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ON CERTAIN CONCEPTS OF UNIFORM LACUNARY STATISTICAL CONVERGENCE AND BOUNDEDNESS ON TIME SCALES

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Abstract. In this paper, we first define new kinds of uniform lacunary statistical convergence and uniform lacunary statistical boundedness of Δ-measurable real valued functions on time scales. We have also included a fundamental theorem in connection to uniform lacunary statistical convergence. Some theorems relating uniform lacunary statistical convergence and uniform lacunary statistical boundedness on time scales have also been included. Furthermore, we established some inclusion theorems relating statistical boundedness and uniform lacunary statistical boundedness on time scales. The findings are expected to provide some concrete connections between the studied notions and aid in better understanding their structures.

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1. Introduction

The notion of statistical convergence was first studied by Zygmund ([28]). Statistical convergence of number sequences was formally introduced by Fast ([10]) and Steinhaus ([22]) independently. Statistical convergence has been discussed in different fields of mathematics over the years. Later, it was investigated from the sequence space point of view and linked to summability theory by many researchers (see, for examples, [8, 11, 12, 16, 18, 25]).

A time scale $\mathbb{T}$ is an arbitrary non-empty closed subset of real numbers. The calculus of time scales was introduced by Stefen Hilger in his Ph.D. thesis that can unify discrete and continuous analysis (see, e.g., [14, 15]). Over the years, the study of time scale theory has received a worldwide attention in engineering, physics, economics, population dynamics and other fields. The basic concepts of time scale calculus can be obtained from the works of Agarwal and Bohner ([1]), Bohner and Peterson ([6]), Hilger ([14]), and the references given therein.

For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is given by

$$\sigma(t) = \inf \{s \in \mathbb{T} : s > t\},$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is given by

$$\rho(t) = \sup \{s \in \mathbb{T} : s < t\}.$$

The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is given by

$$\mu(t) = \sigma(t) - t.$$

Here, we put $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$, if $\mathbb{T}$ has a maximum t) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(t) = t$, if $\mathbb{T}$ has a minimum t), where $\emptyset$ is the empty set.

A closed interval, open interval and semi-closed (or semi-open) interval on a time scale $\mathbb{T}$ for $a, b \in \mathbb{T}$, are given by $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} \; : a \leq t \leq b\}$, $(a, b)_{\mathbb{T}} = \{t \in \mathbb{T} \; : a < t < b\}$ and $[a, b)_{\mathbb{T}} = \{t \in \mathbb{T} \; : a \leq t < b\}$ respectively.

Next, let S be the collection of all left closed and right open intervals of the form $[a, b)_{\mathbb{T}}$ with $a, b \in \mathbb{T}$ and $a \leq b$. Then the set function $m : S \to [0, \infty)$ defined by $m\left([a, b)_{\mathbb{T}}\right) = b - a$ is a countably additive measure. The Carathéodory extension of the set function m associated with the family S is called the Lebesgue Δ-measure on $\mathbb{T}$ and it is denoted by μ_{Δ} ([13]).

We say that a function $f : \mathbb{T} \to \mathbb{R}$ is Δ-measurable if the set $f^{-1}(A)$ is Δ-measurable for every open subset A of $\mathbb{R}$. For each $a \in \mathbb{T} - \{\max \mathbb{T}\}$, the singleton point set $\{a\}$ is Δ-measurable and its Δ-measure is given by $\mu_{\Delta}(a) = \sigma(a) - a$. If $a, b \in \mathbb{T}$ and $a \leq b$, then $\mu_{\Delta}\left([a, b)_{\mathbb{T}}\right) = b - a$ and $\mu_{\Delta}\left((a, b)_{\mathbb{T}}\right) = b - \sigma(a)$. If $a, b \in \mathbb{T} - \{\max \mathbb{T}\}$ and $a \leq b$, then $\mu_{\Delta}\left((a, b]_{\mathbb{T}}\right) = \sigma(b) - \sigma(a)$, and $\mu_{\Delta}\left([a, b]_{\mathbb{T}}\right) = \sigma(b) - a$.

The idea of statistical convergence on time scales was first introduced by Seyyidoğlu and Tan ([19]) and Turan and Duman ([23]) independently. Seyyidoğlu and Tan ([19]) put forwarded some important notions such as Δ-convergence and Δ-Cauchy using Δ-density and investigated their relations on $\mathbb{T}$. Turan and Duman([23]) introduced density and statistical convergence of Δ-measurable real-valued functions defined on $\mathbb{T}$. The notion of m-uniform statistical convergence was first introduced by Nuray ([17]). m-uniform and (λ, m)-uniform density of a set and m-uniform and (λ, m)-uniform convergence on $\mathbb{T}$ was introduced by Altin et al. ([2]). Also, Yilmaz et al. ([26]) defined λ-statistical convergence on time scale $\mathbb{T}$. The notion of lacunary sequence and lacunary statistical convergence on $\mathbb{T}$ was introduced by Turan and Duman ([24]). Yilmaz et al. ([27]) introduced uniform lacunary statistical convergence on time scale $\mathbb{T}$. The idea of lacunary statistical boundedness on time scales was recently studied by Sözbir and Altundağ ([21]).

Measure theory on time scales was constructed by Guseinov ([13]) and Lebesque Δ-integral on time scales was introduced by Cabada and Vivero ([7]). Several other works relating to statistical boundedness may be found in [3, 4, 5, 9] and so on.

