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EXISTENCE AND UNIQUENESS ANALYSIS FOR THE GENERALIZED CAPUTO-TYPE FRACTIONAL-ORDER BOUNDARY VALUE PROBLEM

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Abstract. Over the years, several results related to the solution existence of the fractional-order boundary value problems have come using different techniques. In this paper, we apply Banach's contraction theorem to demonstrate the existence of a unique solution for a fractional boundary value problem that is subject to the generalized Caputo fractional derivative. In this regard, we propose a novel lemma, proposition, and theorem to justify our approach. We complete our findings with an illustrative example.

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1. Introduction

The fractional Boundary Value Problems (BVPs) have been shown great interest by researchers as a consequence of the fast advances in the literature on fractional calculus, [1, 2, 3]. Out of the core areas of mathematics, BVPs in terms of fractional derivatives have many applications in the various scientific fields [4, 5, 6, 7]. Several works related to the existence of unique solutions for fractional BVPs are conducted and driven by numerous applications [8, 9]. Concurrently, integer order BVPs are commonly studied, and even fractional BVPs have arisen as an important field of investigation until quite recently. In this way, intensive studies related to the unique solution existence of fractional order BVPs have been carried out in favor of methods of nonlinear analysis such as topological degree theory, fixed point theory, monotone iterative methods, and lower and upper solutions theory. Readers are advised to see the literature review on the topic.

The following result is taken from one theorem proved in [10], which is also stated without proof in [11] (Theorem 3.3) as well as left as a problem in [12] (Problem [41.13]) to its readers.

Theorem 1.1. *([3]) Assume $j : [\eta, \zeta] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous function and satisfying a uniform Lipschitz constraint respect to ψ*

$$|j(\tau, \psi) - j(\tau, \phi)| \le Y|\psi - \phi|,$$

for $(\tau, \psi),\ (\tau, \phi) \in [\eta, \zeta] \times \mathbb{R}$, where $Y > 0$ is a constant. If

$$Y\frac{(\zeta-\eta)^2}{8}<1,$$

then the BVP

$$\psi''(\tau)=-j(\tau,\psi(\tau)),\quad \psi(\eta)=\xi_1,\quad \psi(\zeta)=\xi_2,$$

has a unique solution.

In our paper, we desire to further increase the results mentioned above in the theorem by incorporating a generalized Caputo fractional derivative (we advise the reader to see [13] for fundamental aspects of fractional-order operators) at the place of integer-order operator ψ'', i.e we analyze the solution existence along with its uniqueness for the following fractional BVP

$$ {}^{\varrho}_{c}D^{\delta}_{0^+}\psi(\tau)=-j(\tau,\psi(\tau)),\quad 0<\tau<1 \tag{1}$$

$$\psi(0)=\xi_1,\qquad \psi(1)=\xi_2. \tag{2}$$

Here $1<\delta\leq 2$. As aforementioned, the solution existence along with uniqueness for the different types fractional BVPs have been studied by many researchers (see, for example, [14, 15, 16, 17, 18] and the references cited therein). As far as we know, our results are the first to report the fractional counterpart of the problem given in Theorem 1.1.

2. Basic notions

Firstly, we present some literature materials from the theory of fractional calculus to analyze the study, see [13, 19].

Definition 2.1. *The generalized left-sided fractional integral* ${}^{\varrho}I^{\delta}_{a^+}\psi$ *of order* $\delta\in\mathbb{C}(Re(\delta)>0)$ *is defined by*

$$({}^{\varrho}I^{\delta}_{a^+}\psi)(\tau)=\frac{\varrho^{1-\delta}}{\Gamma(\delta)}\int_a^{\tau}s^{\varrho-1}(\tau^{\varrho}-s^{\varrho})^{\delta-1}\psi(s)ds \tag{3}$$

where $\tau>a,\ \varrho>0$.
Corresponding to the generalized fractional integrals (3)*, the generalized fractional derivative for* $\tau>a$*, is defined by*

$$({}^{\varrho}D^{\delta}_{a^+}\psi)(\tau)=\left(\tau^{1-\varrho}\frac{d}{d\tau}\right)^m({}^{\varrho}I^{m-\delta}_{a^+}\psi)(\tau)$$

$$=\frac{\varrho^{\delta-m+1}}{\Gamma(m-\delta)}\left(\tau^{1-\varrho}\frac{d}{d\tau}\right)^m\int_a^{\tau}s^{\varrho-1}(\tau^{\varrho}-s^{\varrho})^{m-1-\delta}\psi(s)ds \tag{4}$$

