Advanced Studies in Contemporary Mathematics
33 (2023), No. 2, pp. 181 - 186

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2023.33.2.181

ON THE DEGENERATE MULTI-POLY-GENOCCHI POLYNOMIALS AND NUMBERS

JIN-WOO PARK

Abstract. Recently, the degenerate version of special polynomials are defined by many researcher and found some new and interesting identities by using Carlitz's degenerate exponential function.

In this paper, we define the degenerate multi-poly-Genocchi polynomials and numbers and found some interesting relationships between Genocchi polynomials, the falling factorial polynomials and the Stirling numbers of the second kind.

2010 Mathematics Subject Classification. 33E20, 05A30, 11B83, 11S80.

Keywords and phrases. Poly-logarithm function, Genocchi polynomials, degenerate exponential function.

1. Introduction

The *Genocchi numbers* which are defined by the A. Genocchi are defined by the generating function to be

$$\frac{2t}{e^t+1}e^x(t)=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!},\ \text{(see [4, 17])}.$$

When $x=0$. $G_n=G_n(0)$ are called the *Genocchi numbers.*

The properties and applications of the Genocchi polynomials and numbers have bee investigated in the various field by many researchers. In [1], authors investigated some interesting properties of weighted q-Genocchi polynomials by using fermionic p-adic q-integral on $\mathbb{Z}_p$. Luo generalized the Genocchi polynomials which are called the q-Apostol-Genocchi polynomials and derived an explicit formula and a variety of relations for these polynomials including a differential equation, integral formula and recursive formulas in [14]. Belbachir-Hadj-Rachid defined a mixed polynomials which are called the Euler-Genocchi polynomials and found some properties including the expression of the power of a variable, the Raabe-like formula, the linear recurrence and the difference equations (see [2]). In [12], authors investigated the properties of degenerate Genocchi polynomials of higher-order by using λ-umbral calculus, and in [6], authors defined a new type generalized Genocchi polynomials which were called the degenerate poly-Genocchi polynomials with the degenerate polylogarithm function and derived some new explicit expressions and identities of those polynomials.

For a given $\lambda\in\mathbb{R}-\{0\}$, the *degenerate exponential function* is defined to be

$$e_\lambda^x(t)=(1+\lambda t)^{\frac{x}{\lambda}},\ e_\lambda(t)=(1+\lambda t)^{\frac{1}{\lambda}},\ \text{(see [3])}. \tag{1}$$

In [13], Lim defined the degenerate Genocchi polynomials of order r as follows:

$$\sum_{n=0}^{\infty} G_{n,\lambda}^{(r)}(x)\frac{t^n}{n!} = \left(\frac{2t}{e_\lambda(t)+1}\right)^r e_\lambda^x(t), \ (|t| < \pi).$$

For $k_1, k_2, \ldots, k_r \in \mathbb{Z}$, the *multiple polylogarithm function* is defined by

$$Li_{k_1,\ldots,k_r}(x) = \sum_{0<n_1<\cdots<x_r} \frac{x^{n_r}}{n_1^{k_1} n_2^{k_2} \cdots n_r^{k_r}}, \ \text{(see [8, 18])}. \tag{2}$$

In particular, $Li_1(x) = -\log(1-x)$.

By (2), we note that

$$\begin{aligned} \frac{d}{dx}Li_{k_1,\ldots,k_r}(x) =& \frac{d}{dx}\sum_{o<n_1<\cdots<n_r} \frac{x^{n_r}}{n_1^{k_1} n_2^{k_2} \cdots n_r^{k_r}} \\ =& \frac{1}{x}\sum_{0<n_1<\cdots<n_r} \frac{x^{n_r}}{n_1^{k_1} n_2^{k_2} \cdots n_r^{k_r-1}} \\ =& \frac{1}{x}Li_{k_1,\ldots,k_r-1}(x), \end{aligned} \tag{3}$$

and by (3), we see that

$$\begin{aligned} \frac{d}{dx}Li_{k_1,\ldots,k_{r-1},1}(x) =& \sum_{0<n_1<\cdots<n_r} \frac{1}{n_1^{k_1} n_2^{k_2} \cdots n_{r-1}^{k_{r-1}}} \sum_{n_r=n_{r-1}+1}^{\infty} x^{n_r-1} \\ =& \sum_{0<n_1<\cdots<n_{r-1}} \frac{1}{n_1^{k_1} n_2^{k_2} \cdots n_{r-1}^{k_{r-1}}} \frac{x_{n_{r-1}}}{1-x} \\ =& \frac{1}{1-x}Li_{k_1,\ldots,k_{r-1}}, \end{aligned} \tag{4}$$

and thus

$$Li_{k_1,1,}(x) = \int \frac{1}{1-x}Li_{k_1}(x)dx, \ \text{(see [8])}. \tag{5}$$

By (5) and induction, we see that

$$Li_{\underbrace{1,1,\ldots,1}_{r-times}}(x) = \frac{(-1)^r}{r!}\left(\log(1-x)\right)^r, \ (r \in \mathbb{N}). \tag{6}$$

(see [8]).

