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SOME SYMMETRIC IDENTITIES FOR SOME NEW NUMBERS AND POLYNOMIALS ARISING FROM p-ADIC FERMIONIC INTEGRAL ON $\mathbb{Z}_p$

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ABSTRACT. Recently, we introduced new numbers of polynomials which derived from the Fermionic p-adic integral on $\mathbb{Z}_p$, called the generalized Changhee numbers and polynomials of order r, and the generalized Cartalan numbers and polynomials of order r ($r \in \mathbb{N}$), respectively. In this paper, we explore several symmetric identities of some sums involving the number and polynomials dependent on $\xi \in \mathbb{N}$ for these numbers and polynomials, respectively.

1. INTRODUCTION

There are many methods and techniques for investigating and constructing generating functions for special polynomials and numbers. One of the most important techniques is the p-adic Volkenborn integral on $\mathbb{Z}_p$. In [9], Kim constructed the p-adic q-Volkenborn integration. When $q = -1$, it is called the p-adic Fermionic integral on $\mathbb{Z}_p$. In this paper, we introduce new numbers and polynomials which derived from the Fermionic p-adic integral on $\mathbb{Z}_p$. The p-adic analysis and their applications utilize p-adic distributions and p-adic measure, p-adic integrals, p-adic L-function, and other generalized functions. The p-adic integral and its applications are very important in finding solutions to special (differential) equations, real problems in both mathematics, physics, and engineering ([3-29]). Recently, in [12], we introduced new numbers and polynomials of a generalization of Changhee numbers and polynomials of order r ($r \in \mathbb{N}$), called generalized Changhee numbers and polynomials of order r, respectively. We explore some interesting identities and explicit formulas. We also introduced new numbers and polynomials, respectively, for one of generalizations of Catalan numbers and polynomials of order r ($r \in \mathbb{N}$), called generalized Catalan numbers and polynomials of order r, respectively. We also studied some combinatorial identities and explicit formulas. In this paper, we consider the number and polynomials dependent on $\xi \in \mathbb{N}$ for these numbers and polynomials, respectively. we investigate several symmetric identities of some sums involving the number and polynomials dependent on $\xi \in \mathbb{N}$ for these numbers and polynomials, respectively.

Let p be a prime number with $p \equiv 1 \pmod 2$. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let$|\cdot|_p$ be the p-adic norm with $|p|_p = \frac{1}{p}$.

The p-adic logarithm and exponential function are given by the following infinite series:

$$\log(1+t) = -\sum_{n=1}^{\infty} \frac{(-t)^n}{n}, \quad (s \in \mathbb{C}_p,\ |t|_p < 1),$$

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and

$$e^t=\sum_{n=1}^{\infty}\frac{t^n}{n!},\quad (s\in\mathbb{C}_p,\ \ |t|_p<p^{\frac{p}{p-1}})\quad \text{(see [8, 11, 12, 14])}.$$

For a $\mathbb{C}_p$-valued continuous function f on $\mathbb{Z}_p$, Kim [14, 15] introduced the p-adic fermionic integral on $\mathbb{Z}_p$ as follows:

$$\begin{aligned}I_{-1}(f)=\int_{\mathbb{Z}_p}f(x)d\mu_{-1}(x)&=\lim_{\mathbb{N}\to\infty}\sum_{x=0}^{p^{\mathbb{N}}-1}f(x)\mu_{-1}(x+p^{\mathbb{N}}\mathbb{Z}_p)\\&=\lim_{\mathbb{N}\to\infty}\sum_{x=0}^{p^{\mathbb{N}}-1}f(x)(-1)^x,\quad \text{(see [14, 15])}.\end{aligned}\tag{1}$$

Let $f_n(x)=f(x+n)$ for $n\in\mathbb{N}$. From (1), we observe that

$$I_{-1}(f_n)+(-1)^{n-1}I_{-1}(f)=2\sum_{l=0}^{n-1}(-1)^{n-1-l}f(l),\quad \text{(see [14, 15])}.\tag{2}$$