Definition 2.2. *The generalized Caputo fractional derivative with the operator notation* ${}^{\varrho}_{c}D^{\delta}_{a^+}$ *is defined by*

$${}^{\varrho}_{c}D^{\delta}_{a^+}\psi(\tau)=\left({}^{\varrho}D^{\delta}_{a^+}\left[\psi(\tau)-\sum_{i=0}^{m-1}\frac{\psi^{(i)}(a)}{i!}(t-a)^i\right]\right)(\tau) \tag{5}$$

where $m=[\delta]$.

3. Main Results

We give the integral formula for the boundary value problem (1) – (2) in terms of the Green function.

Lemma 3.1. *Let us take j as a continuous mapping. A function $\psi \in C[0,1]$ solves the problem (1) – (2) if and only if ψ is the solution to the integral equation*

$$\psi(\tau) = (\xi_2 - \xi_1)\tau^{\varrho} + \xi_1 + \int_0^1 G(\tau, s) j(s, \psi(s)) ds$$

where

$$G(\tau, s) = \frac{\varrho^{1-\delta}}{\Gamma(\delta)} \begin{cases} \tau^{\varrho} s^{\varrho-1}(1-s^{\varrho})^{\delta-1} - s^{\varrho-1}(\tau^{\varrho} - s^{\varrho})^{\delta-1}, & 0 \le s \le \tau \le 1, \\ \\ \tau^{\varrho} s^{\varrho-1}(1-s^{\varrho})^{\delta-1}, & 0 \le \tau \le s \le 1. \end{cases}$$

Proof. Consider the problem

$${}^{\varrho}_{c}D^{\delta}_{0+}\psi(\tau) = -\Upsilon(\tau)$$

$${}^{\varrho}_{c}I^{\delta}_{0+}\ {}^{\varrho}_{c}D^{\delta}_{0+}\psi(\tau) = -{}^{\varrho}I^{\delta}_{0+}\Upsilon(\tau) + c_0 + c_1\frac{\tau^{\varrho}}{\varrho}$$

$$\psi(\tau) = -{}^{\varrho}I^{\delta}_{0+}\Upsilon(\tau) + c_0 + c_1\frac{\tau^{\varrho}}{\varrho}$$

$$\psi(\tau) = -\frac{\varrho^{1-\delta}}{\Gamma(\delta)}\int_0^{\tau} s^{\varrho-1}(\tau^{\varrho} - s^{\varrho})^{\delta-1}\Upsilon(s) ds + c_0 + c_1\frac{\tau^{\varrho}}{\varrho}.$$

Using the boundary conditions

$$\psi(0) = \xi_1 \Longrightarrow c_0 = \xi_1,$$

$$\psi(1) = \xi_2 \Longrightarrow c_1 = \varrho(\varrho_2 - \varrho_1) + \frac{\varrho^{2-\delta}}{\Gamma(\delta)}\int_0^1 s^{\varrho-1}(1-s^{\varrho})^{\delta-1}\Upsilon(s) ds.$$

Replacing in $\psi(\tau)$, we get

$$\psi(\tau) = -\frac{\varrho^{1-\delta}}{\Gamma(\delta)}\int_0^{\tau} s^{\varrho-1}(t^{\varrho} - s^{\varrho})^{\delta-1}\Upsilon(s) ds$$
$$+ \left[\varrho(\xi_2 - \xi_1) + \frac{\varrho^{2-\delta}}{\Gamma(\delta)}\int_0^1 s^{\varrho-1}(1-s^{\varrho})^{\delta-1}\Upsilon(s) ds\right]\frac{\tau^{\varrho}}{\varrho} + \xi_1.$$

$$\psi(\tau) = -\frac{\varrho^{1-\delta}}{\Gamma(\delta)}\int_0^{\tau} s^{\varrho-1}(\tau^{\varrho}-s^{\varrho})^{\delta-1}\Upsilon(s) ds + \frac{\varrho^{1-\delta}\tau^{\varrho}}{\Gamma(\delta)}\int_0^1 s^{\varrho-1}(1-s^{\varrho})^{\delta-1}\Upsilon(s) ds$$
$$+(\xi_2 - \xi_1)\tau^{\varrho} + \xi_1.$$

