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GENRALIZED OF THE FUBINI-EULER-GENOCCHI POLYNOMIALS

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ABSTRACT. Recently, Kim-Kim-Kim introduced generalized degenerate Euler-Genocchi polynomials and investigated some properties and identities of those polynomials. The purpose of this paper is to introduce the generalized Fubini-Euler-Genocchi polynomials and derived some properties and identities. In addition, we define their higher-order version that is the generalized Fubini-Euler-Genocchi polynomials of order α and obtain some basic relationship between Euler, Genocchi, and Fubini polynomials and Fubini-Euler-Genocchi polynomials.

1. INTRODUCTION

In 2019, Belbachir and Hadj-Brahim introduced the Euler-Genocchi polynomials and investigated some properties and identities of those polynomials and Goubi studied the generalized Euler-Genocchi polynomials and higher-order the generalized Euler-Genocchi polynomials in [6]. Recently, Kim-Kim-Kim introduced the generalized degenerate Euler-Genocchi polynomials and investigated some properties and identities of those polynomials. From the idea of the generalized degenerate Euler-Genocchi polynomials, we introduce the generalized Fubini-Euler-Genocchi polynomials and the generalized Fubini-Euler-Genocchi polynomials of order α.

The outline of this paper is as follows. In Section 1, we recall the Stirling numbers of the second kind, the Euler polynomials, the Genocchi polynomials, the Fubini polynomials and higher-order version of those polynomials. Also we remind the two varialbe Fubini polynomials of order α and the gamma function. In Section 2, we obtain the main results of this paper. We introduce the generalized Fubini-Euler-Genocchi polynomials $C_n^{(r)}(x,y)$ and the generalized Fubini-Euler-Genocchi polynomials of order α $C_n^{(r,\alpha)}(x,y)$ and obtain the relationshop with the special polynomials introduced in Section 1. In the last section, we recall the results that are needed throughout this paper.

The Stirling numbers of the second kind are given by

$$x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k, \quad (\text{see}[8]).$$

and

$$\frac{1}{k!}(e^t-1)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}, \quad (\text{see}[8]). \tag{1}$$

Where

$$(x)_0 = 1(x)_n = (x)(x-1)\cdots(x-n+1), (n \geq 1).$$

It is well known that the Euler polynomials are defined by

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \quad (\text{see}[4,6,7,11]).$$

When $x=0$, $E_n = E_n(0)$ are called the Euler numbers.

The Genocchi polynomials are given by

$$\frac{2t}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}, \quad (\text{see}[4,6,10]).$$

When x=0, $G_n = G_n(0)$ are called the Genocchi numbers.

The Fubini polynomialws are defined by

$$\frac{1}{1-x(e^t-1)} = \sum_{n=0}^{\infty} F_n(x)\frac{t^n}{n!}, \quad (\text{see}[3,9]). \tag{2}$$

From (1) and (2) we note that

$$F_n(x) = \sum_{k=0}^{n} k!S_2(n,k)x^k.$$

For any nonzero $\alpha \in \mathbb{Z}$, the Euler polynomials of order α are defined by

$$\left(\frac{2}{e(t)+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{t^n}{n!}, \quad (\text{see}[11]).$$

When $x = 0$, $E_n^{(\alpha)} = E_n^{(\alpha)}(0)$ are called the Euler numbers of order α.

The Genocchi polynomials of order α are defined by

$$\left(\frac{2t}{e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x)\frac{t^n}{n!}, \quad (\text{see}[10,11]).$$

When $x = 0, G_n^{(\alpha)} = G_n^{(\alpha)}(0)$ are called the Genocchi numbers of order α.

The Fubini polynomials of order α are defined by

$$\left(\frac{1}{1-y(e^t-1)}\right)^{\alpha} = \sum_{n=0}^{\infty} F_n^{(\alpha)}(y)\frac{t^n}{n!}, \quad (\text{see}[3,9]).$$

The two variable Fubini polynomials of order α are defined by

$$\left(\frac{1}{1-y(e^t-1)}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} F_n^{(\alpha)}(x;y)\frac{t^n}{n!}, \quad (\text{see}[2]).$$

In particular, when α=1, then $F_n(x;y) = F_n^{(1)}(x;y)$ are called two variable Fubini polynomials. For $x = 0, F_n^{(\alpha)}(y) = F_n^{(r)}(0;y)$ and $F_n^{(\alpha} = F_n^{(\alpha}(1) = F_n^{(\alpha}(0;1)$ are called the Fubini polynomials of order α and the Fubini numbers of order α.

