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THE NEW FUBINI-TYPE NUMBERS AND POLYNOMIALS

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Abstract. The nth Fubini number counts the ordered partitions of a set with n elements. There are many variants of Fubini numbers and polynomials. One of them is the degenerate Fubini polynomials which were introduced as a degenerate version of the Fubini polynomials. In this paper, we consider the new two-variables Fubini-type numbers and polynomials arising from p-adic fermionic integrals on $\mathbb{Z}_p$ and those degenerate version. We derive some interesting properties, recurrence relations and explicit formulas for those numbers and polynomials.

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1. Introduction

For any $\lambda \in \mathbb{R}$, the degenerate exponential function is defined by

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}, \quad \text{(see [8, 11-13])}, \tag{1}$$

where $(x)_{0,\lambda}=1$ and $(x)_{n,\lambda} = x(x-\lambda)\cdots\big(x-(n-1)\lambda)\big)$, $(n \geq 1)$.
The degenerate Stirling numbers of the second kind are given by

$$(x)_{n,\lambda} = \sum_{l=0}^{n} S_{2,\lambda}(n,l)(x)_l \quad (n \geq 0), \quad \text{(see [8, 11-13])}, \tag{2}$$

and the generating function

$$\frac{1}{k!}\big(e_\lambda(t)-1\big)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!}, \quad (k \geq 0), \quad \text{(see [8, 11-13])}, \tag{3}$$

where $(x)_0 = 1$ and $(x)_n = x(x-1)\cdots\big(x-(n-1))\big)$, $(n \geq 1)$.
When $\lambda \to 0$, $S_2(n,k)$ are the Stirling numbers of the second kind.
By (3), we easily get

$$S_{2,\lambda}(n,k) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(j)_{n,\lambda}, \quad \text{(see [8])}. \tag{4}$$

It is well known that the degenerate Bell polynomials and are given by

$$\phi_{n,\lambda}(x) = \sum_{k=0}^{n} S_{2,\lambda}(n,k)x^k,$$

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and the generating function of them

$$e^{x(e_\lambda(t)-1)} = \sum_{n=0}^{\infty} \phi_{n,\lambda}(x)\frac{t^n}{n!}, \quad \text{(see [11])}. \tag{5}$$

When $\lambda \to 0$, $\phi_n(x)$ are the Bell polynomials.

The degenerate versions of some special numbers and polynomials' interests were not only in combinatorial and arithmetical properties but also in applications to differential equations, identities of symmetry and probability theory [2, 9, 10, 12-16]. These degenerate versions include the degenerate Stirling numbers of the first and second kinds, degenerate Bernoulli numbers of the second kind and degenerate Bell numbers and polynomials [9-16, 19].

Cayley(1859) [3] considered the ordered Bell numbers which used them to count certain plane trees with $n+1$ totally ordered leaves. The ordered Bell numbers are given by

$$\beta_n = \sum_{k=0}^{n} k! S_2(n,k), \quad (n \geq 0) \quad \text{(see [3, 6])}. \tag{6}$$

By (6), we have the generating function of F_n as

$$\frac{1}{2-e^t} = \sum_{n=0}^{\infty} \beta_n \frac{t^n}{n!}, \quad \text{(see [3])}. \tag{7}$$

The two variables Fubini polynomials are given by

$$\frac{1}{1-y(e^t-1)}e^{xt} = \sum_{n=0}^{\infty} F_n(y \mid x)\frac{t^n}{n!}, \quad \text{(see [12, 13, 16])}, \tag{8}$$

when $x=0$, $F_n(y) = F_n(y \mid 0)$ are called the Fubini numbers.

When $x=0$ and $y=1$ in (8), we have the ordered Bell numbers.

Moreover, the case $y = -\frac{1}{2}$ in (8) gives the Euler polynomials $E_n(x) = F_n(-\frac{1}{2}, x)$. Recently, some authors have been very interested in arithmetic properties for the Fubini polynomials [5, 6, 7, 12, 13].