Throughout the paper, we shall consider $\mathbb{T}$ as a time scale satisfying $\inf \mathbb{T} = t_0 > 0$ and $\sup \mathbb{T} = \infty$. Next, we recall some concepts existing in the literature.

Definition 1.1. ([23]) *Let $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function. Then f is said to be statistically convergent to a real number L on $\mathbb{T}$ if, for every $\varepsilon > 0$,*

$$\lim_{t\to\infty} \frac{\mu_\Delta\left(\{s \in [t_0, t]_\mathbb{T} : |f(s) - L| \geq \varepsilon\}\right)}{\mu_\Delta([t_0, t]_\mathbb{T})} = 0.$$

Definition 1.2. ([20]) *Let $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function. Then f is said to be statistically bounded on $\mathbb{T}$ if there exists a real number $L > 0$ such that*

$$\lim_{t\to\infty} \frac{\mu_\Delta\left(\{s \in [t_0, t]_\mathbb{T} : |f(s)| \geq L\}\right)}{\mu_\Delta([t_0, t]_\mathbb{T})} = 0.$$

The set of all statistically bounded functions on $\mathbb{T}$ is denoted by $S_\mathbb{T}(B)$.

Definition 1.3. ([24]) *Let $\theta = (k_r)$ be an increasing sequence of non-negative integers with $k_0 = 0$ and $\sigma(k_r) - \sigma(k_{r-1}) \to \infty$ as $r \to \infty$, where $\sigma : \mathbb{T} \to \mathbb{T}$ is the forward jump operator defined as $\sigma(s) = \inf\{t \in \mathbb{T} : t > s\}$. Then θ is called a lacunary sequence with respect to $\mathbb{T}$.*

Using the Definition 1.3, the notions of lacunary statistical convergence and lacunary statistical boundedness were defined in the following manner:

Definition 1.4. ([24]) *Let $\theta = (k_r)$ be a lacunary sequence on $\mathbb{T}$. A Δ-measurable function $f : \mathbb{T} \to \mathbb{R}$ is said to be lacunary statistical convergent to a number L if, for every $\varepsilon > 0$,*

$$\lim_{r\to\infty} \frac{\mu_\Delta\left(\{s \in (k_{r-1}, k_r]_\mathbb{T} : |f(s) - L| \geq \varepsilon\}\right)}{\mu_\Delta((k_{r-1}, k_r]_\mathbb{T})} = 0.$$

Definition 1.5. ([21]) *Let $\theta = (k_r)$ be a lacunary sequence on $\mathbb{T}$. A Δ-measurable function $f : \mathbb{T} \to \mathbb{R}$ is said to be lacunary statistically bounded if there exists a number $L > 0$ such that*

$$\lim_{r\to\infty} \frac{\mu_\Delta\left(\{s \in (k_{r-1}, k_r]_\mathbb{T} : |f(s)| \geq L\}\right)}{\mu_\Delta((k_{r-1}, k_r]_\mathbb{T})} = 0,$$

i.e., $|f(s)| \leq L$ (for almost all s with respect to θ).

In the next section, we define the notions of uniform lacunary statistical convergence, uniform lacunary statistical boundedness on time scales and study some inclusion theorems relating statistical boundedness and uniform lacunary statistical boundedness on time scales. The goal is to extend, generalise, and unify many findings in the literature, including the most recent and relevant findings of Sözbir and Altundağ ([21]).

2. (θ, m)-Uniform Lacunary Statistical Convergence and Boundedness on $\mathbb{T}$

In this section, we introduce and study the notions of (θ, m)-density, (θ, m)-uniform lacunary statistical convergence and (θ, m)-uniform lacunary statistical boundedness on time scales.

Definition 2.1. *Let Ω be Δ-measurable subset of $\mathbb{T}$ and $\theta = (k_r)$ be lacunary sequence on $\mathbb{T}$. Then we can define the set $\Omega(\theta, m)$ by*

$$\Omega(\theta, m) = \{s \in (k_{r+m-1}, k_{r+m}]_\mathbb{T} : s \in \Omega\}.$$

Then (θ, m)-density of the set Ω on $\mathbb{T}$ is defined as

$$\delta_{\mathbb{T}}^{\theta,m}(\Omega) = \lim_{r\to\infty} \frac{\mu_\Delta(\Omega(\theta,m))}{\mu_\Delta((k_{r+m-1}, k_{r+m}]_{\mathbb{T}})},$$

provided the limit exists for each $m = 0,\ 1,\ 2,\ \ldots$.

Definition 2.2. *Let $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function and $\theta = (k_r)$ be a lacunary sequence on $\mathbb{T}$. Then, f is (θ, m)-uniform lacunary statistically convergent to a real number L on $\mathbb{T}$ if, for every $\varepsilon > 0$,*

$$\lim_{r\to\infty} \frac{\mu_\Delta(\{s \in (k_{r+m-1}, k_{r+m}]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon\})}{\mu_\Delta((k_{r+m-1}, k_{r+m}]_{\mathbb{T}})} = 0,$$

for each $m = 0,\ 1,\ 2,\ \ldots$.

We denote it by $S_{\mathbb{T}}^{\theta,m} - \lim_{t\to\infty} f(t) = L$. The set of all (θ, m)-uniform lacunary statistically convergent functions on time scale $\mathbb{T}$ will be denoted by $S_{\mathbb{T}}^{\theta,m}$.