In (2), when $n=1$, we have

$$I_{-1}(f_1)+I_{-1}(f)=2f(0).\tag{3}$$

For $t\in\mathbb{C}_p$ with $|t|_p<p^{-\frac{1}{p-1}}$, from (1), for $r\in\mathbb{N}$, Kim-Kim-Seo introduced the Changhee numbers $Ch_n^{(r)}$ and polynomials $Ch_n^{(r)}(x)$ of the first kind of order r, respectively, as follows:

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r)_n d\mu_{-1}(x_1)\cdots d_{\mu-1}(x_r)=Ch_n^{(r)},\quad \text{(see [11])},\tag{4}$$

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{x_1+\cdots+x_r+x}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\&\quad=\left(\frac{2}{2+t}\right)^r(1+t)^x=\sum_{n=0}^{\infty}Ch_n^{(r)}(x)\frac{t^n}{n!},\qquad \text{(see [11])}.\end{aligned}\tag{5}$$

When $x=0$, $Ch_n^{(r)}=Ch_n(0)$, which are called the Changhee numbers of order r.

When $r=1$, $Ch_n=Ch_n^{(1)}$ and $Ch_n(x)=Ch_n^{(1)}(x)$, which are called the Changhee numbers and Changhee polynomials, respectively.

For $t\in\mathbb{C}_p$ with $|t|_p<p^{-\frac{1}{p-1}}$, from (3), the Catalan numbers C_n are given by the generating function

$$\int_{\mathbb{Z}_p}(1-4t)^{\frac{x}{2}}d\mu_{-1}(x)=\frac{2}{\sqrt{1-4t}+1}=\sum_{n=0}^{\infty}C_nt^n,\quad \text{(see [10, 19])},\tag{6}$$

and the Catalan number $C_n^{(r)}$ of order r ($r\in\mathbb{N}$) given by the generating function

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}(x_1+x_2+\cdots+x_r)}d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_r)\\&\qquad=\left(\frac{2}{\sqrt{1-4t}+1}\right)^r=\sum_{n=0}^{\infty}C_n^{(r)}t^n\quad \text{(see [10, 19])}.\end{aligned}\tag{7}$$

Throughout this paper, assume that $p\equiv 1(\mathrm{mod}\ 2)$, $t\in\mathbb{C}_p$ with $|t|_p<p^{-\frac{1}{p-1}}$, $a\in\mathbb{Q}^+$, $b\in\mathbb{Q}-\{0\}$ with $(a,p)=1=(b,p)$ and $(b,t)=1$, where (m,n) is the greatest common divisor of m and n. Let $f(x)=a+bt$.

Recently, in [12], we introduced new numbers of polynomials as one generalization of Changhee numbers and polynomials which derived from the Fermionic p-adic integral on $\mathbb{Z}_p$, called the generalized Changhee numbers and polynomials, respectively, as follows:

$$\int_{\mathbb{Z}_p}(a+bt)^x d\mu_{-1}(x)=\frac{2}{(a+1)+bt}=\sum_{n=0}^{\infty}A_n(a,b)t^n \quad (\text{see [12]}), \tag{8}$$

and

$$\int_{\mathbb{Z}_p}(a+bt)^{y+x}\, d\mu_{-1}(y)=\sum_{n=0}^{\infty}A_n(a,b|x)t^n \quad (\text{see [12]}). \tag{9}$$

When $x=0$, $A_n(a,b)=A_n(a,b|0)$. From (3), we have

$$\sum_{n=0}^{\infty}A_n(a,b|x)t^n=\frac{2}{(a+1)+bt}(a+bt)^x.$$

In particular, when $a=1,\ b=1$, the generating function of Changhee numbers of the first kind are given by

$$\int_{\mathbb{Z}_p}(1+t)^x d\mu_{-1}(x)=\frac{2}{2+t} \text{ and } n!A_n(1,1)=Ch_n.$$

We note that $n!A_n(1,1|x)=Ch_n(x)$.