The degenerate two variables Fubini polynomials are given by

$$\frac{1}{1-y(e_\lambda(t)-1)}e_\lambda^x(t) = \sum_{n=0}^{\infty} F_{n,\lambda}(y \mid x)\frac{t^n}{n!}, \quad \text{(see [12, 13])}. \tag{9}$$

The p-adic analysis and their applications utilize p-adic distributions and p-adic measure, p-adic integrals, p-adic L-function, and other generalized functions. Among these, the p-adic integral and its applications are very important in finding solutions to special (differential) equations, real problems in both physics and engineering ([7, 9, 10, 14, 15, 17]).

Let p be a prime number with $p \equiv 1 \pmod 2$. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let$|\cdot|_p$ be the p-adic norm with $|p|_p = \frac{1}{p}$.

For a $\mathbb{C}_p$-valued continuous function f on $\mathbb{Z}_p$, the p-adic fermionic integral on $\mathbb{Z}_p$ is given by

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \lim_{\mathbb{N}\to\infty} \sum_{x=0}^{p^{\mathbb{N}}-1} f(x)\mu_{-1}(x+p^{\mathbb{N}}\mathbb{Z}_p)$$
$$= \lim_{\mathbb{N}\to\infty} \sum_{x=0}^{p^{\mathbb{N}}-1} f(x)(-1)^x, \quad \text{(see [9, 10])}. \tag{10}$$

Let $f_n(x) = f(x+n)$ for $n \in \mathbb{N}$. From (5), we observe that

$$I_{-1}(f_n) + (-1)^{n-1} I_{-1}(f) = 2\sum_{l=0}^{n-1} (-1)^{n-1-l} f(l), \quad \text{(see [9, 10])}. \tag{11}$$

In (11), when $n = 1$, we have

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0). \tag{12}$$

We note that

$$(1-t)^{-m} = \sum_{l=0}^{\infty} \binom{-m}{l} (-1)^l t^l = \sum_{l=0}^{\infty} < m >_l \frac{t^l}{l!}, \quad \text{(see [4])}. \tag{13}$$

where $< x >_0 = 1$ and $< x >_n = x(x+1)(x+2)\cdots(x+n-1)$, $(n \geq 1)$.

In this paper, we consider the new two-variables Fubini-type numbers and polynomials arising from p-adic fermionic integrals on $\mathbb{Z}_p$ and those degenerate version. We derived some interesting properties, recurrence relations and explicit formulas for those numbers and polynomials.

2. A new Fubini-type numbers and polynomials

We naturally consider

$$\int_{\mathbb{Z}_p} (y(e^t+1))^z e^{xt} d\mu_{-1}(z) = \frac{2e^{xt}}{1+y(e^t+1)}, \tag{14}$$

and

$$\int_{\mathbb{Z}_p} (y(e_\lambda(t)+1))^z e_\lambda^x(t) d\mu_{-1}(z) = \frac{2e_\lambda^x(t)}{1+y(e_\lambda(t)+1)}. \tag{15}$$

When $x = 0$, we have

$$\int_{\mathbb{Z}_p} (y(e^t+1))^z d\mu_{-1}(z) = \frac{2}{1+y(e^t+1)}, \tag{16}$$

and

$$\int_{\mathbb{Z}_p} (y(e_\lambda(t)+1))^z d\mu_{-1}(z) = \frac{2}{1+y(e_\lambda(t)+1)}. \tag{17}$$

From (14), we consider the two variables Fubini-type polynomials given by

$$\frac{1}{1+y(e^t+1)} e^{xt} = \sum_{n=0}^{\infty} J_n(y \mid x) \frac{t^n}{n!}. \tag{18}$$