Remark 2.3. *If we take $m = 0$, then $(k_{r+m-1}, k_{r+m}]_{\mathbb{T}}$ becomes $(k_{r-1}, k_r]_{\mathbb{T}}$. This implies that (θ, m)-uniform lacunary statistically convergence reduces to lacunary statistically convergence on $\mathbb{T}$ ([24]).*

Theorem 2.4. *Let $\theta = (k_r)$ be lacunary sequence on $\mathbb{T}$ and $f, g : \mathbb{T} \to \mathbb{R}$ be two Δ-measurable functions. Then for each $m = 0,\ 1,\ 2,\ \ldots$, the following statements hold.*

(i) *If $S_{\mathbb{T}}^{\theta,m} - \lim_{t\to\infty} f(t) = L_1$ and $S_{\mathbb{T}}^{\theta,m} - \lim_{t\to\infty} f(t) = L_2$, then $L_1 = L_2$.*

(ii) *If $S_{\mathbb{T}}^{\theta,m} - \lim_{t\to\infty} f(t) = L$ and $c \in \mathbb{R}$, then $S_{\mathbb{T}}^{\theta,m} - \lim_{t\to\infty} cf(t) = cL$.*

(iii) *If $S_{\mathbb{T}}^{\theta,m} - \lim_{t\to\infty} f(t) = L_1$ and $S_{\mathbb{T}}^{\theta,m} - \lim_{t\to\infty} g(t) = L_2$, then $S_{\mathbb{T}}^{\theta,m} - \lim_{t\to\infty} [f(t) + g(t)] = L_1 + L_2$.*

Proof. (i) Let $S_{\mathbb{T}}^{\theta,m} - \lim_{t\to\infty} f(t) = L_1$ and $S_{\mathbb{T}}^{\theta,m} - \lim_{t\to\infty} f(t) = L_2$. For $\varepsilon > 0$, we can construct two sets

$$A^{\theta,m}(\varepsilon) = \{t \in (k_{r+m-1}, k_{r+m}]_{\mathbb{T}}\ :\ |f(t) - L_1| \geq \varepsilon/2\},$$

and

$$B^{\theta,m}(\varepsilon) = \{t \in (k_{r+m-1}, k_{r+m}]_{\mathbb{T}}\ :\ |f(t) - L_2| \geq \varepsilon/2\}.$$

Then, for $m = 0,\ 1,\ 2,\ \ldots,$

$$\lim_{r\to\infty} \frac{\mu_\Delta(A^{\theta,m}(\varepsilon))}{\mu_\Delta((k_{r+m-1}, k_{r+m}]_{\mathbb{T}})} = 0 = \lim_{r\to\infty} \frac{\mu_\Delta(B^{\theta,m}(\varepsilon))}{\mu_\Delta((k_{r+m-1}, k_{r+m}]_{\mathbb{T}})}.$$

Using the fact that,

$$\mu_\Delta\left(A^{\theta,m}(\varepsilon) \cup B^{\theta,m}(\varepsilon)\right) \leq \mu_\Delta\left(A^{\theta,m}(\varepsilon)\right) + \mu_\Delta\left(B^{\theta,m}(\varepsilon)\right),$$

we get

$$\lim_{r\to\infty} \frac{\mu_\Delta(A^{\theta,m}(\varepsilon) \cup B^{\theta,m}(\varepsilon))}{\mu_\Delta((k_{r+m-1}, k_{r+m}]_{\mathbb{T}})} = 0.$$

For infinitely many $t \in (k_{r+m-1}, k_{r+m}]_{\mathbb{T}} \setminus \left(A^{\theta,m}(\varepsilon) \cup B^{\theta,m}(\varepsilon)\right)$, we get

$$|L_1 - L_2| \leq |f(t) - L_1| + |f(t) - L_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This implies $L_1 = L_2$.

(ii) Let $S_{\mathbb{T}}^{\theta,m} - \lim_{t\to\infty} f(t) = L$, then

$$\lim_{r\to\infty} \frac{\mu_\Delta\left(\{\ t\ \in (k_{r+m-1}, k_{r+m}]_{\mathbb{T}} : |f(t) - L| \geq \varepsilon\}\right)}{\mu_\Delta((k_{r+m-1}, k_{r+m}]_{\mathbb{T}})} = 0.$$

Now, for $\alpha \in \mathbb{R}$,

$$\lim_{r\to\infty} \frac{\mu_\Delta\left(\{\ t\ \in (k_{r+m-1}, k_{r+m}]_{\mathbb{T}}\ :\ |\alpha f(t) - \alpha L| \geq \varepsilon\}\right)}{\mu_\Delta((k_{r+m-1}, k_{r+m}]_{\mathbb{T}})}$$
$$= \lim_{r\to\infty} \frac{\mu_\Delta\left(\{\ t\ \in (k_{r+m-1}, k_{r+m}]_{\mathbb{T}}\ :\ |f(t) - L| \geq \frac{\varepsilon}{\alpha}\}\right)}{\mu_\Delta((k_{r+m-1}, k_{r+m}]_{\mathbb{T}})}$$

Considering $\varepsilon' = \varepsilon/\alpha$, we get $S_{\mathbb{T}}^{\theta,m} - \lim_{t\to\infty} \alpha f(t) = \alpha L$.

