P^5Q^{20} + 6308P^5Q^{19}n$$
$$- 18779P^4Q^{21} + 1437P^4Q^{20}n + 11011P^3Q^{22} - 1101P^3Q^{21}n - 1291P^2Q^{23} + 188P^2Q^{22}n - 840PQ^{24}$$
$$+ 53PQ^{23}n + 375Q^{25} - 25Q^{24}n - P^{14}Q^{10} + 12P^{13}Q^{11} - P^{13}Q^{10}n - 130P^{12}Q^{12} + 34P^{12}Q^{11}n$$
$$- 1488P^{11}Q^{13} + 109P^{11}Q^{12}n + 7167P^{10}Q^{14} - 980P^{10}Q^{13}n - 6882P^9Q^{15} + 2489P^9Q^{14}n - 71014P^8Q^{16}$$
$$+ 3925P^8Q^{15}n + 219762P^7Q^{17} - 32180P^7Q^{16}n + 62811P^6Q^{18} - 285P^6Q^{17}n - 45075P^5Q^{19}$$
$$- 6007P^5Q^{18}n + 83514P^4Q^{20} - 11328P^4Q^{19}n - 36719P^3Q^{21} + 5566P^3Q^{20}n - 2426P^2Q^{22} - 366P^2Q^{21}n$$
$$+ 5945PQ^{23} - 575PQ^{22}n - 2410Q^{24} + 250Q^{23}n + 10P^{14}Q^9 - P^{14}Q^8n - 28P^{13}Q^{10} + 7P^{13}Q^9n$$
$$- 597P^{12}Q^{11} + 25P^{12}Q^{10}n + 4801P^{11}Q^{12} - 737P^{11}Q^{11}n - 387P^{10}Q^{13} + 1137P^{10}Q^{12}n - 71284P^9Q^{14}$$
$$+ 5936P^9Q^{13}n + 256557P^8Q^{15} - 32620P^8Q^{14}n + 97374P^7Q^{16} + 20583P^7Q^{15}n - 374107P^6Q^{17}$$
$$+ 41717P^6Q^{16}n + 547884P^5Q^{18} - 51856P^5Q^{17}n - 59042P^4Q^{19} + 26487P^4Q^{18}n + 10776P^3Q^{20}$$
$$- 10508P^3Q^{19}n + 49386P^2Q^{21} - 3900P^2Q^{20}n - 21468PQ^{22} + 3100PQ^{21}n + 7065Q^{23} - 1175Q^{22}n$$
$$- 25P^{14}Q^8 + 5P^{14}Q^7n - 150P^{13}Q^9 + 7P^{13}Q^8n + 2325P^{12}Q^{10} - 296P^{12}Q^9n + 2758P^{11}Q^{11} + 368P^{11}Q^{10}n$$
$$- 51199P^{10}Q^{12} + 5133P^{10}Q^{11}n + 157838P^9Q^{13} - 22019P^9Q^{12}n + 41709P^8Q^{14} + 17586P^8Q^{13}n$$
$$- 1420537P^7Q^{15} + 127904P^7Q^{14}n + 406814P^6Q^{16} - 84661P^6Q^{15}n - 1064437P^5Q^{17} + 179279P^5Q^{16}n$$
$$- 795526P^4Q^{18} + 51872P^4Q^{17}n + 373046P^3Q^{19} - 24134P^3Q^{18}n - 184904P^2Q^{20} + 26823P^2Q^{19}n$$