When $x = 0$, $J_n(y) = J_n(y \mid 0)$ are called the Fubini-type numbers as

$$\frac{1}{1+y(e^t+1)}=\sum_{n=0}^{\infty}J_n(y)\frac{t^n}{n!}. \tag{19}$$

From (15), we naturally the degenerate two variables Fubini-type polynomials of the first kind given by

$$\frac{1}{1+y(e_\lambda(t)+1)}e_\lambda^x(t)=\sum_{n=0}^{\infty}J_{n,\lambda}(y|x)\frac{t^n}{n!}. \tag{20}$$

When $x=0$, $J_{n,\lambda}(y)=J_{n,\lambda}(y\mid 0)$ are called the degenerate Fubini-type numbers as

$$\frac{1}{1+y(e_\lambda(t)+1)}e_\lambda^x(t)=\sum_{n=0}^{\infty}J_{n,\lambda}(y\mid x)\frac{t^n}{n!}. \tag{21}$$

When $y=-\frac{1}{3}$, we note that $\dfrac{3}{2-e^t}=3\beta_n=J_n\Big(-\dfrac{1}{3}\Big)$.
When $y=1$, we observe that

$$\begin{aligned}\frac{1}{2+e^t}&=\sum_{n=0}^{\infty}J_n(1)\frac{t^n}{n!}=\frac{1}{2}\sum_{k=0}^{\infty}\Big(-\frac{e^t}{2}\Big)^k\\&=\frac{1}{2}\sum_{k=0}^{\infty}\Big(-\frac{1}{2}\Big)^k\sum_{n=0}^{\infty}k^n\frac{t^n}{n!}=\sum_{n=0}^{\infty}\Big(\frac{1}{2}\sum_{k=0}^{\infty}\Big(-\frac{1}{2}\Big)^k k^n\Big)\frac{t^n}{n!}.\end{aligned} \tag{22}$$

From (22), we have

$$J_n(1)=\frac{1}{2}\sum_{k=0}^{\infty}\Big(-\frac{1}{2}\Big)^k k^n.$$

For $y\neq-\frac{1}{2}$, we observe that

$$\begin{aligned}\frac{e_\lambda^x(t)}{1+y(e_\lambda(t)+1)}&=\frac{e_\lambda^x(t)}{(1+2y)+y(e_\lambda(t)-1)}\\&=\frac{1}{1+2y}\frac{e_\lambda^x(t)}{1+\frac{y}{1+2y}(e_\lambda(t)-1)}\\&=\frac{1}{1+2y}\sum_{n=0}^{\infty}F_{n,\lambda}\Big(-\frac{y}{1+2y}\ \Big|\ x\Big)\frac{t^n}{n!}.\end{aligned} \tag{23}$$

Comparing the coefficients of (21) and (23), we have

$$J_{n,\lambda}(y\mid x)=\frac{1}{1+2y}F_{n,\lambda}\Big(-\frac{y}{1+2y}\ \Big|\ x\Big). \tag{24}$$

When $x=0$, we have

$$J_{n,\lambda}(y)=\frac{1}{1+2y}F_{n,\lambda}\Big(-\frac{y}{1+2y}\Big). \tag{25}$$

When $\lambda\to 0$, we have

$$J_n(y\mid x)=\frac{1}{1+2y}F_n\Big(-\frac{y}{1+2y}\mid x\Big), \tag{26}$$

and

$$J_n(y)=\frac{1}{1+2y}F_n\Big(-\frac{y}{1+2y}\Big). \tag{27}$$

From (1) and (21), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}J_{n,\lambda}(y\mid x)\frac{t^n}{n!}&=\frac{e_\lambda^x(t)}{1+y(e_\lambda(t)+1)}\\&=\sum_{j=0}^{\infty}J_{j,\lambda}(y)\frac{t^j}{j!}\sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{t^l}{l!}\\&=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}J_{n-l,\lambda}(y)(x)_{l,\lambda}\frac{t^n}{n!}.\end{aligned} \tag{28}$$