(iii) Let $S_{\mathbb{T}}^{\theta,m} - \lim_{t\to\infty} f(t) = L_1$ and $S_{\mathbb{T}}^{\theta,m} - \lim_{t\to\infty} g(t) = L_2$. So, for each $m = 0,\ 1,\ 2,\ \ldots$, we get

$$\lim_{r\to\infty} \frac{\mu_\Delta\left(\{\ t \in (k_{r+m-1}, k_{r+m}]_{\mathbb{T}}\ :\ |f(t) - L_1| \geq \varepsilon\}\right)}{\mu_\Delta((k_{r+m-1}, k_{r+m}]_{\mathbb{T}})} = 0\ ,$$

and

$$\lim_{r\to\infty} \frac{\mu_\Delta\left(\{\ t \in (k_{r+m-1}, k_{r+m}]_{\mathbb{T}}\ : |g(t) - L_2| \geq \varepsilon\}\right)}{\mu_\Delta((k_{r+m-1}, k_{r+m}]_{\mathbb{T}})} = 0.$$

Therefore,

$$\lim_{r\to\infty} \frac{\mu_\Delta\left(\{\ t \in (k_{r+m-1}, k_{r+m}]_{\mathbb{T}} : |(f(t) + g(t)) - (L_1 + L_2)| \geq \varepsilon\}\right)}{\mu_\Delta((k_{r+m-1}, k_{r+m}]_{\mathbb{T}})}$$
$$= \lim_{r\to\infty} \frac{\mu_\Delta\left(\{\ t \in (k_{r+m-1}, k_{r+m}]_{\mathbb{T}} : |(f(t) - L_1) + (g(t) - L_2)| \geq \varepsilon\}\right)}{\mu_\Delta((k_{r+m-1}, k_{r+m}]_{\mathbb{T}})}$$
$$\leq \lim_{r\to\infty} \frac{\mu_\Delta\left(\{\ t \in (k_{r+m-1}, k_{r+m}]_{\mathbb{T}} : |(f(t) - L_1)| \geq \varepsilon\}\right)}{\mu_\Delta\left((k_{r+m-1}, k_{r+m}]_{\mathbb{T}}\right)} +$$
$$\lim_{r\to\infty} \frac{\mu_\Delta\left(\{\ t \in (k_{r+m-1}, k_{r+m}]_{\mathbb{T}} : |(g(t) - L_2)| \geq \varepsilon\}\right)}{\mu_\Delta((k_{r+m-1}, k_{r+m}]_{\mathbb{T}})}$$
$$= 0.$$

This implies, $S_{\mathbb{T}}^{\theta,m} - \lim_{t\to\infty} [f(t) + g(t)] = L_1 + L_2$.

□

Definition 2.5. *Let $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function and $\theta = (k_r)$ be a lacunary sequence on $\mathbb{T}$. Then, f is (θ, m)-uniform lacunary statistically bounded if there exists a real number $L > 0$ such that*

$$\lim_{r\to\infty} \frac{\mu_\Delta\left(\{s \in (k_{r+m-1}, k_{r+m}]_{\mathbb{T}} : |f(s)| \geq L\}\right)}{\mu_\Delta\left((k_{r+m-1}, k_{r+m}]_{\mathbb{T}}\right)} = 0,$$

for each $m = 0,\ 1,\ 2,\ \ldots.$

We denote the set of all (θ, m)-uniform lacunary statistically bounded functions on $\mathbb{T}$ by $S_{\mathbb{T}}^{\theta,m}(B)$.

Remark 2.6. *If we take $m = 0$, then $(k_{r+m-1}, k_{r+m}]_{\mathbb{T}}$ becomes $(k_{r-1}, k_r]_{\mathbb{T}}$, which is nothing but lacunary statistical boundedness on $\mathbb{T}$, introduced in* ([21]).

Theorem 2.7. *Every (θ, m)-uniform lacunary statistically convergent function on $\mathbb{T}$ is (θ, m)-uniform lacunary statistically bounded on $\mathbb{T}$. However, the converse does not need to be true.*

Proof. Let $f : \mathbb{T} \to \mathbb{R}$ be a (θ, m)-uniform lacunary statistically convergent to L. Then for each $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{\mu_\Delta\left(\{s \in (k_{r+m-1}, k_{r+m}]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon\}\right)}{\mu_\Delta((k_{r+m-1}, k_{r+m}]_{\mathbb{T}})} = 0,$$

for each $m = 0,\ 1,\ 2,\ \ldots$. Using the fact that

$$\{s \in (k_{r+m-1}, k_{r+m}]_{\mathbb{T}} : |f\,(s)| \geq \varepsilon + L\}$$
$$\subseteq \{s \in (k_{r+m-1}, k_{r+m}]_{\mathbb{T}} : |f\,(s) - L| \geq \varepsilon\},$$

and considering $\varepsilon + L = M$, a positive real number, we obtain our desired result.

For the converse part, we consider the example as follows: Let $\mathbb{T} = \mathbb{N}$, $\theta = (k_r) = (2^r)$ and let $f\ : \mathbb{T} \to \mathbb{R}$ be a function such that $f\,(s) = (-1)^s$, then f is (θ, m)-uniform lacunary statistically bounded but not (θ, m)-uniform lacunary statistically convergent. ☐

Remark 2.8. *If we take $m = 0$, then the above Theorem 2.7 becomes the Theorem 2.1 of* [21], *for an instance.*

Theorem 2.9. *Let $\theta = (k_r)$ be a lacunary sequence on $\mathbb{T}$ and $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function. Then f is (θ, m)-uniform lacunary statistically bounded if and only if there exists a bounded function $g : \mathbb{T} \to \mathbb{R}$ such that $f(s) = g(s)$ for almost all s with respect to θ.*

Proof. We assume that f is (θ, m)-uniform lacunary statistically bounded on $\mathbb{T}$. Then there exists some $L > 0$ such that for the set

$$K = \{s \in (k_{r+m-1}, k_{r+m}]_{\mathbb{T}} : |f(s)| \geq L\},$$

we have $\delta_{\mathbb{T}}^{\theta,m}(K) = 0$. Next, we consider the function $g : \mathbb{T} \to \mathbb{R}$ defined in the following manner:

$$g(s) = \begin{cases} f(s), & \text{if } s \notin K, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear from the definition of the function that g is Δ-measurable bounded function and $f(s) = g(s)$ for almost all s with respect to θ.