The gamma function is defined by

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt, \quad (\text{see}[12]).$$

Where $x \in \mathbb{C}$, $Re(x) > 0$ Thus , we note that

$$\Gamma(x+1) = x\Gamma(x)$$

and

$$\Gamma(k+1) = k!, (k \in \mathbb{N}).$$

2. Generalized Fubini-Euler-Genocchi numbers and polynomials

For $r \in \mathbb{Z}$, we consider the generalized Fubini-Euler-Genocchi polynomials given by

$$\frac{t^r}{1-y(e^t-1)}e^{xt} = \sum_{n=0}^{\infty} C_n^{(r)}(x,y)\frac{t^n}{n!}. \tag{3}$$

When $x = 0, C_n^{(r)}(y) = C_n^{(r)}(0;y)$ are called the generalized Fubini-Euler-Genocchi numbers.

Note that

$$C_0^{(r)}(x;y) = C_1^{(r)}(x;y) = \cdots = C_{r-1}^{(r)}(x;y) = 0.$$

Remark 1. *We get the some special type of Fubini-Euler-Genocchi polynomials, which is obtained by putting particular value in (3) as follows:*
1. When $x=0$ and $r=0$, then $G_n^{(0)}(0;y)=F_n(y)$.
2. When $x=0$ and $y=-1/2$, then $r=1$, $G_n^{(1)}(0;-\frac{1}{2})=G_n(0)$.
3. When $x=0$ and $r=0, y=-1/2$, then $G_n^{(0)}(0;-\frac{1}{2})=E_n(0)$.

By (2) and (3), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} C_n^{(0)}(0;y)\frac{t^n}{n!} &= \frac{1}{1-y(e^t-1)} = \sum_{k=0}^{\infty} y^k(e^t-1)^k \\
&= \sum_{k=0}^{\infty} y^k k! \frac{1}{k!}(e^t-1)^k = \sum_{k=0}^{\infty} y^k k! \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} y^k k! S_2(n,k)\frac{t^n}{n!}.
\end{aligned} \tag{4}$$

Therefore, by comparing the coefficients on both sides of (4), we obtain the following theorem.

Theorem 1. *For $n \geq 0$, we have*

$$C_n^{(0)}(0;y) = \sum_{k=0}^{n} y^k k! S_2(n,k).$$

From (3), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{x^n t^n}{n!} &= \frac{1}{t^r}\sum_{n=0}^{\infty} C_n^{(r)}(x;y)\frac{t^n}{n!}(1-y(e^t-1)) \\
&= \sum_{l=0}^{\infty} C_l^{(r)}(x;y)\frac{t^{l-r}}{l!}(1-y)\left(\sum_{m=0}^{\infty}\frac{t^m}{m!}-1\right) \\
&= \sum_{l=0}^{\infty} C_{l+r}^{(r)}(x;y)\frac{t^l}{(l+r)!}\left(-y\sum_{m=0}^{\infty}\frac{t^m}{m!}+(1+y)\right) \\
&= -y\sum_{l=0}^{\infty} C_l^{(r)}(x;y)\frac{l!}{(l+r)!}\frac{t^l}{l!}\sum_{m=0}^{\infty}\frac{t^m}{m!} + (1+y)\sum_{n=0}^{\infty} C_{n+r}^{(r)}(x;y)\frac{n!}{(n+r)!}\frac{t^n}{n!} \\
&= -y\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n+r}{l+r}\frac{n!}{(n+r)!}C_{l+r}^{(r)}(x;y)\frac{t^n}{n!} + (1+y)\sum_{n=0}^{\infty}\frac{n!}{(n+r)!}C_{n+r}^{(r)}(x;y)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(-y\sum_{l=0}^{n}\binom{n+r}{l+r}\frac{n!}{(n+r)!}C_{l+r}^{(r)}(x;y) + (1+y)\frac{n!}{(n+r)!}C_{n+r}^{(r)}(x;y)\right)\frac{t^n}{n!}.
\end{aligned} \tag{5}$$

Therefore, by comparing the coefficients on both sides of (5), we obtain the following theorem.