For $r\in\mathbb{N}$, in [12], we introduced the generating functions of $A_n(a,b)$ and $A_n(a,b|x)$ of order r, called generalized Changhee numbers and polynomials of order r, which are derived from the multivariate Fermionic p-adic integral on $\mathbb{Z}_p$, respectively, as follows:

$$\begin{aligned}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}&(a+bt)^{x_1+x_2+\cdots+x_r}\, d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_r)\\ &=\left(\frac{2}{(a+1)+bt}\right)^r=\sum_{n=0}^{\infty}A_n^{(r)}(a,b)t^n \quad (\text{see [12]}),\end{aligned} \tag{10}$$

and

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(a+bt)^{x_1+x_2+\cdots+x_r+x}\, d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_r)\\ &\qquad=\left(\frac{2}{(a+1)+bt}\right)^r(a+bt)^x=\sum_{n=0}^{\infty}A_n^{(r)}(a,b|x)t^n \quad (\text{see [12]}).\end{aligned} \tag{11}$$

For $r\in\mathbb{N}$, we also introduced the generating functions of $W_n(a,b)$ and $W_n(a,b|x)$ of order r, called generalized Catalan numbers and polynomials of order r, which are derived from the multivariate Fermionic p-adic integral on $\mathbb{Z}_p$, respectively, given by

$$\int_{\mathbb{Z}_p}(a+bt)^{\frac{x}{2}}d\mu_{-1}(x)=\frac{2}{\sqrt{a+bt}+1}=\sum_{n=0}^{\infty}W_n(a,b)t^n \quad (\text{see [12]}), \tag{12}$$

and

$$\int_{\mathbb{Z}_p}(a+bt)^{\frac{y+x}{2}}\, d\mu_{-1}(y)=\sum_{n=0}^{\infty}W_n(a,b|x)t^n \quad (\text{see [12]}). \tag{13}$$

When $x=0$, $W_n(a,b)=W_n(a,b|0)$.

In particular, when $a=1,\ b=-4$, we get the generating function of Catalan numbers as follows:

$$\int_{\mathbb{Z}_p}(1-4t)^{\frac{x}{2}}d\mu_{-1}(x)=\frac{2}{\sqrt{1-4t}+1}\ \text{ and }\ W_n(1,-4)=C_n.$$

From (3), we have

$$\sum_{n=0}^{\infty}W_n(a,b|x)t^n=\frac{2}{\sqrt{a+bt}+1}(a+bt)^{\frac{x}{2}}.$$

For $r\in\mathbb{N}$, we introduced the generating functions of $W(a,b)$ and $W(a,b|x)$ of order r, which are derived from the multivariate Fermionic p-adic integral on $\mathbb{Z}_p$, respectively as follows:

$$\begin{aligned}(14)\qquad &\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(a+bt)^{\frac{x_1+x_2+\cdots+x_r}{2}}\,d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_r)\\ &\qquad=\left(\frac{2}{\sqrt{a+bt}+1}\right)^r=\sum_{n=0}^{\infty}W_n^{(r)}(a,b)t^n\quad\text{(see [12])},\end{aligned}$$

and

$$\begin{aligned}(15)\qquad &\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(a+bt)^{\frac{x_1+x_2+\cdots+x_r+x}{2}}\,d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_r)\\ &\qquad=\left(\frac{2}{\sqrt{a+bt}+1}\right)^r(a+bt)^{\frac{x}{2}}=\sum_{n=0}^{\infty}W_n^{(r)}(a,b|x)t^n\quad\text{(see [12])}.\end{aligned}$$

2. Symmetric identities of ξ-new numbers and polynomials arising rom the Fermionic p-adic integral on $\mathbb{Z}_p$

In this section, we consider ξ-generalized Changhee numbers and polynomials, and ξ-generalized Catalan numbers and polynomials, respectively. We study several symmetric identities of some sums involving the number and polynomials dependent on $\xi\in\mathbb{N}$ for these numbers and polynomials, respectively.