Therefore,

$$\psi(\tau) = (\xi_2 - \xi_1)\tau^{\varrho} + \xi_1 + \frac{\varrho^{1-\delta}}{\Gamma(\delta)}\int_0^{\tau}(\tau^{\varrho}s^{\varrho-1}(1-s^{\varrho})^{\delta-1} - s^{\varrho-1}(\tau^{\varrho}-s^{\varrho})^{\delta-1})\Upsilon(s) ds$$
$$+\frac{\varrho^{1-\delta}}{\Gamma(\delta)}\int_{\tau}^1 \tau^{\varrho}s^{\varrho-1}(1-s^{\varrho})^{\delta-1}\Upsilon(s) ds,$$

and the proof is completed. □

Now, we will present the following result which is very necessary to prove our main simulation.

Proposition 3.2. *Consider G be the Green function mentioned in Lemma 3.1. Then,*

$$\int_0^1 |G(\tau,s)|ds \le \frac{1}{\varrho^\delta \Gamma(\delta+1)}\left(\delta^{1/(1-\delta)} - \delta^{\delta/(1-\delta)}\right).$$

Proof. It is obvious that

$$\int_0^1 |G(\tau,s)|ds,\ G(\tau,s)\ge 0,\ \forall\ 0\le\tau,\ s\le 1.$$

Thus

$$\int_0^1 |G(\tau,s)|ds = \frac{\varrho^{1-\delta}}{\Gamma(\delta)}\int_0^\tau (\tau^\varrho s^{\varrho-1}(1-s^\varrho)^{\delta-1} - s^{\varrho-1}(\tau^\varrho - s^\varrho)^{\delta-1})ds$$
$$+\frac{\varrho^{1-\delta}}{\Gamma(\delta)}\int_\tau^1 \tau^\varrho s^{\varrho-1}(1-s^\varrho)^{\delta-1}ds,$$

we calculate the primitives by integration by a change of variable

$$\int_0^1 |G(\tau,s)|ds = \frac{\varrho^{1-\delta}\tau^\varrho}{\Gamma(\delta)}[\frac{1}{\varrho\delta}(1-(1-\tau^\varrho)^\delta)] - \frac{\varrho^{1-\delta}}{\Gamma(\delta)}(\frac{\tau^{\varrho\delta}}{\varrho\delta}) + \frac{\varrho^{1-\delta}\tau^\varrho}{\Gamma(\delta)}[\frac{1}{\varrho\delta}(1-\tau^\varrho)^\delta].$$

Then

$$\int_0^1 |G(\tau,s)|ds = \frac{1}{\varrho^\delta\Gamma(\delta+1)}(\tau^\varrho - \tau^{\varrho\delta}).$$

We define $\varphi : [0,1] \longrightarrow \mathbb{R}$ by

$$\varphi(\tau) = \frac{1}{\varrho^\delta}(\tau^\varrho - \tau^{\varrho\delta}).$$

Differentiating the function φ, we calculate that it's maximum is received at the point

$$\tau^* = \delta^{1/\varrho(1-\delta)}.$$

Moreover

$$\varphi(\tau^*) = \frac{1}{\varrho^\delta}\left(\delta^{1/(1-\delta)} - \delta^{\delta/(1-\delta)}\right),$$

which finishes the proof. □

Theorem 3.3. *Let $j : [0,1]\times\mathbb{R} \longrightarrow \mathbb{R}$ is a continuous mapping satisfying the uniform Lipschitz property for the second variable defined on $[0,1]\times\mathbb{R}$ with Lipschitz constant Y, such that,*

$$|j(\tau,\psi) - j(\tau,\phi)| \le Y|\psi - \phi|,$$

for $(\tau,\psi),\ (\tau,\phi)\in[0,1]\times\mathbb{R}$, where $Y>0$ is a constant. If

$$\frac{Y}{\varrho^\delta\Gamma(1+\delta)}\left[\delta^{1/(1-\delta)} - \delta^{\delta/(1-\delta)}\right] < 1, \tag{6}$$

then the BVP

$${}^{\varrho}_{c}D^{\delta}_{0+}\psi(\tau) = -j(\tau,\psi(\tau)),\quad 0<\tau<1 \tag{7}$$

$$\psi(0) = \xi_1,\qquad \psi(1) = \xi_2, \tag{8}$$

has a unique solution.