$$
\begin{aligned}
&+13455PQ^{21}-6785PQ^{20}n+5220Q^{22}+1576Q^{21}n-30P^{14}Q^{7}+655P^{13}Q^{8}-100P^{13}Q^{7}n-33P^{12}Q^{9}\\
&+224P^{12}Q^{8}n-33392P^{11}Q^{10}+3334P^{11}Q^{9}n+67418P^{10}Q^{11}-10142P^{10}Q^{10}n+146061P^{9}Q^{12}\\
&-2006P^{9}Q^{11}n-1225375P^{8}Q^{13}+108721P^{8}Q^{12}n+1691025P^{7}Q^{14}-236850P^{7}Q^{13}n+1318629P^{6}Q^{15}\\
&-55148P^{6}Q^{14}n-1479438P^{5}Q^{16}-16869P^{5}Q^{15}n+2582112P^{4}Q^{17}-350929P^{4}Q^{16}n-1208482P^{3}Q^{18}\\
&+157625P^{3}Q^{17}n+10760P^{2}Q^{19}-43411P^{2}Q^{18}n+204460PQ^{20}-11282PQ^{19}n-107900Q^{21}\\
&+9085Q^{20}n+170P^{14}Q^{6}-15P^{14}Q^{5}n+92P^{13}Q^{7}+39P^{13}Q^{6}n-10863P^{12}Q^{8}+1026P^{12}Q^{7}n\\
&+30248P^{11}Q^{9}-4488P^{11}Q^{8}n+146376P^{10}Q^{10}-9687P^{10}Q^{9}n+12P^{14}Q^{3}n-779867P^{9}Q^{11}\\
&+68730P^{9}Q^{10}n+1190526P^{8}Q^{12}-149815P^{8}Q^{11}n+2880463P^{7}Q^{13}-102350P^{7}Q^{12}n-4114844P^{6}Q^{14}\\
&+407269P^{6}Q^{13}n+8466747P^{5}Q^{15}-834230P^{5}Q^{14}n+127956P^{4}Q^{16}+317854P^{4}Q^{15}n+382050P^{3}Q^{17}\\
&-208151P^{3}Q^{16}n+1862925P^{2}Q^{18}-150871P^{2}Q^{17}n-756480PQ^{19}+95655PQ^{18}n+279840Q^{20}\\
&-42300Q^{19}n-78P^{14}Q^{5}+9P^{14}Q^{4}n-3019P^{13}Q^{6}+249P^{13}Q^{5}n+9761P^{12}Q^{7}-1008P^{12}Q^{6}n\\
&+79218P^{11}Q^{8}-4569P^{11}Q^{7}n-355006P^{10}Q^{9}+30213P^{10}Q^{8}n+335818P^{9}Q^{10}-47296P^{9}Q^{9}n\\
&+2462454P^{8}Q^{11}-123049P^{8}Q^{10}n-8225108P^{7}Q^{12}+617356P^{7}Q^{11}n+1508311P^{6}Q^{13}-409427P^{6}Q^{12}n\\
&-6170742P^{5}Q^{14}+1092696P^{5}Q^{13}n-14274564P^{4}Q^{15}+1179873P^{4}Q^{14}n+6036648P^{3}Q^{16}\\
&-392453P^{3}Q^{15}n-4538703P^{2}Q^{17}+664975P^{2}Q^{16}n+388740PQ^{18}-145870PQ^{17}n+140340Q^{19}\\
&+33245Q^{18}n-285P^{14}Q^{4}+2337P^{13}Q^{5}-177P^{13}Q^{4}n+21773P^{12}Q^{6}-1395P^{12}Q^{5}n-143788P^{11}Q^{7}\\
&+9111P^{11}Q^{6}n-67655P^{10}Q^{8}-3288P^{10}Q^{7}n+1503288P^{9}Q^{9}-70101P^{9}Q^{8}n-5157664P^{8}Q^{10}\\
&+358110P^{8}Q^{9}n+2047693P^{7}Q^{11}-387564P^{7}Q^{10}n+10439854P^{6}Q^{12}-481695P^{6}Q^{11}n\\
&-21860709P^{5}Q^{13}+1202232P^{5}Q^{12}n+20412366P^{4}Q^{14}-2565702P^{4}Q^{13}n-12704715P^{3}Q^{15}\\
&+1460544P^{3}Q^{14}n-2923209P^{2}Q^{16}-301650P^{2}Q^{15}n+3664737PQ^{17}-234045PQ^{16}n-2113380Q^{18}\\
&+183846Q^{17}n+270P^{14}Q^{3}+3551P^{13}Q^{4}-33P^{13}Q^{3}n-30546P^{12}Q^{5}+1182P^{12}Q^{4}n-30606P^{11}Q^{6}\\
&-39P^{11}Q^{5}n+708401P^{10}Q^{7}-28812P^{10}Q^{6}n-1821342P^{9}Q^{8}+94377P^{9}Q^{7}n-187953P^{8}Q^{9}\\
&-115008P^{8}Q^{8}n+13466788P^{7}Q^{10}-493017P^{7}Q^{9}n-17366655P^{6}Q^{11}+1367256P^{6}Q^{10}n\\
&+41554521P^{5}Q^{12}-3354468P^{5}Q^{11}n+22979551P^{4}Q^{13}-472263P^{4}Q^{12}n-2789919P^{3}Q^{14}\\
&-787395P^{3}Q^{13}n+28372536P^{2}Q^{15}-2549325P^{2}Q^{14}n-8230680PQ^{16}+939225PQ^{15}n+3235950Q^{17}\\
&-467745Q^{16}n+101P^{14}Q^{2}-4904P^{13}Q^{3}-10487P^{12}Q^{4}-270P^{12}Q^{3}n+189259P^{11}Q^{5}-2124P^{11}Q^{4}n\\
&-431939P^{10}Q^{6}+20799P^{10}Q^{5}n-581516P^{9}Q^{7}-15687P^{9}Q^{6}n+8070705P^{8}Q^{8}-259533P^{8}Q^{7}n\\
&-15347485P^{7}Q^{9}+757773P^{7}Q^{8}n-1401820P^{6}Q^{10}-626481P^{6}Q^{9}n+8516054P^{5}Q^{11}+644751P^{5}Q^{10}n\\
&-81699883P^{4}Q^{12}+5636025P^{4}Q^{11}n+43451738P^{3}Q^{13}-2701890P^{3}Q^{12}n-28871525P^{2}Q^{14}\\
&+4135275P^{2}Q^{13}n-1693980PQ^{15}-435690PQ^{14}n+3883140Q^{16}+2P^{15}-112P^{14}Q+1102P^{13}Q^{2}
\end{aligned}
$$