Comparing the coefficients of the both side of (28), we get

$$J_{n,\lambda}(y\mid x)=\sum_{l=0}^{n}\binom{n}{l}J_{n-l,\lambda}(y)(x)_{l,\lambda}. \tag{29}$$

When $\lambda\to 0$, we have

$$J_n(y\mid x)=\sum_{l=0}^{n}\binom{n}{l}J_{n-l}(y)x^l. \tag{30}$$

The degenerate ordered Bell polynomials are given by the generating function

$$\frac{1}{2-e_\lambda(t)}e_\lambda^x(t)=\sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!}. \tag{31}$$

When $x=0$, $\beta_{n,\lambda}^{(r)}=\beta_{n,\lambda}^{(r)}(0)$ are called the degenerate ordered Bell numbers. When $y=-\frac{1}{3}$, we have

$$J_{n,\lambda}\big(-\frac{1}{3}|x\big)=3F_{n,\lambda}(1|x)=3\beta_{n,\lambda}(x). \tag{32}$$

When $\lambda\to 0$ at (32),

$$J_n\big(-\frac{1}{3}|x\big)=3F_n(1|x)=3\beta_n(x). \tag{33}$$

Theorem 2.1. *For $n\geq 0$, we have*

$$J_{n,\lambda}(y)=\frac{1}{1+2y}\sum_{k=0}^{n}\Big(-\frac{y}{1+2y}\Big)^k k!S_{2,\lambda}(n,k).$$

When $\lambda\to 0$, we have

$$J_n(y)=\frac{1}{1+2y}\sum_{k=0}^{n}\Big(-\frac{y}{1+2y}\Big)^k k!S_2(n,k).$$

Proof. From (3) and (20), we observe that

$$
\begin{aligned}
\frac{1}{1+y(e_\lambda(t)+1)} &= \frac{1}{1+2y}\frac{1}{1+\frac{y}{1+2y}(e_\lambda(t)-1)} \\
&= \frac{1}{1+2y}\sum_{k=0}^{\infty}\Big(-\frac{y}{1+2y}\Big)^k (e_\lambda(t)-1)^k\frac{k!}{k!} \\
&= \frac{1}{1+2y}\sum_{k=0}^{\infty}k!\Big(-\frac{y}{1+2y}\Big)^k\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!} \\
&= \frac{1}{1+2y}\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}k!\Big(-\frac{y}{1+2y}\Big)^k S_{2,\lambda}(n,k)\Big)\frac{t^n}{n!}.
\end{aligned} \tag{34}
$$

By (20) and (34), we have

$$
J_{n,\lambda}(y) = \frac{1}{1+2y}\sum_{k=0}^{n}\Big(-\frac{y}{1+2y}\Big)^k k!S_{2,\lambda}(n,k).
$$

□

From (3), it is easy to see that

$$
S_{2,\lambda}(n,k) = \frac{1}{k!}\sum_{c=0}^{k}\binom{k}{c}(-1)^{k-c}(c)_{n,\lambda}. \tag{35}
$$

From (35), we obtain the following corollary.

Corollary 2.2. *For $n \geq 0$, we get*

$$
J_{n,\lambda}(y) = \frac{1}{1+2y}\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{k}{l}(-1)^l(l)_{n,\lambda}\Big(\frac{y}{1+2y}\Big)^k.
$$

When $\lambda \to 0$, we have

$$
J_n(y) = \frac{1}{1+2y}\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{k}{l}(-1)^l l^n\Big(\frac{y}{1+2y}\Big)^k.
$$

Theorem 2.3. *For $n \geq 0$, we have*

$$
J_{n,\lambda}(y \mid x+1) = \sum_{l=0}^{n}\binom{n}{l}(1)_{n-l,\lambda}J_{l,\lambda}(y|x).
$$