Proof. Let Σ be the Banach space of continuous mappings derived on $[0,1]$ under the norm

$$\|\psi\| = \max_{\tau\in[0,1]} |\psi(\tau)|.$$

By Lemma 3.1, $\psi \in C[0,1]$ becomes a solution to (7) – (8) if and only if it satisfies the integral equation

$$\psi(\tau) = (\xi_2 - \xi_1)\tau^{\varrho} + \xi_1 + \int_0^1 G(\tau,s)j(s,\psi(s))ds.$$

Define the operator $\Pi : \Sigma \longrightarrow \Sigma$ by

$$\Pi\psi(\tau) = (\xi_2 - \xi_1)\tau^{\varrho} + \xi_1 + \int_0^1 G(\tau,s)j(s,\psi(s))ds,$$

for $\tau \in [0,1]$. Now we will demonstrate that for the defined operator Π there exists a unique fixed-point. Consider $\psi, \phi \in \Sigma$, then

$$\begin{aligned}
|\Pi\psi(\tau) - \Pi\phi(\tau)| &\le \int_0^1 |G(\tau,s)|\,|j(s,\psi(s)) - j(s,\phi(s))|\,ds \\
&\le \int_0^1 |G(\tau,s)|\,(Y|\psi(\tau) - \phi(\tau)|)\,ds \\
&\le Y\int_0^1 |G(\tau,s)|\|\psi - \phi\|\ ds \\
&\le Y\frac{1}{\varrho^{\delta}\Gamma(\delta+1)}\left(\delta^{1/(1-\delta)} - \delta^{\delta/(1-\delta)}\right)\|\psi - \phi\|,
\end{aligned}$$

here we have applied Proposition 3.2. From (6), we get that Π is a contracting mapping on Σ, and by applying the Banach contraction theorem, we receive the expected output. So, we get that Π has a unique fixed point in $C[0,1]$, this implies that the BVP (7) – (8) has a unique solution. □

Remark We note that when $\delta = 2, \varrho = 1$ in Theorem 3.3 in condition (6), one immediately obtains Theorem 1, which we need to consider in order to take continuous solutions on $[0,1]$ to (7).

4. Application

Here we solve an example to check the correctness of the above simulated results.

Example: We consider the following fractional BVP

$$ {}^{1}_{c}D^{4/3}_{0^+}\psi(\tau) = 1 + \psi(\tau) - \cos(t), \quad 0 < \tau < 1 \tag{9}$$

$$\psi(0) = 2, \qquad \psi(1) = 3. \tag{10}$$

Here

$$j(\tau,\psi(\tau)) = 1 + \psi(\tau) - \cos(\tau),$$

therefore

$$|j(\tau,\psi) - j(\tau,\phi)| \le Y|\psi - \phi|.$$

Since $\delta = 4/3,\ \varrho\ = 1$, we have

$$\frac{Y}{\varrho^{\delta}\Gamma(1+\delta)}\left[\delta^{1/(1-\delta)} - \delta^{\delta/(1-\delta)}\right] = \frac{1}{4/9\ \Gamma(1/3)}\ \frac{27}{256} \cong 0.08858 < 1,$$

So Eq. (6) is satisfied. Now by the application of Theorem 3.3, we get that (9) − (10) has a unique solution.

Conclusion

In this manuscript, we have successfully obtained novel results regarding the existence of a unique solution for a generalized Caputo-type fractional-order boundary value problem. We have proposed our results by using a lemma, proposition, and theorem. Also, we have solved an example to check the correctness of the derived results. Given analyses are fully novel and correct for use in future simulations.

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