$$+ 30295P^{12}Q^3 - 86931P^{11}Q^4 + 855P^{11}Q^3n - 372465P^{10}Q^5 - 5841P^{10}Q^4n + 2231081P^9Q^6$$
$$- 32562P^9Q^5n - 6349915P^8Q^7 + 263331P^8Q^6n - 3522797P^7Q^8 - 134964P^7Q^7n + 31175560P^6Q^9$$
$$- 989217P^6Q^8n - 81370305P^5Q^{10} + 3456495P^5Q^9n + 32462307P^4Q^{11} - 3946635P^4Q^{10}n$$
$$- 46192789P^3Q^{12} + 4250475P^3Q^{11}n - 58897525P^2Q^{13} + 2500875P^2Q^{12}n + 26736160PQ^{14}$$
$$- 1668015PQ^{13}n - 17443460Q^{15} + 1403235Q^{14}n + 303P^{13}Q - 14712P^{12}Q^2 - 31461P^{11}Q^3$$
$$+ 567777P^{10}Q^4 + 702P^{10}Q^3n - 1180797P^9Q^5 + 18225P^9Q^4n - 2518827P^8Q^6 - 68310P^8Q^5n$$
$$+ 17357652P^7Q^7 - 274806P^7Q^6n - 28504821P^6Q^8 + 1254420P^6Q^7n + 52373850P^5Q^9$$
$$- 2590758P^5Q^8n + 111709563P^4Q^{10} - 3391335P^4Q^9n - 39680490P^3Q^{11} + 273645P^3Q^{10}n$$
$$+ 142797300P^2Q^{12} - 10506375P^2Q^{11}n - 24884475PQ^{13} + 2036745PQ^{12}n + 11649420Q^{14}$$
$$-1654614Q^{13}n+2430P^{12}Q+31959P^{11}Q^2-274914P^{10}Q^3-323973P^9Q^4-4590P^9Q^3n+6294528P^8Q^5$$
$$- 1350P^8Q^4n - 9636273P^7Q^6 + 152010P^7Q^5n - 6976179P^6Q^7 - 442530P^6Q^6n + 50033439P^5Q^8$$
$$- 549423P^5Q^7n - 139152600P^4Q^9 + 5460210P^4Q^8n + 114595155P^3Q^{10} - 4658580P^3Q^9n$$
$$- 13881825P^2Q^{11} + 4478625P^2Q^{10}n - 24533280PQ^{12} + 417825PQ^{11}n + 29123550Q^{13} - 897615Q^{12}n$$
$$-7695P^{11}Q+63099P^{10}Q^2+566487P^9Q^3-3407211P^8Q^4+2430P^8Q^3n-2757861P^7Q^5-28350P^7Q^4n$$
$$+27719982P^6Q^6+5265P^6Q^5n-75825180P^5Q^7+1104840P^5Q^6n-8502705P^4Q^8-1526850P^4Q^7n$$
$$- 47615175P^3Q^9 + 2776680P^3Q^8n - 230459175P^2Q^{10} + 8168850P^2Q^9n + 51663420PQ^{11}$$
$$- 1211760PQ^{10}n - 57061260Q^{12} + 3426300Q^{11}n - 6318P^{10}Q - 232875P^9Q^2 + 594540P^8Q^3$$
$$+ 4723191P^7Q^4 + 6075P^7Q^3n - 18058140P^6Q^5 + 24300P^6Q^4n + 16382655P^5Q^6 - 266085P^5Q^5n$$
$$+116601120P^4Q^7-749655P^4Q^6n-75809115P^3Q^8+504225P^3Q^7n+205847325P^2Q^9-8511075P^2Q^8n$$
$$-12145140PQ^{10}-89910PQ^9n+11367540Q^{11}-2207655Q^{10}n+41310P^9Q+48600P^8Q^2-1793340P^7Q^3$$
$$+ 4098195P^6Q^4 - 7290P^6Q^3n + 21618495P^5Q^5 - 19683P^5Q^4n - 73794240P^4Q^6 + 349920P^4Q^5n$$
$$+87928335P^3Q^7-827415P^3Q^6n+78628725P^2Q^8+1057050P^2Q^7n-17143650PQ^9-554040PQ^8n$$
$$+68001120Q^{10}-1148904Q^9n-21870P^8Q+346275P^7Q^2+317115P^6Q^3-13945770P^5Q^4+2187P^5Q^3n$$
$$+2496825P^4Q^5-18301545P^3Q^6+174960P^3Q^5n-209441700P^2Q^7+1530900P^2Q^6n-1545480PQ^8$$
$$+ 1913625PQ^7n - 78659100Q^9 + 2569725Q^8n - 54675P^7Q - 328050P^6Q^2 + 2099520P^5Q^3$$
$$+ 11995695P^4Q^4 - 16949250P^3Q^5 + 94915800P^2Q^6 - 492075P^2Q^5n - 415530PQ^7 - 1345005PQ^6n$$
$$+ 11416140Q^8 - 1640250Q^7n + 65610P^6Q + 209952P^5Q^2 - 3149280P^4Q^3 + 10071135P^3Q^4$$
$$+ 13450050P^2Q^5 + 32214510PQ^6 + 295245PQ^5n + 46353465Q^7 + 492075Q^6n - 19683P^5Q$$
$$- 1574640P^3Q^3 - 21159225P^2Q^4 - 37397700PQ^5 - 47436030Q^6 - 59049Q^5n + 4428675P^2Q^3$$
$$+ 16533720PQ^4 + 22143375Q^5 - 2657205PQ^3 - 5314410Q^4 + 531441Q^3.$$