First, from (8) and (9), we consider ξ-generalized Changhee numbers and polynomials, respectively, given by

$$(16)\qquad \int_{\mathbb{Z}_p}(a+bt)^{\xi x}\,d\mu_{-1}(x)=\frac{2}{1+(a+bt)^{\xi}}=\sum_{n=0}^{\infty}A_{n,\xi}(a,b)t^n,$$

and

$$(17)\qquad \int_{\mathbb{Z}_p}(a+bt)^{\xi(x+y)}\,d\mu_{-1}(y)=\frac{2}{1+(a+bt)^{\xi}}(a+bt)^{\xi x}=\sum_{n=0}^{\infty}A_{n,\xi}(a,b|x)t^n.$$

For positive integer ξ, l with $\xi \equiv 1$ (mod 2), by (16), we get

(18)
$$\begin{aligned}\frac{\int_{\mathbb{Z}_p}(a+bt)^{lx}\,d\mu_{-1}(x)}{\int_{\mathbb{Z}_p}(a+bt)^{l\xi x}\,d\mu_{-1}(x)} &= \frac{1+(a+bt)^{l\xi}}{1+(a+bt)^{l}} = \sum_{j=0}^{\xi-1}(a+bt)^{lj}(-1)^j\\ &= \sum_{j=0}^{\xi-1}\sum_{k=0}^{\infty}\binom{lj}{k}(-1)^j a^{lj-k}b^k t^k\\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{\xi-1}\binom{lj}{k}(-1)^j a^{lj-k}b^k t^k.\end{aligned}$$

For the convenience of calculation in this paper, put

(19)
$$H_{k,l}(\xi,a) = \sum_{j=0}^{\xi-1}\binom{lj}{k}(-1)^j a^{lj-k}.$$

From (18) and (19), we have

(20)
$$\frac{\int_{\mathbb{Z}_p}(a+bt)^{lx}\,d\mu_{-1}(x)}{\int_{\mathbb{Z}_p}(a+bt)^{l\xi x}\,d\mu_{-1}(x)} = \sum_{k=0}^{\infty}H_{k,l}(\xi,a)b^k t^k.$$

For $\xi_1,\xi_2 \in \mathbb{N}$ with $\xi_1 \equiv 1$ (mod 2) and $\xi_2 \equiv 1$ (mod 2), by (16), we observe that

(21)
$$\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}(a+bt)^{\xi_1 x+\xi_2 y}\,d\mu_{-1}(x)\,d\mu_{-1}(y)}{\int_{\mathbb{Z}_p}(a+bt)^{\xi_1\xi_2 x}\,d\mu_{-1}(x)} = \frac{2((a+bt)^{\xi_1\xi_2}+1)}{((a+bt)^{\xi_1}+1)((a+bt)^{\xi_2}+1)}.$$

For $\xi_1,\xi_2 \in \mathbb{N}$ with $\xi_1 \equiv 1$ (mod 2) and $\xi_2 \equiv 1$ (mod 2), we consider

(22)
$$\begin{aligned}I(\xi_1,\xi_2) &= \frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}(a+bt)^{\xi_1 x_1+\xi_2 x_2+\xi_1\xi_2 x}\,d\mu_{-1}(x_1)\,d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}(a+bt)^{\xi_1\xi_2 x}\,d\mu_{-1}(x)}\\ &= \frac{2(a+bt)^{\xi_1\xi_2 x}((a+bt)^{\xi_1\xi_2}+1)}{((a+bt)^{\xi_1}+1)((a+bt)^{\xi_2}+1)}.\end{aligned}$$

It is easy to see that $I(\xi_1,\xi_2)$ is symmetric in ξ_1 and ξ_2.

Theorem 1. *For $n \geq 0$, $\xi_1,\xi_2 \in \mathbb{N}$ with $\xi_1 \equiv 1$ (mod 2) and $\xi_2 \equiv 1$ (mod 2), we have*

$$\sum_{j=0}^{n} b^{n-j}H_{n-j,\xi_2}(\xi_1,a)A_{j,\xi_1}(a,b|\xi_2 x) = \sum_{j=0}^{n} b^{n-j}H_{n-j,\xi_1}(\xi_2,a)A_{j,\xi_2}(a,b|\xi_1 x).$$

