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Characteristic Polynomials of Oriented Graphs

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Abstract

Graph energy is defined as the sum of the absolute values of all eigenvalues and it has important applications related to molecular graphs. Directed graphs play important role in some applications in social sciences and network studies. There are some studies on several aspects related to directed graphs. But there is a special class of directed graphs called oriented graphs for which there is hardly nothing done. In this paper, we study the spectral properties of oriented graphs. We determine characteristic polynomials of several graph classes, we determine the effect of edge addition to characteristic polynomial. We show that the orientation of pendant edges is not important, but the orientation of the edges on a cycle effects the characteristic polynomial which shows difference from the classical graphs.

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1 Introduction

For a given graph G, the set of all eigenvalues is called the spectrum of G. The sum of absolute values is defined as the graph energy by Gutman and Trinajstić in [6]. In [2], spectra of some graph types are obtained. As the eigenvalues of cycles and paths being trigonometric algebraic numbers are different than other classes of graphs, these two classes need special attention when there is the spectral study of graphs. In [3], the polynomials and recurrence relations for the spectral polynomials of these two graph classes were obtained. In particular, it was shown that we can obtain the spectra of the cycle graph C_{2n} and path graph P_{2n+1} without detailed calculations just in terms of the spectra of C_n and P_n, respectively. In [1], graph energy is calculated for prime graphs

A directed graph (or digraph) is a graph that is a set of vertices connected by edges where each edge has a direction associated with it. The direction of an edge is shown by an arrow put on the edge. The main difference between a graph and digraph is that the edges in the latter one are ordered. An edge connecting two vertices u and v in a graph is denoted by uv and in this case, we say u and v are adjacent to each other. If we denote an edge connecting two vertices u and v in a directed graph, we have to choose either uv or vu and the former one reads as u is adjacent to v and v is adjacent from u, where the latter one reads as v is adjacent to u and u is adjacent from v. Although very rare, there are situations that allow an edge to be be bidirected in a directed graph. That is, both of the edges uv and vu may be amongst the edges of the directed graph.

A directed graph is called oriented graph if no pair of vertices in it is linked by two symmetrical directed edges. That is, an oriented graph can have no bidirected edges (i.e. in an oriented graph, at most one of uv and vu for every vertex pair u and v may be an edge of the graph). There are results obtained for the energy of directed graphs in [7, 8, 9]. Throughout this paper, we take G to be a finite, simple, connected, oriented graph unless stated otherwise.

Recall that the number of non-isomorphic oriented graphs having $n = 1, 2, 3, \cdots$ vertices are $1, 2, 7, 42, 582, 21480, 21422888, \cdots$ (see the OEIS A001174), respectively, and the number of connected oriented graphs on $n = 1, 2, 3, \cdots$ vertices are $1, 1, 5, 34, 535, 20848, 2120098, \cdots$, respectively, (see OEIS A086345).

Similarly to classical adjacency matrix for classical graphs, in oriented graphs, we can define an adjacency matrix $OA(G) = [a_{ij}]$ where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j \\ 0 & \text{if not.} \end{cases}$$

Once again, note that we do not have a symmetrical matrix as in the ordinary adjacency matrix as v_i is adjacent to v_j does not mean v_j is adjacent to v_i. In this paper, the adjacency matrices of oriented graphs are calculated. Let G be an oriented graph and let λ_i's be the eigenvalues of the adjacency matrix A. As well-known, λ_i's are the roots of

$$\det(\lambda I_n - A) = 0. \tag{1}$$

The left hand side of Equation (1) is a polynomial of λ of degree n and will be denoted by $P(G)$.

2 Characteristic polynomials of some oriented graph classes

In this section, we compute the characteristic polynomials $P(G)$ of some oriented graph classes G. First we have a useful result:

Lemma 2.1. *Let G_1 and G_2 be two connected oriented graphs of order 1 and 2, respectively. Then*

$$P(G_1) = \lambda \qquad \textit{and} \qquad P(G_2) = \lambda^2.$$

Proof. The characteristic matrix of G_1 is $[\lambda]$ with determinant λ. The characteristic matrix of the oriented graph G_2 is $\begin{pmatrix} \lambda & -1 \\ 0 & \lambda \end{pmatrix}$ or $\begin{pmatrix} \lambda & 0 \\ -1 & \lambda \end{pmatrix}$ both having determinant λ^2. □

Theorem 2.2. *Let G be a connected oriented graph, $v \in G$ and let $G + e = G + vu$ be the graph obtained by adding the oriented pendant edge $e = vu$ to G, see Fig. 1. Then*

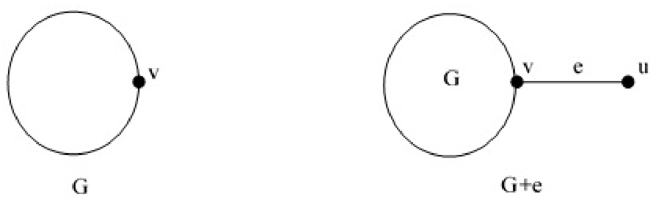

Figure 1: Adding an edge to G

$$P(G + e) = \lambda \cdot P(G).$$

This is independent from the choice of the orientation of e.