As $q \to 0$, the first factor vanishes in the neighbourhood of 0 where as the second factor does not vanish. Hence, by the identity theorem, we obtain (21). □

Theorem 3.2. *If* $P = \dfrac{A_1}{A_7}$ *and* $Q = \dfrac{A_2}{A_{14}}$, *then*

$$P^3 - P^2Q^2 - 2P^2Q - PQ - 2PQ^2 + Q^3 = 0. \tag{27}$$

Proof. Solving the equation (17) for M^3, we find that

$$M^3 = \frac{c+r}{2}, \tag{28}$$

where $c = P^4 + \dfrac{1}{P^4} + 7\left(P^3 + \dfrac{1}{P^3}\right) + 28\left(P^2 + \dfrac{1}{P^2}\right) + 56\left(P + \dfrac{1}{P}\right) + 77$ and $r^2 = c^2 + 108$.
Substituting $M^3 = \left(\dfrac{A_1^2}{P}\right)^3$ in the above equation, we get

$$A_1^6 = P^3\left(\frac{c+r}{2}\right). \tag{29}$$

Replacing q by q^2 in the above equation, we find that

$$A_2^6 = Q^3\left(\frac{d+s}{2}\right), \tag{30}$$

where $d = Q^4 + \dfrac{1}{Q^4} + 7\left(Q^3 + \dfrac{1}{Q^3}\right) + 28\left(Q^2 + \dfrac{1}{Q^2}\right) + 56\left(Q + \dfrac{1}{Q}\right) + 77$ and $s^2 = d^2 + 108$.
Using (29) and (30) in the equation (18), we deduce that

$$\begin{aligned}
&-Q^6P^6c^2d^2 - Q^6P^6c^2ds - Q^6P^6cd^2r - Q^6P^6cdrs - 54Q^6P^6c^2 + 27Q^3P^3cd - 54Q^6P^6d^2 \\
&-54Q^6P^6ds + 2Q^9d^3 + 2Q^9d^2s - 2916Q^6P^6 + 12Q^6P^3cd^2 + 12Q^6P^3cds + 12Q^6P^3d^2r \\
&+12Q^6P^3drs + 12Q^3P^6c^2d + 12Q^3P^6c^2s + 12Q^3P^6cdr + 12Q^3P^6crs + 2P^9c^3 + 2P^9c^2r \\
&+162Q^9d + 54Q^9s + 648Q^6P^3c + 648Q^6P^3r + 648Q^3P^6d + 648Q^3P^6s + 162P^9c + 54P^9r \\
&+27Q^3P^3cs + 27Q^3P^3dr + 27Q^3P^3rs - 54Q^6P^6cr = 0.
\end{aligned} \tag{31}$$

Collecting the terms having r on one side of the above equation and rest on the other side, squaring both sides and then substituting for r^2, s^2, c and d, we obtain (27). □

4. Some new Ramanujan's type modular equations of composite degrees 45 and 63.

In this section, we establish the Ramanujan's type mixed modular equations of composite degrees 45 and 63 using P - Q modular equations obtained in the previous section.

Theorem 4.1. *Let α, β, γ, λ, μ and ν be of the first, third, ninth, fifth, fifteenth and fortyfifth degrees respectively. Let $m=\dfrac{z_1}{z_3}, m'=\dfrac{z_3}{z_9}, m_1=\dfrac{z_5}{z_{15}}$ and $m_1'=\dfrac{z_{15}}{z_{45}}$. Then*

$$3\sqrt{\frac{mm_1}{m'm_1'}}=\left(\frac{\beta^2\mu^2}{\alpha\gamma\lambda\nu}\right)^{1/8}+\left(\frac{(1-\beta)^2(1-\mu)^2}{(1-\alpha)(1-\gamma)(1-\lambda)(1-\nu)}\right)^{1/8}$$

$$+2\left(\frac{\beta^2\mu^2(1-\beta)^2(1-\mu)^2}{\alpha\gamma\lambda\nu(1-\alpha)(1-\gamma)(1-\lambda)(1-\nu)}\right)^{1/12}-\left(\frac{\beta^2\mu^2(1-\beta)^2(1-\mu)^2}{\alpha\gamma\lambda\nu(1-\alpha)(1-\gamma)(1-\lambda)(1-\nu)}\right)^{1/8} \tag{32}$$

and

$$\sqrt{\frac{m'm_1'}{mm_1}}=\left(\frac{\alpha\gamma\lambda\nu}{\beta^2\mu^2}\right)^{1/8}+\left(\frac{(1-\alpha)(1-\gamma)(1-\lambda)(1-\nu)}{(1-\beta)^2(1-\mu)^2}\right)^{1/8}$$

$$+2\left(\frac{\alpha\gamma\lambda\nu(1-\alpha)(1-\gamma)(1-\lambda)(1-\nu)}{\beta^2\mu^2(1-\beta)^2(1-\mu)^2}\right)^{1/12}-\left(\frac{\alpha\gamma\lambda\nu(1-\alpha)(1-\gamma)(1-\lambda)(1-\nu)}{\beta^2\mu^2(1-\beta)^2(1-\mu)^2}\right)^{1/8}. \tag{33}$$