Theorem 2. *For $n \geq 0$,we have*

$$x^n = -y\sum_{l=0}^{n}\binom{n+r}{l+r}\frac{n!}{(n+r)!}C_{l+r}^{(r)}(x;y) + (1+y)\frac{n!}{(n+r)!}C_{n+r}^{(r)}(x;y).$$

By (3), we obtain

$$\begin{aligned}
\sum_{n=r}^{\infty} C_n^{(r)}(x;y)\frac{t^n}{n!} &= \frac{t^r}{1-y(e^t-1)}e^{xt} \\
&= t^r \sum_{l=0}^{\infty} F_l(y)\frac{t^l}{l!}\sum_{m=0}^{\infty}\frac{(xt)^m}{m!} \\
&= \sum_{n=0}^{\infty}\sum_{l=0}^{n} x^{n-l}F_l(y)\frac{(l+n-r)!}{l!(n-r)!}\frac{t^{n+r}}{n!} \\
&= \sum_{n=r}^{\infty}\sum_{l=0}^{n-r} x^{n-r-l}\binom{l+n-r}{n-r}F_l(y)\frac{n!}{(n-r)!}\frac{t^n}{n!} \\
&= \sum_{n=r}^{\infty}\sum_{l=0}^{n-r} x^{n-r-1}(n)_r\binom{l+n-r}{n-r}F_l(y)\frac{t^n}{n!}.
\end{aligned} \tag{6}$$

Thus, by comparing the coefficients on the both sides of (6),we obtain the following theorem.

Theorem 3. *For $n \geq 0$, we have*

$$C_n^{(r)}(x;y) = \sum_{l=0}^{n-r} x^{n-r-l}(n)_r\binom{l+n-r}{n-r}F_l(y).$$

For nonzero $\alpha \in \mathbb{Z}$ and $r \in \mathbb{Z}$ with $r \geq 0$, we consider the generalized Fubini-Euler-Genocchi polynomials of order α which are given by

$$t^r\left(\frac{1}{1-y(e^t-1)}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} C_n^{(r,\alpha)}(x;y)\frac{t^n}{n!}. \tag{7}$$

when $x = 0$, $C_n^{(r,\alpha)}(y) = C_n^{(r,\alpha)}(0;y)$ are called the generalized Fubini-Euler-Genocchi numbers of order α.

Remark 2. *We get the some special type of Fubini-Euler-Genocchi polynomials, which is obtained by putting particular value in (7) as follows:*
1. When $x = 0$ and $r = 0$, then $C_n^{(r,\alpha)}(0;y) = F_n^{(\alpha)}(y)$.
2. When $r = 0$ and $y = -\frac{1}{2}$, then $C_n^{(0,\alpha)}(x;-\frac{1}{2}) = E_n^{(\alpha)}(x)$.
3. When $r = \alpha$ and $y = -\frac{1}{2}$, then $C_n^{(\alpha,\alpha)}(x;-\frac{1}{2}) = G_n^{(\alpha)}(x)$.
4. When $r = 0$ and $\alpha = 1$, then $C_n^{(0,1)}(x,y) = F_n^{(1)}(x;y)$.