When $\lambda \to 0$, we have

$$
J_n(y \mid x+1) = \sum_{l=0}^{n}\binom{n}{l}J_l(y|x).
$$

Proof. From (3) and (20), we note that

$$\begin{aligned}\frac{e_\lambda^{x+1}(t)-e_\lambda^x(t)}{1+y(e_\lambda(t)+1)} &= \frac{e_\lambda^x(t)(e_\lambda(t)-1)}{1+y(e_\lambda(t)+1)}\\ &= \sum_{l=0}^{\infty} J_{l,\lambda}(y\mid x)\frac{t^l}{l!}\sum_{j=1}^{\infty}(1)_{j,\lambda}\frac{t^j}{j!}\\ &= \sum_{n=1}^{\infty}\sum_{l=0}^{n-1}\binom{n}{l}J_{l,\lambda}(y\mid x)(1)_{n-l,\lambda}\frac{t^n}{n!}.\end{aligned} \tag{36}$$

On the other hand, by (20), we have

$$\frac{e_\lambda^{x+1}(t)-e_\lambda^x(t)}{1+y(e_\lambda(t)+1)} = \sum_{n=0}^{\infty}\{J_{n,\lambda}(y\mid x+1)-J_{n,\lambda}(y\mid x)\}\frac{t^n}{n!}. \tag{37}$$

Comparing the coefficients of (36) and (37), for $n\geq 1$, we have

$$J_{n,\lambda}(y\mid x+1)-J_{n,\lambda}(y\mid x) = \sum_{l=0}^{n-1}\binom{n}{l}(1)_{n-l,\lambda}J_{l,\lambda}(y\mid x). \tag{38}$$

From (38), we get

$$\begin{aligned}J_{n,\lambda}(y\mid x+1) &= J_{n,\lambda}(y\mid x)+\sum_{l=0}^{n-1}\binom{n}{l}(1)_{n-l,\lambda}J_{l,\lambda}(y\mid x)\\ &= \sum_{l=0}^{n}\binom{n}{l}(1)_{n-l,\lambda}J_{l,\lambda}(y|x).\end{aligned} \tag{39}$$

□

Theorem 2.4. *For $n\geq 0$ and $y\neq 1$, we have*

$$\frac{1}{1-y}J_{n,\lambda}\left(\frac{y}{1-y}\;\middle|\; x\right) = \sum_{l=0}^{\infty}(-1)^l y^l (l+x)_{n,\lambda}.$$

When $\lambda\to 0$, we have

$$\frac{1}{1-y}J_{n}\left(\frac{y}{1-y}\;\middle|\; x\right) = \sum_{l=0}^{\infty}(-1)^l y^l (l+x)^{n}.$$

Proof. By (1). when $y\neq 1$, we have

$$\begin{aligned}\frac{1}{1-y}\frac{e_\lambda^x(t)}{1+\frac{y}{1-y}(e_\lambda(t)+1)} &= \frac{e_\lambda^x(t)}{1+ye_\lambda(t)} = \sum_{l=0}^{\infty}(-1)^l y^l e_\lambda^{l+x}(t)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}(-1)^l y^l (l+x)_{n,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{40}$$

On the other hand, by (19), we get

$$\frac{1}{1-y}\frac{e_\lambda^x(t)}{1+\frac{y}{1-y}(e_\lambda(t)+1)} = \frac{1}{1-y}\sum_{n=0}^{\infty}J_{n,\lambda}\left(\frac{y}{1-y}\;\middle|\; x\right)\frac{t^n}{n!}. \tag{41}$$

Comparing the coefficients of the both side of (40) and (41), we have the desired result.

□

When $x = 0$ in Theorem 5, we obtain the following corollary .