Proof. Replacing q by q^2 in the equation (21), we find that

$$R^3+Q^3-R^2Q^2+2R^2Q-3RQ+2RQ^2=0, \tag{34}$$

where $R=A_4A_{20}$.

Subtracting R^2 times of the equation (21) from P^2 times of the equation (34), we obtain

$$(P-R)(P^2R^2-PQ^3-2PQ^2R-Q^3R+3PQR)=0.$$

Since $P-R\neq 0$ implies

$$P^2R^2-PQ^3-2PQ^2R-Q^3R+3PQR=0. \tag{35}$$

Replacing q by $-q$ in the above equation (35), we find that

$$M^2R^2+MQ^3+2MQ^2R-Q^3R-3MQR=0, \tag{36}$$

where $M=\dfrac{f^2(q^3)f^2(q^{15})}{qf(q)f(q^9)f(q^5)f(q^{45})}$.

Using the equations (10), (11) and (12) in M, Q and R, we obtain

$$
\begin{aligned}
M &= \sqrt{\frac{z_3^2 z_{15}^2}{z_1 z_5 z_9 z_{45}}}\left(\frac{\beta^2\mu^2(1-\beta)^2(1-\mu)^2}{\alpha\gamma\lambda\nu(1-\alpha)(1-\gamma)(1-\lambda)(1-\nu)}\right)^{1/24} \\
&= \sqrt{\frac{m'm_1'}{mm_1}}\left(\frac{\beta^2\mu^2(1-\beta)^2(1-\mu)^2}{\alpha\gamma\lambda\nu(1-\alpha)(1-\gamma)(1-\lambda)(1-\nu)}\right)^{1/24}, \qquad (37)
\end{aligned}
$$

$$Q = \sqrt{\frac{m'm_1'}{mm_1}}\left(\frac{\beta^2\mu^2(1-\beta)^2(1-\mu)^2}{\alpha\gamma\lambda\nu(1-\alpha)(1-\gamma)(1-\lambda)(1-\nu)}\right)^{1/12}, \qquad (38)$$

$$R = \sqrt{\frac{m'm_1'}{mm_1}}\left(\frac{\beta^2\mu^2}{\alpha\gamma\lambda\nu}\right)^{1/6}\left(\frac{(1-\beta)^2(1-\mu)^2}{(1-\alpha)(1-\gamma)(1-\lambda)(1-\nu)}\right)^{1/24}, \qquad (39)$$

which implies

$$\frac{Q}{M} = \left(\frac{\beta^2\mu^2(1-\beta)^2(1-\mu)^2}{\alpha\gamma\lambda\nu(1-\alpha)(1-\gamma)(1-\lambda)(1-\nu)}\right)^{1/24}, \qquad (40)$$

$$\frac{Q}{M^2} = \sqrt{\frac{mm_1}{m'm_1'}}, \qquad (41)$$

$$\frac{R}{M} = \left(\frac{\beta^2\mu^2}{\alpha\gamma\lambda\nu}\right)^{1/8} \qquad (42)$$

and

$$\frac{Q^3}{M^2R} = \left(\frac{(1-\beta)^2(1-\mu)^2}{(1-\alpha)(1-\gamma)(1-\lambda)(1-\nu)}\right)^{1/8}. \qquad (43)$$

Multiplying the equation (36) by $\frac{1}{M^3R}$, we find that

$$\frac{R}{M} + \frac{Q^3}{M^2R} + 2\frac{Q^2}{M^2} - \frac{Q^3}{M^3} - 3\frac{Q}{M^2} = 0. \qquad (44)$$

Using (40)–(43) in the equation (44), we obtain (32).

Subtracting R times of the equation (21) from P times of the equation (36), we obtain

$$(P-R)(-PQ^2R + P^2R + 2PQR + PR^2 - Q^3) = 0. \qquad (45)$$

Since $P - R \neq 0$ implies

$$-PQ^2R + P^2R + 2PQR + PR^2 - Q^3 = 0. \qquad (46)$$

Replacing q by $-q$ in the above equation (46), we obtain

$$MQ^2R + M^2R - 2MQR - MR^2 - Q^3 = 0. \qquad (47)$$

Multiplying the above equation (47) by $\dfrac{M}{Q^3R}$, we find that

$$\frac{M^2}{Q}+\frac{M^3}{Q^3}-2\frac{M^2}{Q^2}-\frac{M^2R}{Q^3}-\frac{M}{R}=0. \tag{48}$$