For nonzero integers n and k, the *Stirling numbers of the first kind* $S_1(n,k)$ and *Stirling numbers of the second kind* $S_2(n,k)$, respectively, are given by

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k, \ \text{(see [4, 17])}, \tag{7}$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \geq 1)$ is the falling factorial sequences. By (7), we can derive easily the followings (see [4, 17])

$$\frac{1}{k!}\left(\log(1+t)\right)^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}, \text{ and } \frac{1}{k!}\left(e^t-1\right)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}. \tag{8}$$

The generalizations of various special functions using the multi polylogarithm function have been attempted actively by many researchers (see [5, 7, 8, 9, 10, 11, 15, 16, 18]).

In this paper, we defined a degenerate multi-poly-Genocchi polynomials and numbers by using multiple polylogarithm function and degenerate exponential function and derived some interesting identities which are related to the falling factorial functions, Stirling numbers of the second kind and Genocchi polynomials.

2. Multi-poly-Genocchi polynomials and numbers

By (2), we define the *degenerate multi-poly-Genocchi polynomials* which are defined by the generating function to be

$$\sum_{n=0}^{\infty} G_{n,\lambda}^{(k_1,\ldots,k_r)}(x)\frac{t^n}{n!} = \frac{2^r Li_{k_1,\ldots,k_r}\left(1-e^{-t}\right)}{\left(e_\lambda(t)+1\right)^r} e_\lambda^x(t). \tag{9}$$

When $x=0$, $G_{n,\lambda}^{(k_1,\ldots,k_r)} = G_{n,\lambda}^{(k_1,\ldots,k_r)}(0)$ are called the *degenerate multi-poly-Gencochi numbers*. Note that, by (6), $G_{n,\lambda}^{(\overbrace{1,\ldots,1}^{r-times})}(x) = \frac{1}{r!}G_{n,\lambda}^{(r)}(x)$.

By the definition of degenerate multi-poly-Genocchi polynomials, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} G_{n,\lambda}^{(k_1,\ldots,k_r)}(x+y)\frac{t^n}{n!} &= \frac{2^r Li_{k_1,\ldots,k_r}\left(1-e^{-t}\right)}{\left(e_\lambda(t)+1\right)^r} e_\lambda^{x+y}(t) \\
&= \frac{2^r}{\left(e_\lambda(t)+1\right)^r} Li_{k_1,\ldots,k_r}\left(1-e^{-t}\right) e_\lambda^x(t) e_\lambda^y(t) \\
&= \left(\sum_{n=0}^{\infty} G_{n,\lambda}^{(k_1,\ldots,k_r)}(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(y)_{n,\lambda}\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m} G_{n-m,\lambda}^{(k_1,\ldots,k_r)}(x)(y)_{m,\lambda}\right)\frac{t^n}{n!}.
\end{aligned} \tag{10}$$

By (10), we obtain the following theoem.

Theorem 2.1. *For each nonnegative integer n, we have*

$$G_{n,\lambda}^{(k_1,\ldots,k_r)}(x+y) = \sum_{m=0}^{n}\binom{n}{m} G_{n-m,\lambda}^{(k_1,\ldots,k_r)}(x)(y)_{m,\lambda}.$$

In the special case $y=0$ of Theorem 2.1, we obtain the following corollary.

Corollary 2.2. *For each nonnegative integer n, we have*

$$G_{n,\lambda}^{(k_1,\ldots,k_r)}(x) = \sum_{m=0}^{n}\binom{n}{m} G_{n-m,\lambda}^{(k_1,\ldots,k_r)}(x)_{m,\lambda}.$$