For the converse part, since g is bounded, there exists a real number $M > 0$, such that $|g(s)| \leq M$ for all $s \in \mathbb{T}$. Also $f(s) = g(s)$ for almost all s with respect to θ, so for the set $P = \{s \in (k_{r+m-1}, k_{r+m}]_{\mathbb{T}} : |f(s)| \geq M\}$, we have

$\delta_{\mathbb{T}}^{\theta,m}(P) = 0$. This implies that f is (θ,m)-uniform lacunary statistically bounded. □

Theorem 2.10. *Let $\theta = (k_r)$ be a lacunary sequence on $\mathbb{T}$ and $m = 0,\ 1,\ 2, \ldots$. Then*

$$S_{\mathbb{T}}(B) \subset S_{\mathbb{T}}^{\theta,m}(B) \text{ if and only if } \lim_{r\to\infty}\inf\left(\frac{\sigma(k_{r+m})}{\sigma(k_{r+m-1})}\right) > 1.$$

Proof. Sufficient part: Suppose that $\lim_{r\to\infty}\inf\left(\frac{\sigma(k_{r+m})}{\sigma(k_{r+m-1})}\right) > 1$, then for sufficiently large r, we get

$$\frac{\sigma(k_{r+m})}{\sigma(k_{r+m-1})} \geq 1+\delta,$$

for some δ. Hence,

$$\frac{\sigma(k_{r+m})-\sigma(k_{r+m-1})}{\sigma(k_{r+m})} \geq \frac{\delta}{1+\delta}.$$

Let, $f \in S_{\mathbb{T}}(B)$, then there exists a real number $L > 0$, such that

$$\begin{aligned}
&\frac{\mu_\Delta(\{s \in [t_0, k_{r+m}]_{\mathbb{T}} \ : \ |f(s)| > L\})}{\mu_\Delta([t_0, k_{r+m}]_{\mathbb{T}})} \\
&\geq \frac{\mu_\Delta(\{s \in (k_{r+m-1}, k_{r+m}]_{\mathbb{T}} \ : \ |f(s)| > L\})}{\sigma(k_{r+m}) - t_0} \\
&\geq \frac{\sigma(k_{r+m}) - \sigma(k_{r+m-1})}{\sigma(k_{r+m})} \frac{\mu_\Delta(\{s \in (k_{r+m-1}, k_{r+m}]_{\mathbb{T}} \ : \ |f(s)| > L\})}{\sigma(k_{r+m}) - \sigma(k_{r+m-1})} \\
&\geq \frac{\delta}{1+\delta} \frac{\mu_\Delta(\{s \in (k_{r+m-1}, k_{r+m}]_{\mathbb{T}} \ : \ |f(s)| > L\})}{\mu_\Delta((k_{r+m-1}, k_{r+m}]_{\mathbb{T}})}.
\end{aligned}$$

Since, $f \in S_{\mathbb{T}}(B)$, so

$$\lim_{r\to\infty} \frac{\mu_\Delta(\{s \in [t_0, k_{r+m}]_{\mathbb{T}} : |f(s)| > L\})}{\mu_\Delta([t_0, k_{r+m}]_{\mathbb{T}})} = 0.$$

This implies

$$\lim_{r\to\infty} \frac{\mu_\Delta(\{s \in (k_{r+m-1}, k_{r+m}]_{\mathbb{T}} : |f(s)| > L\})}{\mu_\Delta((k_{r+m-1}, k_{r+m}]_{\mathbb{T}})} = 0.$$

Thus, this proves the sufficiency of the theorem.

Necessary part: Suppose $S_{\mathbb{T}}(B) \subset S_{\mathbb{T}}^{\theta,m}(B)$. We assume that

$$\lim_{r\to\infty}\inf\left(\frac{\sigma(k_{r+m})}{\sigma(k_{r+m-1})}\right) = 1$$

For our convenience, we put $r+m=n$ from here on the rest of the proof, then

$$\lim_{n\to\infty}\inf\left(\frac{\sigma(k_n)}{\sigma(k_{n-1})}\right) = 1.$$

Then we can select a subsequence $\left(k_{n(j)}\right)$ of the sequence $\theta = (k_n)$ such that

$$\frac{\sigma\left(k_{n(j)}\right) - t_0}{\sigma\left(k_{n(j)-1}\right) - t_0} < 1 + \frac{1}{j}$$

and,

$$\frac{\sigma\left(k_{n(j)-1}\right)-t_0}{\sigma\left(k_{n(j-1)}\right)-t_0}>j, \quad \text{where,} \quad n\left(j\right)>n\left(j-1\right)+1.$$

Now, we define a Δ-measurable function $f:\mathbb{T}\to\mathbb{R}$ by

$$f(s)=\begin{cases} s, \text{ if } s\in(k_{n(j)-1},k_{n(j)}]_{\mathbb{T}}, & j=1,2,3,\dots \\ 0, & \text{otherwise.}\end{cases}$$