Proof. Let the orientation of the new edge vu be from u to v (the proof is similar if the orientation is from v to u). Let the characteristic matrix of G be $P(G)$. The characteristic matrix of $G+e$ would be

$$P(G+e) = \begin{vmatrix} P(G) & A \\ B & \lambda \end{vmatrix}$$

where A is an $(n \times 1)-$ column matrix and B is a $(1 \times n)-$ row matrix. Also all entries in A and B are zero except for only one entry which is -1. Therefore the result follows. □

Theorem 2.2 guarantees that the orientation of any pendant edge added to a connected graph is not important and just multiplies the polynomial by λ. Therefore adding k oriented pendant edges to a connected graph G means that the characteristic polynomial of G will be multiplied by λ^k. The following results are just some applications of Theorem 2.2.

Now we know that adding a new oriented pendant edge to an oriented graph G multiplies the characteristic polynomial of G by λ. Applying this consecutively, we can give characteristic polynomials of some important oriented graph classes. The following are some consequences:

Corollary 2.3. *Let G' be the oriented graph obtained by adding k oriented pendant edges to an oriented graph G. Then*

$$P(G') = \lambda^k \cdot P(G).$$

Corollary 2.4. *Let G be a graph (not necessarily an oriented graph) obtained by joining a graph G_1 with a path P_{k+1} by identifying one of two endpoints of the path with a vertex of G_1, see Fig. 2. Then*

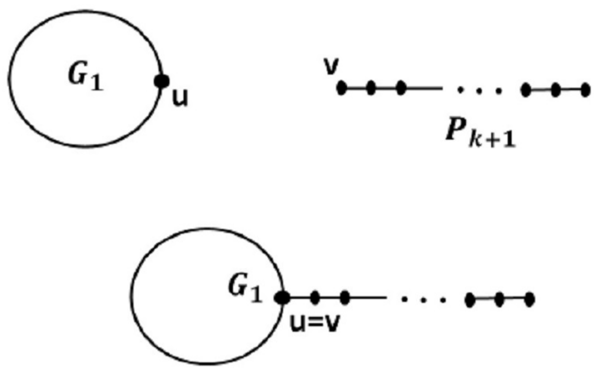

Figure 2: Adding a path to a graph

$$P(G) = P(G_1) \cdot \lambda^k.$$

Corollary 2.5. *For an oriented tadpole graph $T_{r,s}$, we have*

$$P(T_{r,s}) = P(C_r) \cdot P(P_s) = P(C_r) \cdot \lambda^s.$$

Corollary 2.6. *The characteristic polynomial of an oriented path graph P_n is*

$$P(P_n) = \lambda^n.$$

Proof. First we prove the result for a path graph P_n with all the edges $v_{i-1}v_i$ having the same orientation from v_{i-1} to v_i , for $i = 2, 3, 4, \cdots, n$. The characteristic polynomial of this oriented graph is obtained by means of the determinant

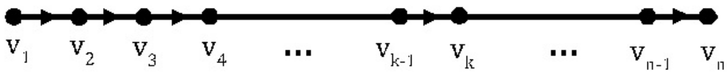

Figure 3: An oriented path graph

$$P(P_n) = \begin{array}{c|cccccccccc|}
 & v_1 & v_2 & v_3 & \cdots & v_{k-1} & v_k & \cdots & v_{n-1} & v_n \\
v_1 & \lambda & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
v_2 & 0 & \lambda & -1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
v_3 & 0 & 0 & \lambda & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
v_{k-1} & 0 & 0 & 0 & \cdots & \lambda & -1 & \cdots & 0 & 0 \\
v_k & 0 & 0 & 0 & \cdots & 0 & \lambda & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
v_{n-1} & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \lambda & -1 \\
v_n & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \lambda
\end{array}_{n\times n}.$$

This determinant is an upper triangular determinant and therefore the result is obtained.

It is not difficult to see that there are $2^{n-1}/2 = 2^{n-2}$ different orientations of P_n. We now want to show that $P(P_n)$ is independent from the choice of orientation of the edges of P_n. We do this by showing that the change in the orientation of any edge does not effect the characteristic polynomial.