Note that

$$
\begin{aligned}
&\sum_{a=1}^{\infty}\frac{\left(1-e^{-t}\right)^a}{(a+b)^k}=\sum_{a=1}^{\infty}\frac{a!}{a+b)^k}\frac{1}{a!}\left(1-e^{-t}\right)^a\\
&=\sum_{a=1}^{\infty}a!(a+b)^{-k}\sum_{l=a}(-1)^{l-a}S_2(l,a)\frac{t^l}{l!}\\
&=\sum_{a=1}^{\infty}a!\sum_{m=0}^{\infty}(-1)^m\binom{k+m-1}{m}b^m a^{-k-m}\sum_{l=a}(-1)^{l-a}S_2(l,a)\frac{t^l}{l!}\\
&=\sum_{m=0}^{\infty}(-1)^m\binom{k+m-1}{m}\sum_{l=0}^{\infty}\sum_{a=1}^{l+1}\frac{a!(-1)^{l+1-a}S_2(l+1,a)}{l+1}a^{-k-m}\frac{t^l}{l!}.
\end{aligned}\tag{11}
$$

By (11), we get

(12)

$$
\begin{aligned}
&\sum_{n=0}^{\infty}G_{n,\lambda}^{(k_1,\ldots,k_r)}(x)\frac{t^n}{n!}=\frac{2^r Li_{k_1,\ldots,k_r}\left(1-e^{-t}\right)}{\left(e_\lambda(t)+1\right)^r}e_\lambda^x(t)\\
=&\frac{2^r e_\lambda^x(t)}{\left(e_\lambda(t)+1\right)^r}\sum_{0<n_1<n_2<\cdots<n_{r-1}}\frac{1}{n_1^{k_1}n_2^{k_2}\cdots n_{r-1}^{k_{r-1}}}\sum_{n_r=n_{r-1}+1}^{\infty}\frac{\left(1-e^{-t}\right)^{n_r}}{n_r^{k_r}}\\
=&\frac{2^r e_\lambda^x(t)}{\left(e_\lambda(t)+1\right)^r}\sum_{0<n_1<n_2<\cdots<n_{r-1}}\frac{\left(1-e^{-t}\right)^{n_{r-1}}}{n_1^{k_1}n_2^{k_2}\cdots n_{r-1}^{k_{r-1}}}\sum_{n_r=1}^{\infty}\frac{\left(1-e^{-t}\right)^{n_r}}{\left(n_r+n_{r-1}\right)^{k_r}}\\
=&\frac{2t}{e_\lambda(t)+1}e_\lambda^x(t)\sum_{m=0}^{\infty}(-1)^m\binom{k_r+m-1}{m}\sum_{i=0}^{\infty}G_{i,\lambda}^{(k_1,\ldots,k_{r-1}-m)}(x)\frac{t^i}{i!}\\
&\times\sum_{l=0}^{\infty}\sum_{n_r=1}^{l+1}\frac{n_r!(-1)^{l+1-n_r}S_2(l+1,n_r)}{l+1}n_r^{-k_r-m}\frac{t^l}{l!}\\
=&\left(\sum_{n=0}^{\infty}G_{n,\lambda}\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{n_r=1}^{l+1}\sum_{m=0}^{\infty}(-1)^m\binom{k_r+m-1}{m}\binom{k}{l}\right.\\
&\left.\times\frac{n_r!(-1)^{l+1-n_r}S_2(l+1,n_r)}{l+1}n_r^{-k_r-m}G_{k-l,\lambda}^{(k_1,\ldots,k_{r-1}-m)}(x)\frac{t^k}{k!}\right)\\
=&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}\sum_{n_r=1}^{l+1}\sum_{m=0}^{\infty}(-1)^m\binom{k_r+m-1}{m}\binom{k}{l}\binom{n}{k}\right.\\
&\left.\times\frac{n_r!(-1)^{l+1+n_r}S_2(l+1,n_r)}{l+1}n_r^{-k_r-m}G_{k-l,\lambda}^{(k_1,\ldots,k_{r-1}-m)}(x)G_{n-k,\lambda}(x)\right)\frac{t^n}{n!}.
\end{aligned}
$$

By (12), we obtain the following theorem.

Theorem 2.3. *For each nonnegative integer n, we have*

$$
\begin{aligned}
G_{n,\lambda}^{(k_1,\ldots,k_r)}(x)=&\sum_{k=0}^{n}\sum_{l=0}^{k}\sum_{n_r=1}^{l+1}\sum_{m=0}^{\infty}(-1)^m\binom{k_r+m-1}{m}\binom{k}{l}\binom{n}{k}\\
&\times\frac{n_r!(-1)^{l+1+n_r}S_2(l+1,n_r)}{l+1}n_r^{-k_r-m}G_{k-l,\lambda}^{(k_1,\ldots,k_{r-1}-m)}(x)G_{n-k,\lambda}(x).
\end{aligned}
$$

If we replace k_r by $-k_r$, then we obtain the following corollary.