Now, for any $M>0$, there exists $j_0\in\mathbb{N}$ such that $k_{n(j_0)-1}>M$. Then we have

$$\frac{\mu_\Delta(\{s\in\left(k_{n(j_0)-1},k_{n(j_0)}\right]_{\mathbb{T}}:|f\left(s\right)|>M\})}{\mu_\Delta((k_{n(j_0)-1},k_{n(j_0)}]_{\mathbb{T}})}$$
$$\geq\frac{\mu_\Delta\left(\left\{s\in\left(k_{n(j_0)-1},k_{n(j_0)}\right]_{\mathbb{T}}:|f\left(s\right)|>k_{n(j_0)-1}\right\}\right)}{\mu_\Delta\left(\left(k_{n(j_0)-1},k_{n(j_0)}\right]_{\mathbb{T}}\right)}=1.$$

For all $j>j_0$, we get

$$\frac{\mu_\Delta\left(\left\{s\in\left(k_{n(j)-1},k_{n(j)}\right]_{\mathbb{T}}:|f(s)|>M\right\}\right)}{\mu_\Delta\left(\left(k_{n(j)-1},k_{n(j)}\right]_{\mathbb{T}}\right)}=1.$$

So, for $n\neq n(j)$, we get

$$\frac{\mu_\Delta\left(\{s\in(k_{n-1},k_n]_{\mathbb{T}}:|f\left(s\right)|>M\}\right)}{\mu_\Delta((k_{n-1},k_n]_{\mathbb{T}})}=0.$$

This implies $f\notin S_{\mathbb{T}}^{\theta,m}(B)$. Again, for sufficiently large $t\in\mathbb{T}$, we can find a unique $j\in\mathbb{N}$ for which $k_{n(j)-1}<t\leq k_{n(j+1)-1}$. Then we can write

$$\begin{aligned}
&\frac{1}{\mu_\Delta\left([t_0,t]_{\mathbb{T}}\right)}\mu_\Delta\left(\left\{\ s\in[t_0,t]_{\mathbb{T}}:|f\left(s\right)|>\frac{t_0}{2}\right\}\right)\\
&\leq\frac{1}{\mu_\Delta\left(\left[t_0,k_{n(j)-1}\right]_{\mathbb{T}}\right)}\mu_\Delta\left(\left\{\{s\in[t_0,k_{n(j-1)}]_{\mathbb{T}}:|f(s)|>\frac{t_0}{2}\right\}\right)\\
&+\frac{1}{\mu_\Delta\left(\left[t_0,k_{n(j)-1}\right]_{\mathbb{T}}\right)}\mu_\Delta\left(\left\{s\in(k_{n(j)-1},k_{n(j)}]_{\mathbb{T}}:|f(s)|>\frac{t_0}{2}\right\}\right)\\
&=\frac{\sigma\left(k_{n(j-1)}\right)-t_0}{\sigma\left(k_{n(j)-1}\right)-t_0}+\frac{\sigma\left(k_{n(j)}\right)-\sigma\left(k_{n(j)-1}\right)}{\sigma\left(k_{n(j)-1}\right)-t_0}\\
&\leq\frac{1}{j}+\frac{\sigma\left(k_{n(j)}\right)-t_0}{\sigma\left(k_{n(j)-1}\right)-t_0}-1\\
&<\frac{1}{j}+\frac{1}{j}\\
&=\frac{2}{j}
\end{aligned}$$

As $j\to\infty$, $t\to\infty$, and we get $f\in S_{\mathbb{T}}(B)$. Then we obtain $S_{\mathbb{T}}(B)\nsubseteq S_{\mathbb{T}}^{\theta,m}(B)$, a contradiction. This proves the theorem. □

Theorem 2.11. *Let $\theta = (k_r)$ be a lacunary sequence on $\mathbb{T}$ such that $\mu(t) \le Mt$ for some $M \ge 0$ and for every $t \in \mathbb{T}$. Then for each $m = 0, 1, 2, \ldots,$ we have*

$$S_{\mathbb{T}}^{\theta,m}(B) \subset S_{\mathbb{T}}(B) \text{ if and only if } \lim_{r\to\infty} \sup \left(\frac{\sigma(k_{r+m})}{\sigma(k_{r+m-1})} \right) < \infty.$$

Proof. Sufficient part: Suppose that $\lim_{r\to\infty} \sup \left(\frac{\sigma(k_{r+m})}{\sigma(k_{r+m-1})} \right) < \infty$ holds. For our convenience, we take $r + m = n$ in the later part of the proof. So, $\lim_{n\to\infty} \sup \left(\frac{\sigma(k_n)-t_0}{\sigma(k_{n-1})-t_0} \right) < \infty$. This gives, for some $K > 0$ and for all n,

$$\frac{\sigma(k_n) - t_0}{\sigma(k_{n-1}) - t_0} \le K.$$

Let, $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function belonging to $S_{\mathbb{T}}^{\theta,m}(B)$, then there exists some real number $L > 0$ such that

$$\lim_{n\to\infty} \frac{U_n}{\mu_\Delta((k_{n-1}, k_n]_{\mathbb{T}})} = 0,$$

where $U_n = \mu_\Delta(\{s \in (k_{n-1}, k_n]_{\mathbb{T}} : |f(s)| > L\})$. This implies that there exists a natural number $N = N(\varepsilon)$ such that

$$\frac{U_n}{\sigma(k_n) - \sigma(k_{n-1})} < \varepsilon, \text{ for all } n > N.$$