Let us change the orientation of an arbitrary edge $v_{k-1}v_k$ from v_k to v_{k-1} as in Fig. 4: The

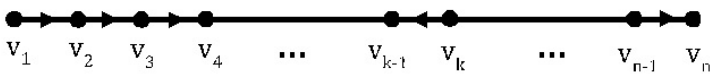

Figure 4: An oriented path graph with only one orientation reversed

characteristic polynomial of this graph is

$$P(P_n) = \begin{array}{c|cccccccccc|}
 & v_1 & v_2 & v_3 & \cdots & v_{k-1} & v_k & \cdots & v_{n-1} & v_n \\
v_1 & \lambda & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
v_2 & 0 & \lambda & -1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
v_3 & 0 & 0 & \lambda & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
v_{k-1} & 0 & 0 & 0 & \cdots & \lambda & 0 & \cdots & 0 & 0 \\
v_k & 0 & 0 & 0 & \cdots & -1 & \lambda & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
v_{n-1} & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \lambda & -1 \\
v_n & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \lambda
\end{array}_{n\times n}.$$

Note that the only difference between (1) and (2) is that the two entries $a_{k-1,k}$ and $a_{k,k-1}$ are transpozed. So both determinants are equal as required. □

This result can easily be generalized to any acyclic graph as follows:

Corollary 2.7. *Let T_n be any connected oriented acyclic graph with n vertices. Then*

$$P(T_n) = \lambda^n.$$

So we can say that if an oriented graph having n vertices has no cycles, then its characteristic polynomial is λ^n. This situation is not true in general. There are exceptions as we shall see below:

Theorem 2.8. *The characteristic polynomial of an oriented cycle graph C_n is*

$$P(C_n) = \begin{cases} \lambda^n - 1 & \text{if all orientations are the same} \\ \lambda^n & \text{otherwise.} \end{cases}$$

Proof. If all the orientations are the same, then we have

$$P(C_n) = \begin{array}{c|cccccc|} & v_1 & v_2 & v_3 & \cdots & v_{n-1} & v_n \\ \hline v_1 & \lambda & -1 & 0 & \cdots & 0 & 0 \\ v_2 & 0 & \lambda & -1 & \cdots & 0 & 0 \\ v_3 & 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{n-2} & 0 & 0 & 0 & \cdots & -1 & 0 \\ v_{n-1} & 0 & 0 & 0 & \cdots & \lambda & -1 \\ v_n & -1 & 0 & 0 & \cdots & 0 & \lambda \end{array}_{n\times n}.$$

If we calculate this determinant according to the first column, we get two $(n-1)\times(n-1)$ determinants

$$P(C_n) = \lambda \begin{vmatrix} \lambda & -1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & -1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & \cdots & \lambda & -1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{vmatrix} + (-1)^n \begin{vmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \\ \lambda & -1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & \cdots & \lambda & -1 \end{vmatrix} \qquad (2)$$

$$= \lambda^n - 1.$$

Secondly, let at least one orientation be different than the others. Let us name this edge as $e = v_k v_{k+1}$. Then

$$P(C_n) = \begin{array}{c|cccccccccc|} & v_1 & v_2 & v_3 & \cdots & v_{k-1} & v_k & \cdots & v_{n-1} & v_n \\ \hline v_1 & \lambda & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ v_2 & 0 & \lambda & -1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ v_3 & 0 & 0 & \lambda & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{k-1} & 0 & 0 & 0 & \cdots & \lambda & 0 & \cdots & 0 & 0 \\ v_k & 0 & 0 & 0 & \cdots & -1 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{n-1} & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \lambda & -1 \\ v_n & -1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \lambda \end{array}_{n\times n}$$

The upper right $(k-2)\times(n-k+2)$ matrix is a zero matrix, so $P(C_n) = \lambda^{k-2}\cdot\lambda^{n-(k-2)} = \lambda^n$ giving the result. If more than one edge, say $v_{i_1-1}v_{i_1}$, $v_{i_2-1}v_{i_2}$, $\cdots$, $v_{i_k-1}v_{i_k}$, change their orientations, then

$$P(C_n) = \lambda^{i_1-1}\cdot\lambda^{i_2-1-(i_1-1)}\cdots\lambda^{n-(i_k-1)} = \lambda^n$$

as required. □

Let us continue our study of unicyclic graphs. The following result helps us to obtain the general result for unicyclic graphs and has very important consequences:

Corollary 2.9. *Let G be a connected unicyclic oriented graph with n vertices having a cycle of length $k \leq n$. Then*

$$P(G) = P(C_k) \cdot \lambda^{n-k}.$$

3 Conclusion

Spectral study of graphs is a widely studied subject due to its applications related to chemical energy. In this paper, the spectral polynomials of oriented graphs are studied. Characteristic polynomials of several important oriented graph classes are found. The effect of edge addition on the characteristic polynomial of an oriented graph is determined. The characteristic polynomial of an oriented acyclic graph is given. As there is no spectral study on oriented graphs, this work will play an important role in the area. The target of this paper can also be applied to oriented graphs in terms of omega index [4, 5] as a future project. This would imply a new classification via integers. Note that there are more zero eigenvalues in the graphs having smaller cyclomatic number and therefore smaller omega invariant. But our results show that when the cyclomatic number increases, then the graph will have more non-zero eigenvalues.

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