Using (40)-(43) in the above equation (48), we obtain (33). □

Theorem 4.2. *Let α, β, γ, λ, μ and ν be of the first, third, ninth, seventh, twentyone and sixtythree degrees respectively. Let $m=\dfrac{z_1}{z_3}, m'=\dfrac{z_3}{z_9}, m_1=\dfrac{z_7}{z_{21}}$ and $m_1'=\dfrac{z_{21}}{z_{63}}$. Then*

$$\begin{aligned}\sqrt{\frac{mm_1'}{m'm_1}}&=\left(\frac{\beta^2\lambda\nu}{\alpha\gamma\mu^2}\right)^{1/8}+\left(\frac{(1-\beta)^2(1-\lambda)(1-\nu)}{(1-\alpha)(1-\gamma)(1-\mu)^2}\right)^{1/8}\\&-2\left(\frac{\beta^2\lambda\nu(1-\beta)^2(1-\lambda)(1-\nu)}{\alpha\gamma\mu^2(1-\alpha)(1-\gamma)(1-\mu)^2}\right)^{1/12}-\left(\frac{\beta^2\lambda\nu(1-\beta)^2(1-\lambda)(1-\nu)}{\alpha\gamma\mu^2(1-\alpha)(1-\gamma)(1-\mu)^2}\right)^{1/8}\end{aligned}\tag{49}$$

and

$$\begin{aligned}\sqrt{\frac{m'm_1}{mm_1'}}&=\left(\frac{\alpha\gamma\mu^2}{\beta^2\lambda\nu}\right)^{1/8}+\left(\frac{(1-\alpha)(1-\gamma)(1-\mu)^2}{(1-\beta)^2(1-\lambda)(1-\nu)}\right)^{1/8}\\&-2\left(\frac{\alpha\gamma\mu^2(1-\alpha)(1-\gamma)(1-\mu)^2}{\beta^2\lambda\nu(1-\beta)^2(1-\lambda)(1-\nu)}\right)^{1/12}-\left(\frac{\alpha\gamma\mu^2(1-\alpha)(1-\gamma)(1-\mu)^2}{\beta^2\lambda\nu(1-\beta)^2(1-\lambda)(1-\nu)}\right)^{1/8}.\end{aligned}\tag{50}$$

Proof. Replacing q by q^2 in the equation (27), we find that

$$R^3+Q^3-R^2Q^2-R^2Q^2-2R^2Q-RQ-2RQ^2=0, \tag{51}$$

where $R=\dfrac{A_4}{A_{28}}$.

Subtracting R^2 times of the equation (27) from P^2 times of the equation (51), we obtain

$$(P-R)(P^2R^2-PQ^3+2PQ^2R-Q^3R+PQR)=0.$$

Since $P-R\neq 0$ implies

$$P^2R^2-PQ^3+2PQ^2R-Q^3R+PQR=0. \tag{52}$$

Replacing q by $-q$ in the above equation (52), we find that

$$M^2R^2+MQ^3-2MQ^2R-Q^3R-MQR=0, \tag{53}$$

where $M=\dfrac{f^2(q^3)f(q^7)f(q^{63})}{qf(q)f(q^9)f^2(q^{21})}$.

Using the equations (10), (11) and (12) in M, Q and R, we obtain

$$M = \sqrt{\frac{z_3^2 z_7 z_{63}}{z_1 z_9 z_{21}^2}} \left(\frac{\beta^2\lambda\nu(1-\beta)^2(1-\lambda)(1-\nu)}{\alpha\gamma\mu^2(1-\alpha)(1-\gamma)(1-\mu)^2}\right)^{1/24}$$

(54)
$$= \sqrt{\frac{m'm_1}{mm_1'}} \left(\frac{\beta^2\lambda\nu(1-\beta)^2(1-\lambda)(1-\nu)}{\alpha\gamma\mu^2(1-\alpha)(1-\gamma)(1-\mu)^2}\right)^{1/24},$$

(55)
$$Q = \sqrt{\frac{m'm_1}{mm_1'}} \left(\frac{\beta^2\lambda\nu(1-\beta)^2(1-\lambda)(1-\nu)}{\alpha\gamma\mu^2(1-\alpha)(1-\gamma)(1-\mu)^2}\right)^{1/12},$$

(56)
$$R = \sqrt{\frac{m'm_1}{mm_1'}} \left(\frac{\beta^2\lambda\nu}{\alpha\gamma\mu^2}\right)^{1/6} \left(\frac{(1-\beta)^2(1-\lambda)(1-\nu)}{(1-\alpha)(1-\gamma)(1-\mu)^2}\right)^{1/24},$$

which implies

(57)
$$\frac{Q}{M} = \left(\frac{\beta^2\lambda\nu(1-\beta)^2(1-\lambda)(1-\nu)}{\alpha\gamma\mu^2(1-\alpha)(1-\gamma)(1-\mu)^2}\right)^{1/24},$$

(58)
$$\frac{Q}{M^2} = \sqrt{\frac{mm_1'}{m'm_1}},$$

(59)
$$\frac{R}{M} = \left(\frac{\beta^2\lambda\nu}{\alpha\gamma\mu^2}\right)^{1/8}$$

and

(60)
$$\frac{Q^3}{M^2R} = \left(\frac{(1-\beta)^2(1-\lambda)(1-\nu)}{(1-\alpha)(1-\gamma)(1-\mu)^2}\right)^{1/8}.$$

Multiplying the equation (53) by $\frac{1}{M^3R}$, we find that

(61)
$$\frac{R}{M} + \frac{Q^3}{M^2R} - 2\frac{Q^2}{M^2} - \frac{Q^3}{M^3} - \frac{Q}{M^2} = 0.$$

Using (57)–(60) in the equation (61), we obtain (49).