In addition, when $\xi_2 = 1$, we have

$$A_n(\xi,x) = \sum_{j=0}^{n} b^{n-j}A_{j,\xi_1}(a,b|x)H_{n-j}(\xi_1,a).$$

Proof. From (17), (19) and (22), we have

(23)
$$\begin{aligned}I(\xi_1,\xi_2) &= \frac{\int_{\mathbb{Z}_p}(a+bt)^{\xi_1(x_1+\xi_2 x)}\,d\mu_{-1}(x_1)\int_{\mathbb{Z}_p}(a+bt)^{\xi_2 x_2}\,d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}(a+bt)^{\xi_1\xi_2 x}\,d\mu_{-1}(x)}\\ &= \left(\sum_{j=0}^{\infty}A_{j,\xi_1}(a,b|\xi_2 x)t^j\right)\left(\sum_{k=0}^{\infty}H_{k,\xi_2}(\xi_1,a)b^k t^k\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}b^{n-j}A_{j,\xi_1}(a,b|\xi_2 x)H_{n-j,\xi_2}(\xi_1,a)\right)t^n.\end{aligned}$$

On the other hand, from (17), (19) and (22), we get

$$
\begin{aligned}
I(\xi_1,\xi_2) &= \frac{\int_{\mathbb{Z}_p}(a+bt)^{\xi_2(x_2+\xi_1 x)}\,d\mu_{-1}(x_2)\int_{\mathbb{Z}_p}(a+bt)^{\xi_1 x}\,d\mu_{-1}(x_1)}{\int_{\mathbb{Z}_p}(a+bt)^{\xi_1\xi_2 x}\,d\mu_{-1}(x)} \\
&= \left(\sum_{j=0}^{\infty}A_{j,\xi_2}(a,b|\xi_1 x)t^j\right)\left(\sum_{k=0}^{\infty}H_{k,\xi_1}(\xi_2,a)b^k t^k\right) \qquad (24)\\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}b^{n-j}A_{j,\xi_2}(a,b|\xi_1 x)H_{n-j,\xi_1}(\xi_2,a)\right)t^n.
\end{aligned}
$$

By comparing the coefficients of (23) and (24), we get the desired symmetric identity. □

Corollary 2. *For $\xi_1 \in \mathbb{N}$ with $\xi_1 \equiv 1$ (mod 2), we have*

$$\sum_{j=0}^{n}b^{n-j}A_{j,\xi_1}H_{n-j,\xi_2}(\xi_1,a) = \sum_{j=0}^{n}b^{n-j}A_{j,\xi_2}H_{n-j,\xi_1}(\xi_2,a).$$

Theorem 3. *For $n \geq 0$, we have*

$$\sum_{j=0}^{\xi_1-1}(-1)^j A_{n,\xi_1}(a,b|\xi_2 x+\frac{\xi_2}{\xi_1}j) = \sum_{j=0}^{\xi_2-1}(-1)^j A_{n,\xi_2}\left(a,b|\xi_1 x+\frac{\xi_1}{w_2}j\right).$$

In addition when $\xi_2 = 1$, we have

$$A_n(\xi_1 x) = \sum_{j=0}^{\xi_1-1}(-1)^j A_{n,\xi_1}\left(x+\frac{1}{\xi_1}j\right).$$

Proof. From (17) and (22), we have

$$
\begin{aligned}
J(\xi_1,\xi_2) =&(a+bt)^{\xi_1\xi_2 x}\int_{\mathbb{Z}_p}(a+bt)^{\xi_1 x_1}\,d\mu_{-1}(x_1)\times\frac{\int_{\mathbb{Z}_p}(a+bt)^{\xi_2 x_2}\,d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}(a+bt)^{\xi_1\omega_2 x}\,d\mu_{-1}(x)} \\
&= \left((a+bt)^{\xi_1\xi_2 x}\int_{\mathbb{Z}_p}(a+bt)^{\xi_1 x_1}\,d\mu_{-1}(x_1)\right)\left(\sum_{j=0}^{\xi_1-1}(-1)^j(a+bt)^{\xi_2 j}\right) \qquad (25)\\
&= \sum_{j=0}^{\xi_1-1}(-1)^j\int_{\mathbb{Z}_p}(a+bt)^{\xi_1(x_1+\xi_2 x+\frac{\xi_2}{\xi_1}j)}\,d\mu_{-1}(x_1) \\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{\xi_1-1}(-1)^j A_{n,\xi_1}(a,b|\xi_2 x+\frac{\xi_2}{\xi_1}j)\right)t^n.
\end{aligned}
$$