Corollary 2.5. *When $x = 0$, we have*

$$\frac{1}{1-y} J_{n,\lambda}\left(\frac{y}{1-y}\right) = \sum_{l=0}^{\infty} (-1)^l (l)_{n,\lambda} y^l = 1 - (1)_{n,\lambda} y + (2)_{n,\lambda} y^2 - (3)_{n,\lambda} y^3 + \cdots,$$

and

$$\frac{1}{1-y} J_n\left(\frac{y}{1-y}\right) = \sum_{l=0}^{\infty} (-1)^l y^l l^n = 1 - y + 2^n y^2 - 3^n y^3 + \cdots.$$

Theorem 2.6. *For $n \geq 0$, we have*

$$J_{n,\lambda}(y) = \sum_{k=0}^{n} (-1)^k S_{2,\lambda}(n,k) \left(y \frac{d}{dy}\right)_k \frac{1}{1+2y}.$$

When $\lambda \to 0$, we have

$$J_n(y) = \sum_{k=0}^{n} (-1)^k S_2(n,k) \left(y \frac{d}{dy}\right)_k \frac{1}{1+2y},$$

where $\left(y \frac{d}{dy}\right)_k = y \frac{d}{dy}(y \frac{d}{dy} - 1) \cdots (y \frac{d}{dy} - (k-1))$.

Proof. We note that

$$\left(y \frac{d}{dy}\right)_k \frac{1}{1+2y} = \left(y \frac{d}{dy}\right)_k \left(\sum_{j=0}^{\infty} (-2y)^j\right) = \sum_{j=0}^{\infty} (-2)^j (j)_k y^j. \tag{42}$$

By Theorem 1, (13) and (42), we observe that

$$
\begin{aligned}
J_{n,\lambda}(y) &= \sum_{k=0}^{n}(-1)^k y^k k! S_{2,\lambda}(n,k)\left(\frac{1}{1+2y}\right)^{k+1} \\
&= \sum_{k=0}^{n}(-1)^k k! S_{2,\lambda}(n,k) y^k \sum_{j=0}^{\infty} < k+1 >_j \frac{(-2y)^j}{j!} \\
&= \sum_{k=0}^{n}(-1)^k k! S_{2,\lambda}(n,k) y^k \sum_{j=0}^{\infty} \binom{k+j}{j} (-2)^j y^j \\
&= \sum_{j=0}^{\infty}(-2)^j \sum_{k=0}^{n}(-1)^k k! S_{2,\lambda}(n,k) \binom{k+j}{k} y^{k+j} \\
&= \sum_{j=0}^{\infty}(-2)^j \sum_{k=0}^{n}(-1)^k k! S_{2,\lambda}(n,k) \binom{j}{k} y^{j} \\
&= \sum_{j=0}^{\infty}(-2)^j y^j \sum_{k=0}^{n}(-1)^k S_{2,\lambda}(n,k) (j)_k \\
&= \sum_{k=0}^{n}(-1)^k S_{2,\lambda}(n,k) \sum_{j=0}^{\infty}(-2)^j y^j (j)_k \\
&= \sum_{k=0}^{n}(-1)^k S_{2,\lambda}(n,k) \left(y\frac{d}{dy}\right)_k \frac{1}{1+2y}.
\end{aligned} \tag{43}
$$

From (43), we obtain the desired identity. □

Boyadzhiev [1] introduced

$$
\sum_{n=0}^{\infty}\frac{g^{(n)}(0)}{n!} f(n) x^n = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\sum_{k=0}^{n} S_2(n,k) x^k g^{(k)}(x), \tag{44}
$$

where $f,\ g$ are appropriate functions.