Subtracting R times of the equation (27) from P times of the equation (51), we obtain

$$(P-R)(-PQ^2R + P^2R - 2PQR + PR^2 - Q^3) = 0.$$

Since $P - R \neq 0$ implies

(62)
$$-PQ^2R + P^2R - 2PQR + PR^2 - Q^3 = 0.$$

Replacing q by $-q$ in the above equation, we obtain

$$MQ^2R + M^2R + 2MQR - MR^2 - Q^3 = 0. \tag{63}$$

Multiplying the above equation (63) by $\dfrac{M}{Q^3R}$, we find that

$$\frac{M^2}{Q} + \frac{M^3}{Q^3} + 2\frac{M^2}{Q^2} - \frac{M^2R}{Q^3} - \frac{M}{R} = 0. \tag{64}$$

Using (57)-(60) in the above equation (64), we obtain (50). □

Theorem 4.3. *Let α, β, γ, λ, μ and ν be of the first, third, ninth, fifth, fifteenth and fortyfifth degrees respectively. Then we have*

$$Q^3 + \frac{1}{Q^3} = P^2 + \frac{1}{P^2} - 5\left(P + \frac{1}{P}\right) + 10, \tag{65}$$

where

$$P = \left(\frac{\beta^2\mu^2(1-\beta)^2(1-\mu)^2}{\alpha\gamma\lambda\nu(1-\alpha)(1-\gamma)(1-\lambda)(1-\nu)}\right)^{1/24}$$

and

$$Q = \left(\frac{\beta^2\lambda\nu(1-\beta)^2(1-\lambda)(1-\nu)}{\alpha\gamma\mu^2(1-\alpha)(1-\gamma)(1-\mu)^2}\right)^{1/24}.$$

Proof. Set

$$A = \left(\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/24} \quad \text{and} \quad B = \left(\frac{\mu^2(1-\mu)^2}{\lambda\nu(1-\lambda)(1-\nu)}\right)^{1/24}.$$

Multiplying the equations (13) and (14) by $\{\alpha\gamma(1-\alpha)(1-\gamma)\}^{1/4}$ and $\{\beta^2(1-\beta)^2\}^{1/4}$ respectively to obtain,

$$\{\beta^2(1-\alpha)(1-\gamma)\}^{1/4} + \{\alpha\gamma(1-\beta)^2\}^{1/4} = \{\alpha\gamma(1-\alpha)(1-\gamma)\}^{1/4}(A^6 - 3X) \tag{66}$$

and

$$\{\beta^2(1-\alpha)(1-\gamma)\}^{1/4} + \{\alpha\gamma(1-\beta)^2\}^{1/4} = \{\beta^2(1-\beta)^2\}^{1/4}\left(\frac{1}{A^6} + \frac{1}{X}\right), \tag{67}$$

where $X = \dfrac{m}{m'}$. From the above equations, we arrive at

$$A^6 - 3X = A^6\left(\frac{1}{A^6} + \frac{1}{X}\right). \tag{68}$$

Solving for X, we obtain

$$X = \frac{A^6 - 1 + u}{6}, \tag{69}$$

where $u^2 = A^{12} - 14A^6 + 1$.

Replacing q by q^5 in the above equation (69), we get

$$Y = \frac{B^6 - 1 + v}{6}, \tag{70}$$

where $Y = \dfrac{m_1}{m_1'}$ and $v^2 = B^{12} - 14B^6 + 1$.

Multiplying the equations (32) and (33) by $\{\alpha\gamma\lambda\nu(1-\alpha)(1-\gamma)(1-\lambda)(1-\nu)\}^{1/8}$ and $\{\beta^2\mu^2(1-\beta)^2(1-\mu)^2\}^{1/8}$ respectively to obtain,

$$\begin{aligned} &\{\beta^2\mu^2(1-\alpha)(1-\gamma)(1-\lambda)(1-\nu)\}^{1/8} + \{\alpha\gamma\lambda\nu(1-\beta)^2(1-\mu)^2\}^{1/8} \\ &= \{\alpha\gamma\lambda\nu(1-\alpha)(1-\gamma)(1-\lambda)(1-\nu)\}^{1/8}(3\sqrt{XY} - 2A^2B^2 + A^3B^3) \end{aligned} \tag{71}$$

and

$$\begin{aligned} &\{\beta^2\mu^2(1-\alpha)(1-\gamma)(1-\lambda)(1-\nu)\}^{1/8} + \{\alpha\gamma\lambda\nu(1-\beta)^2(1-\mu)^2\}^{1/8} \\ &= \{\beta^2\mu^2(1-\beta)^2(1-\mu)^2\}^{1/8}\left(\frac{1}{\sqrt{XY}} - \frac{2}{A^2B^2} + \frac{1}{A^3B^3}\right). \end{aligned} \tag{72}$$