For any given $t \in \mathbb{T}$, we can find an interval $(k_{n-1}, k_n]_{\mathbb{T}}$ such that $t \in (k_{n-1}, k_n]_{\mathbb{T}}$. Letting $B = \max\{U_1, U_2, \ldots, U_N\}$, for sufficiently large t's, we get,

$$\begin{aligned}
&\frac{\mu_\Delta(\{s \in [t_0, t]_{\mathbb{T}} : |f(s)| > L\})}{\mu_\Delta([t_0, t]_{\mathbb{T}})} \\
&\le \frac{\mu_\Delta(\{s \in [t_0, k_n]_{\mathbb{T}} : |f(s)| > L\})}{\mu_\Delta([t_0, k_{n-1}]_{\mathbb{T}})} \\
&\le \frac{U_1 + U_2 + \cdots + U_N + U_{N+1} + \cdots + U_n}{\sigma(k_{n-1}) - t_0} \\
&\le \frac{NB}{\sigma(k_{n-1}) - t_0} + \frac{1}{\sigma(k_{n-1}) - t_0} \left\{ \frac{[\sigma(k_{N+1}) - \sigma(k_N)]\, U_{N+1}}{\sigma(k_{N+1}) - \sigma(k_N)} + \cdots + \right. \\
&\quad \left. \frac{[\sigma(k_n) - \sigma(k_{n-1})]U_n}{\sigma(k_n) - \sigma(k_{n-1})} \right\} \\
&\le \frac{NB}{\sigma(k_{n-1}) - t_0} + \varepsilon \frac{\sigma(k_n) - \sigma(k_N)}{\sigma(k_{n-1}) - t_0} \\
&\le \frac{NB}{\sigma(k_{n-1}) - t_0} + \varepsilon \frac{\sigma(k_n) - t_0}{\sigma(k_{n-1}) - t_0} \\
&\le \frac{NB}{\sigma(k_{n-1}) - t_0} + \varepsilon K.
\end{aligned}$$

Taking limit as $n \to \infty$, we get (as ε is arbitrarily small enough),

$$\lim_{t\to\infty} \frac{\mu_\Delta(\{s \in [t_0, t]_{\mathbb{T}} : |f(s)| > L\})}{\mu_\Delta([t_0, t]_{\mathbb{T}})} = 0.$$

This implies $f \in S_{\mathbb{T}}(B)$, which proves the sufficiency part.
Necessary part: This part of the theorem can be proved using a similar technique to necessary part of Theorem 2.4 of [21]. □

Theorem 2.12. *Let $\theta = (k_r)$ and $\theta' = (l_r)$ be two lacunary sequences on $\mathbb{T}$ such that $(k_{r+m-1}, k_{r+m}]_{\mathbb{T}} \subseteq (l_{r+m-1}, l_{r+m}]_{\mathbb{T}}$, for all $r \in \mathbb{N}$ and $m = 0, 1, 2, \ldots$. Then we have the following*

(i) *If* $\lim_{r\to\infty} \inf \frac{\mu_\Delta\left((k_{r+m-1},k_{r+m}]_{\mathbb{T}}\right)}{\mu_\Delta\left((l_{r+m-1},l_{r+m}]_{\mathbb{T}}\right)} > 0$, *then* $S_{\mathbb{T}}^{\theta',m}(B) \subseteq S_{\mathbb{T}}^{\theta,m}(B)$.

(ii) *If* $\lim_{r\to\infty} \frac{\mu_\Delta\left((k_{r+m-1},k_{r+m}]_{\mathbb{T}}\right)}{\mu_\Delta\left((l_{r+m-1},l_{r+m}]_{\mathbb{T}}\right)} = 1$, *then* $S_{\mathbb{T}}^{\theta,m}(B) \subseteq S_{\mathbb{T}}^{\theta',m}(B)$.

Proof. (i) Suppose that $(k_{r+m-1}, k_{r+m}]_{\mathbb{T}} \subseteq (l_{r+m-1}, l_{r+m}]_{\mathbb{T}}$, and

$$\lim_{r\to\infty} \inf \frac{\mu_\Delta((k_{r+m-1}, k_{r+m}]_{\mathbb{T}})}{\mu_\Delta((l_{r+m-1}, l_{r+m}]_{\mathbb{T}})} > 0.$$

For $M > 0$, we have,

$$\{s \in (k_{r+m-1}, k_{r+m}]_{\mathbb{T}} : |f(s)| > M\}$$
$$\subseteq \{s \in (l_{r+m-1}, l_{r+m}]_{\mathbb{T}} : |f(s)| > M\}.$$

And therefore,

$$\frac{1}{\mu_\Delta((k_{r+m-1}, k_{r+m}]_{\mathbb{T}})} \mu_\Delta(\{s \in (k_{r+m-1}, k_{r+m}]_{\mathbb{T}} : |f(s)| > M\})$$
$$\leq \frac{\mu_\Delta((l_{r+m-1}, l_{r+m}]_{\mathbb{T}})}{\mu_\Delta((k_{r+m-1}, k_{r+m}]_{\mathbb{T}})} \frac{\mu_\Delta(\{s \in (l_{r+m-1}, l_{r+m}]_{\mathbb{T}} : |f(s)| > M\})}{\mu_\Delta((l_{r+m-1}, l_{r+m}]_{\mathbb{T}})}$$

for all $r \in \mathbb{N}$, $m = 0, 1, 2, \ldots$. Taking limit as $r \to \infty$, we get $S_{\mathbb{T}}^{\theta',m}(B) \subseteq S_{\mathbb{T}}^{\theta,m}(B)$.