Corollary 2.4. *For each nonnegative integer n, we have*

$$G_{n,\lambda}^{(k_1,\ldots,-k_r)}(x)=\sum_{k=0}^{n}\sum_{l=0}^{k}\sum_{n_r=1}^{l+1}\sum_{m=0}^{\infty}(-1)^m\binom{k_r}{m}\binom{k}{l}\binom{n}{k}\times\frac{n_r!(-1)^{l+1+n_r}S_2(l+1,n_r)}{l+1}n_r^{k_r-m}G_{k-l,\lambda}^{(k_1,\ldots,k_{r-1}-m)}(x)G_{n-k,\lambda}(x).$$

Note that, by (11), we see that

(13)

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(G_{n,\lambda}^{(k_1,\ldots,k_r)}(x+1)+G_{n,\lambda}^{(k_1,\ldots,k_r)}(x)\right)\frac{t^n}{n!}=\frac{2^r Li_{k_1,\ldots,k_r}\left(1-e^{-t}\right)}{(e_\lambda(t)+1)^{r-1}}\\
=&\frac{2^r e_\lambda^x(t)}{(e_\lambda(t)+1)^{r-1}}\sum_{0<n_1<\cdots<n_{r-1}}\frac{1}{n_1^{k_1}n_2^{k_2}\cdots n_{r-1}^{k_{r-1}}}\sum_{n_r=n_{r-1}+1}^{\infty}\frac{\left(1-e^{-t}\right)^{n_r}}{n_r^{k_r}}\\
=&\frac{2^r e_\lambda^x(t)}{(e_\lambda(t)+1)^{r-1}}\sum_{0<n_1<\cdots<n_{r-1}}\frac{\left(1-e^{-t}\right)^{n_{r-1}}}{n_1^{k_1}n_2^{k_2}\cdots n_{r-1}^{k_{r-1}}}\sum_{n_r=1}^{\infty}\frac{\left(1-e^{-t}\right)^{n_r}}{(n_r+n_{r-1})^{k_r}}\\
=&\frac{2^r e_\lambda^x(t)}{(e_\lambda(t)+1)^{r-1}}\sum_{m=0}^{\infty}(-1)^m\binom{k_r+m-1}{m}Li_{k_1,\ldots,k_{r-1}-m}\left(1-e^{-t}\right)\\
&\times\sum_{l=1}^{\infty}\sum_{n_r=1}^{l}n_r!(-1)^{l-n_r}n_r^{-k_r-m}S_2(l,n_r)\frac{t^l}{l!}\\
=&2\sum_{m=0}^{\infty}(-1)^m\binom{k_r+m-1}{m}\sum_{i=0}^{\infty}G_{i,\lambda}^{(k_1,\ldots,k_{r-1}-m)}(x)\frac{t^i}{i!}\sum_{l=1}^{\infty}\sum_{n_r=1}^{l}n_r!(-1)^{l-n_r}n_r^{-k_r-m}S_2(l,n_r)\frac{t^l}{l!}\\
=&\sum_{n=1}^{\infty}\left(2\sum_{m=0}^{\infty}(-1)^m\binom{k_r+m-1}{m}\sum_{l=1}^{n}\sum_{n_r=1}^{l}\binom{n}{l}n_r!(-1)^{n_r}n_r^{-k_r-m}S_2(l,n_r)G_{n-l,\lambda}^{(k_1,\ldots,k_{r-1}-m)}(x)\right)\frac{t^n}{n!}.
\end{aligned}$$

By (13), we obtain the following theorem.

Theorem 2.5. *For each positive integer n, we have*

$$\begin{aligned}
&G_{n,\lambda}^{(k_1,\ldots,k_r)}(x+1)+G_{n,\lambda}^{(k_1,\ldots,k_r)}(x)\\
=&2\sum_{m=0}^{\infty}(-1)^m\binom{k_r+m-1}{m}\sum_{l=1}^{n}\sum_{n_r=1}^{l}\binom{n}{l}n_r!(-1)^{n_r}n_r^{-k_r-m}S_2(l,n_r)G_{n-l,\lambda}^{(k_1,\ldots,k_{r-1}-m)}(x).
\end{aligned}$$

3. Acknowledgements

The authors would like to thank the referees for their valuable and detailed comments which have significantly improved the presentation of this paper.

4. Funding

This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT) NRF-2020R1F1A1A01075658

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Department of Mathematics Education, Daegu University, 38453, Republic of Korea.

Email address: a0417001@daegu.ac.kr