By (7), we have

$$\sum_{n=r}^{\infty} C_n^{(r,\alpha)}(x;y)\frac{t^n}{n!} = t^r\left(\frac{1}{1-y(e^t-1)}\right)^{\alpha} e^{xt} \tag{8}$$
$$= t^r\left(\sum_{l_1=0}^{\infty} F_{l_1}(y)\frac{t^{l_1}}{l_1!}\right)\cdots\left(\sum_{l_\alpha=0}^{\infty} F_{l_\alpha}(y)\frac{t^{l_\alpha}}{l_\alpha!}\right)e^{xt}$$
$$= t^r\sum_{k=0}^{\infty}\sum_{l_1+\cdots+l_\alpha=k}\binom{k}{l_1,\cdots,l_\alpha}F_{l_1}(y)\cdots F_{l_\alpha}(y)\frac{t^k}{k!}\sum_{m=0}^{\infty}\frac{x^m t^m}{m!}$$
$$= t^r\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{l_1+\cdots+l_\alpha=k} x^{n-k}\binom{n}{k}\binom{k}{l_1,\cdots,l_\alpha}F_{l_1}(y)\cdots F_{l_\alpha}(y)\frac{t^n}{n!}$$
$$= \sum_{n=r}^{\infty}\sum_{k=0}^{n-r}\sum_{l_1+\cdots+l_\alpha=k} x^{n-k-r}\frac{n!}{(n-r)!}\binom{n-r}{k}\binom{k}{l_1,\cdots,l_\alpha}F_{l_1}(y)\cdots F_{l_\alpha}(y)\frac{t^n}{n!}.$$

By comparing the coefficients on both sides of (8), we have the following theorem.

Theorem 4. *for $n,r \geq 0$, with $n \geq r$, we have*

$$C_n^{(r,\alpha)}(x;y) = \sum_{l=0}^{n-r}\sum_{k_1+\cdots+k_\alpha=l} x^{n-k-r}(n)_r\binom{n-r}{k}\binom{k}{l_1,\cdots,l_\alpha}F_{l_1}(y)\cdots F_{l_\alpha}(y).$$

By (7), when $x=0$,we have

$$(\frac{1}{1-y(e^t-1)})^{\alpha-1} = \sum_{n=0}^{\infty} F_n^{(\alpha+1)}(y)\frac{t^n}{n!} \tag{9}$$
$$= \sum_{k=0}^{\infty}\binom{k+\alpha}{k}y^k(e^t-1)^k$$
$$= \sum_{k=0}^{\infty}\binom{k+\alpha}{k}y^k k!\sum_{n=k}^{\infty}S_2(n,k)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(k+\alpha)(k+\alpha-1)\cdots(\alpha+1)}{k!}k!y^kS_2(n,k)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}(k+\alpha)(k+\alpha-1)\cdots(\alpha+1)y^kS_(n,k)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(k+\alpha)(k+\alpha-1)\cdots(\alpha+1)}{\Gamma(\alpha+1)}\Gamma(\alpha+1)y^kS_2(n,k)\frac{t^n}{n!}$$
$$= \frac{1}{\Gamma(\alpha+1)}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\Gamma(\alpha+k+1)y^kS_2(n,k)\frac{t^n}{n!}.$$

By comparing the coefficients on both sides of (9),we have the following theorem.

Theorem 5. *For $n \geq 0$,we have*

$$C_n^{(0,\alpha+1)}(0,y) = \frac{1}{\Gamma(\alpha+1)}\sum_{k=0}^{n}\Gamma(\alpha+k+1)y^kS_2(n,k).$$

By (7),we get

$$\sum_{n=r}^{\infty} C_n^{(r,\alpha)}(x;y)\frac{t^n}{n!} = t^t\left(\frac{1}{1-y(e^t-1)}\right)^{\alpha} e^{xt} = t^r \sum_{n=0}^{\infty} F_n^{(\alpha)}(y)\frac{t^n}{n!} \tag{10}$$

$$= \sum_{n=r}^{\infty} F_{n-r}^{(\alpha)}(y)\frac{n!}{(n-r)!}\frac{t^n}{n!} = \sum_{n=r}^{\infty} (n)_r F_{n-r}^{(\alpha)}\frac{t^n}{n!}.$$

By comparing the coefficients on both sides of (10), we have the following theorem.

Theorem 6. *For $n, r \geq 0$ with $n \geq r$,we have*

$$C_n^{(r,\alpha)}(x;y) = (n)_r F_{n-r}^{(\alpha)}(x;y).$$

3. Conclusion

Recently, various generalized versions of special polynomials and numbers have been studied by using generating function and are obtained some properties and identities involving those polynomials and numbers. In this paper, we considered the generalized Fubini-Euler-Genocchi polynomials and Fubini-Euler-Genocchi polynomials of order α. Furthermore, we studied their some identities and properties on those numbers by using generating functions, with several special numbers and polynomials. We focused on the special cases of Fubini, Euler and Genocchi polynomials respectively.

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