From the above equations, we arrive at

$$(3\sqrt{XY} - 2A^2B^2 + A^3B^3) = A^3B^3\left(\frac{1}{\sqrt{XY}} - \frac{2}{A^2B^2} + \frac{1}{A^3B^3}\right). \tag{73}$$

Solving for $\sqrt{XY}$, we find that

$$XY = \left(\frac{1 - A^3B^3 + 2A^2B^2 - 2AB + w}{6}\right)^2, \tag{74}$$

where $w^2 = (A^3B^3 - 2A^2B^2 + 2AB - 1)^2 + 12A^3B^3$.

Using (69) and (70) in the equation (74), we find that

$$\begin{aligned} &-A^6B^6 + 8A^5B^5 - 16A^4B^4 + A^6v + 2A^3B^3w + B^6u - A^6 + 8A^3B^3 - B^6 - 4A^2B^2w \\ &-16A^2B^2 + 4ABw + 8AB + uv - u - v - 2w - 1. \end{aligned} \tag{75}$$

Collecting w on one side of the above equation, squaring both sides and then substituting for w^2, u^2 and v^2, we find that

$$\begin{aligned}
&-4A^{11}B^{11}+8A^{10}B^{10}+7A^{12}B^{6}-4A^{9}B^{9}+7A^{6}B^{12}-4A^{11}B^{5}v-4A^{5}B^{11}u+4A^{11}B^{5}\\
&+8A^{8}B^{8}+4A^{5}B^{11}+8A^{10}B^{4}v+8A^{4}B^{10}u-8A^{10}B^{4}-4A^{7}B^{7}-8A^{4}B^{10}+A^{12}v-4A^{9}B^{3}v\\
&+7A^{6}B^{6}u+7A^{6}B^{6}v-4A^{3}B^{9}u+B^{12}u-A^{12}+4A^{9}B^{3}-98A^{6}B^{6}-4A^{5}B^{5}uv+4A^{3}B^{9}\\
&-B^{12}+8A^{8}B^{2}v+4A^{5}B^{5}u+4A^{5}B^{5}v+8A^{2}B^{8}u-8A^{8}B^{2}-4A^{5}B^{5}+8A^{4}B^{4}uv-8A^{2}B^{8}\\
&-4A^{7}Bv-8A^{4}B^{4}u-8A^{4}B^{4}v-4AB^{7}u+4A^{7}B+A^{6}uv+8A^{4}B^{4}-4A^{3}B^{3}uv+4AB^{7}\\
&+B^{6}uv-A^{6}u-7A^{6}v+4A^{3}B^{3}u+4A^{3}B^{3}v-7B^{6}u-B^{6}v+7A^{6}-4A^{3}B^{3}+8A^{2}B^{2}uv\\
&+7B^{6}-8A^{2}B^{2}u-8A^{2}B^{2}v+8A^{2}B^{2}-4ABuv+4ABu+4ABv-4AB=0. \qquad (76)
\end{aligned}$$

Collecting u on one side of the above equation, squaring both sides and then substituting for u^2 and v^2, we find that

$$\begin{aligned}
&(A^2B^2+AB+1)(A^2B^2-AB+1)^2\\
&\times(-A^5B^5+5A^4B^4+A^6-10A^3B^3+B^6+5A^2B^2-AB)=0. \qquad (77)
\end{aligned}$$

As $q \to 0$ in the above equation, $A^2B^2+AB+1 \neq 0$ and the last factor vanishes in the neighbourhood of 0 where as the second factor does not vanish. Hence, by the identity theorem, we obtain (65). □

Theorem 4.4. *Let α, β, γ, λ, μ and ν be of the first, third, ninth, seventh, twentyfirst and sixtythree degrees respectively. Then we have*

$$P^3+\frac{1}{P^3}=Q^4+\frac{1}{Q^4}+7\left(Q^3+\frac{1}{Q^3}\right)+14\left(Q^2+\frac{1}{Q^2}\right)-21, \qquad (78)$$

where

$$P=\left(\frac{\beta^2\mu^2(1-\beta)^2(1-\mu)^2}{\alpha\gamma\lambda\nu(1-\alpha)(1-\gamma)(1-\lambda)(1-\nu)}\right)^{1/24}$$

and

$$Q=\left(\frac{\beta^2\lambda\nu(1-\beta)^2(1-\lambda)(1-\nu)}{\alpha\gamma\mu^2(1-\alpha)(1-\gamma)(1-\mu)^2}\right)^{1/24}.$$

Proof. Since the proof of the above theorem is similar to the proof of the theorem (4.3), so we omit the details. □

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