Theorem 2.7. *For $n \geq 0$, we have*

$$
(-2)^n n^n y^n = J_n(y),
$$

and

$$
(yD)^n\left(\frac{1}{1+2y}\right) = J_n(y).
$$

Proof. Let us take $g(y) = \frac{1}{1+2y}$, $(|2y| < 1)$. Then

$$
\frac{g^{(n)}(0)}{n!} = (-1)^n 2^n, \quad (n \geq 0), \tag{45}
$$

and

$$
g^{(k)}(x) = \frac{(-1)^k k! 2^k}{(1+2y)^{k+1}}. \tag{46}
$$

By (44), (45), (46) and Theorem 1, we have

$$\begin{aligned} \sum_{n=0}^{\infty}(-2)^n f(n)y^n &= \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\sum_{k=0}^{n}S_2(n,k)y^k(-1)^k\frac{k!}{(1+2y)^{k+1}} \\ &= \frac{1}{1+2y}\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\sum_{k=0}^{n}\Big(-\frac{y}{1+2y}\Big)^k k!S_2(n,k) \\ &= \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}J_n(y). \end{aligned} \tag{47}$$

Comparing the coefficients of the both sides of (47), we have

$$(-2)^n f(n)y^n = \frac{f^{(n)}(0)}{n!}J_n(y). \tag{48}$$

Let $f(x) = x^n$, $n = 0,\ 1,\ 2, \dots$ with $0^0 = 1$. Then, we get $f^{(n)}(0) = n!$. From (48), we obtain

$$(-2)^n n^n y^n = J_n(y). \tag{49}$$

On the other hand, we observe that

$$(yD)^n y^n = n^n y^n \quad \text{and} \quad \frac{1}{1+2y} = \sum_{n=0}^{\infty}(-2)^n y^n, \quad (|2y| < 1). \tag{50}$$

By (49) and (50), we have

$$(yD)^n\Big(\frac{1}{1+2y}\Big) = \sum_{n=0}^{\infty}(-2)^n(yD)^n y^n = \sum_{n=0}^{\infty}(-2)^n n^n y^n = \sum_{n=0}^{\infty}J_n(y). \tag{51}$$

By (49) and (51), we get the desired result. □

Corollary 2.8. *For $l \geq 1$, we have*

$$(1 - 2 + 2^2 2^2 - 2^3 3^3 + \cdots + (-2)^n n^n)y^n = \frac{1}{1-y}J_n(y).$$

Proof. For $l \in \mathbb{N}$, from (51) in Theorem 7, we observe that

$$\begin{aligned} \frac{1}{1-y}\sum_{n=0}^{\infty}J_n(y) = \frac{1}{1-y}\sum_{k=0}^{\infty}(-2)^k k^k y^k &= \sum_{l=0}^{\infty}y^l\sum_{k=0}^{\infty}(-2)^k k^k y^k \\ &= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}(-1)^k 2^k k^k\Big)y^n. \end{aligned} \tag{52}$$

By (52), we obtain the desired identity. □

Theorem 2.9. *For $n \geq 0$, we have*

$$\int_0^{\infty}\phi_{n,\lambda}(-xy)e^{x(2y-1)}dx = J_{n,\lambda}(y).$$

When $\lambda \to 0$, we have

$$\int_0^{\infty}\phi_n(-xy)e^{x(2y-1)}dx = J_n(y).$$

Proof. From (17), we observe that

$$\begin{aligned}\int_0^\infty e^{-xy(e_\lambda(t)+1)}e^{-x}dx &= \int_0^\infty e^{-x(1+y(e_\lambda(t)+1))}dx \\ &= \frac{1}{1+y(e_\lambda(t)+1)} = \sum_{n=0}^\infty J_{n,\lambda}(y)\frac{t^n}{n!}.\end{aligned} \tag{53}$$

On the other hand, by (5), we get

$$\begin{aligned}\int_0^\infty e^{-xy(e_\lambda(t)+1)}e^{-x}dx &= \int_0^\infty e^{-xy(e_\lambda(t)-1)}e^{2xy-x}dx \\ &= \sum_{n=0}^\infty \int_0^\infty \phi_{n,\lambda}(-xy)e^{x(2y-1)}dx.\end{aligned} \tag{54}$$

By (53) and (54), we have the desired identity. □

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The authors declare that there is no ethical problem in the production of this paper.

Competing interests
All authors declare no conflict of interest.

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