(ii) Suppose that $(k_{r+m-1}, k_{r+m}]_{\mathbb{T}} \subseteq (l_{r+m-1}, l_{r+m}]_{\mathbb{T}}$ for $r \in \mathbb{N}$, and

$$\lim_{r\to\infty} \frac{\mu_\Delta((k_{r+m-1}, k_{r+m}]_{\mathbb{T}})}{\mu_\Delta((l_{r+m-1}, l_{r+m}]_{\mathbb{T}})} = 1 \, .$$

For $M > 0$, we can write,

$$\frac{1}{\mu_\Delta\left((l_{r+m-1}, l_{r+m}]_{\mathbb{T}}\right)}\mu_\Delta\left(\{s \in (l_{r+m-1}, l_{r+m}]_{\mathbb{T}} : |f(s)| > M\}\right)$$

$$= \frac{1}{\mu_\Delta\left((l_{r+m-1}, l_{r+m}]_{\mathbb{T}}\right)}\mu_\Delta\left(\{s \in (l_{r+m-1}, k_{r+m-1}]_{\mathbb{T}} : |f(s)| > M\}\right)$$

$$+ \frac{1}{\mu_\Delta\left((l_{r+m-1}, l_{r+m}]_{\mathbb{T}}\right)}\mu_\Delta\left(\{s \in (k_{r+m-1}, k_{r+m}]_{\mathbb{T}} : |f(s)| > M\}\right)$$

$$+ \frac{1}{\mu_\Delta\left((l_{r+m-1}, l_{r+m}]_{\mathbb{T}}\right)}\mu_\Delta\left(\{s \in (k_{r+m}, l_{r+m}]_{\mathbb{T}} : |f(s)| > M\}\right)$$

$$\leq \frac{\mu_\Delta\left((l_{r+m-1}, k_{r+m-1}]_{\mathbb{T}}\right)}{\mu_\Delta\left((l_{r+m-1}, l_{r+m}]_{\mathbb{T}}\right)}$$

$$+ \frac{1}{\mu_\Delta\left((l_{r+m-1}, l_{r+m}]_{\mathbb{T}}\right)}\mu_\Delta\left(\{s \in (k_{r+m-1}, k_{r+m}]_{\mathbb{T}} : |f(s)| > M\}\right)$$

$$+ \frac{\mu_\Delta\left((k_{r+m}, l_{r+m}]_{\mathbb{T}}\right)}{\mu_\Delta\left((l_{r+m-1}, l_{r+m}]_{\mathbb{T}}\right)}$$

$$= \frac{\mu_\Delta\left((l_{r+m-1}, l_{r+m}]_{\mathbb{T}}\right) - \mu_\Delta\left((k_{r+m-1}, k_{r+m}]_{\mathbb{T}}\right)}{\mu_\Delta\left((l_{r+m-1}, l_{r+m}]_{\mathbb{T}}\right)}$$

$$+ \frac{1}{\mu_\Delta\left((l_{r+m-1}, l_{r+m}]_{\mathbb{T}}\right)}\mu_\Delta\left(\{s \in (k_{r+m-1}, k_{r+m}]_{\mathbb{T}} : |f(s)| > M\}\right)$$

$$= \left[1 - \frac{\mu_\Delta\left((k_{r+m-1}, k_{r+m}]_{\mathbb{T}}\right)}{\mu_\Delta\left((l_{r+m-1}, l_{r+m}]_{\mathbb{T}}\right)}\right] + \frac{\mu_\Delta\left(\{s \in (k_{r+m-1}, k_{r+m}]_{\mathbb{T}} : |f(s)| > M\}\right)}{\mu_\Delta\left((l_{r+m-1}, l_{r+m}]_{\mathbb{T}}\right)}$$

$$\leq \frac{\mu_\Delta\left(\{s \in (k_{r+m-1}, k_{r+m}]_{\mathbb{T}} : |f(s)| > M\}\right)}{\mu_\Delta\left((k_{r+m-1}, k_{r+m}]_{\mathbb{T}}\right)}$$

for all $r \in \mathbb{N}$, $m = 0, 1, 2, \ldots$. This implies, if $f \in S_{\mathbb{T}}^{\theta,m}(B)$, then $f \in S_{\mathbb{T}}^{\theta',m}(B)$. This proves our theorem.

□

Remark 2.13. *Putting* $m = 0$*, we get Theorem* 2.5 *of* [21]*, for an instance.*

3. Conclusion

In ([27]), the authors have defined and studied the (θ, m)-density and (θ, m)-uniform lacunary statistical convergence on $\mathbb{T}$ by taking $\theta = \{k_{t-t_0+1}\}$ as a lacunary sequence for $t \in \mathbb{T}$. This definition of lacunary sequence is another form of the Definition 1.3. The results presented in this paper can also be derived by considering the set $\theta = \{k_{t-t_0+1}\}$ in lacunary sense.

The findings of this paper are based on three useful notions: uniform lacunary statistical convergence, uniform lacunary statistical boundedness, and time scale, first two of which are introduced in this paper. The findings reveal some intriguing connections between the said two notions and also between statistical boundedness and uniform lacunary statistical boundedness in the time scale setting. The results are presented in full generality and are capable of drawing far more connections than those mentioned in the previous section.

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