On the other hand, from (17) and (22), we get

(26)
$$\begin{aligned}
J(\xi_1,\xi_2) &= (a+bt)^{\xi_1\xi_2 x}\int_{\mathbb{Z}_p}(a+bt)^{\xi_2 x_2}\, d\mu_{-1}(x_2)\left(\frac{\int_{\mathbb{Z}_p}(a+bt)^{\xi_1 x_1}\, d\mu_{-1}(x_1)}{\int_{\mathbb{Z}_p}(a+bt)^{\xi_1\xi_2 x}\, d\mu_{-1}(x)}\right)\\
&= (a+bt)^{\xi_1\xi_2 x}\int_{\mathbb{Z}_p}(a+bt)^{\xi_2 x_2}\, d\mu_{-1}(x_2)\left(\sum_{j=0}^{\xi_2-1}(-1)^j(a+bt)^{\xi_1 j}\right)\\
&= \sum_{j=0}^{\xi_2-1}(-1)^j\int_{\mathbb{Z}_p}(a+bt)^{\xi_2(x_2+\xi_1 x+\frac{\xi_1}{\xi_2}j)}\, d\mu_{-1}(x_2)\\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{\xi_2-1}(-1)^j A_{n,\xi_2}(a,b|\xi_1 x+\frac{\xi_1}{\xi_2}j)\right)t^n.
\end{aligned}$$

By (25) and (26), we get the desired result.

□

Now, from (12) and (13), we consider ξ-generalized Catalan numbers and polynomials, respectively, as follows:

(27)
$$\int_{\mathbb{Z}_p}(a+bt)^{\frac{\xi x}{2}} d\mu_{-1}(x) = \frac{2}{(a+bt)^{\frac{\xi}{2}}+1} = \sum_{n=0}^{\infty} W_{n,\xi}(a,b)t^n,$$

and

(28)
$$\int_{\mathbb{Z}_p}(a+bt)^{\frac{\xi(x+y)}{2}} d\mu_{-1}(y) = \frac{2}{(a+bt)^{\frac{\xi}{2}}+1}(a+bt)^{\frac{\xi x}{2}} = \sum_{n=0}^{\infty} W_{n,\xi}(a,b|x)t^n,$$

For positive integer l,ξ with $\xi \equiv 1 \pmod 2$, by (19) and (27), we observe that

(29)
$$\begin{aligned}
\frac{\int_{\mathbb{Z}_p}(a+bt)^{\frac{lx}{2}}\, d\mu_{-1}(x)}{\int_{\mathbb{Z}_p}(a+bt)^{\frac{l\xi x}{2}}\, d\mu_{-1}(x)} &= \frac{1+(a+bt)^{\frac{l\xi}{2}}}{1+(a+bt)^{\frac{l}{2}}} = \sum_{j=0}^{\xi-1}(a+bt)^{\frac{lj}{2}}(-1)^j\\
&= \sum_{j=0}^{\xi-1}\sum_{k=0}^{\infty}\binom{\frac{lj}{2}}{k}(-1)^j a^{\frac{lj}{2}-k}b^k t^k\\
&= \sum_{k=0}^{\infty}\sum_{j=0}^{\xi-1}\binom{\frac{lj}{2}}{k}(-1)^j a^{\frac{lj}{2}-k}b^k t^k = \sum_{k=0}^{\infty} H_{k,\frac{l}{2}}(\xi,a)b^k t^k.
\end{aligned}$$

For $\xi_1,\xi_2 \in \mathbb{N}$ with $\xi_1 \equiv 1 \pmod 2$ and $\xi_2 \equiv 1 \pmod 2$, by (27), we get

(30)
$$\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}(a+bt)^{\frac{\xi_1 x+\xi_2 y}{2}}\, d\mu_{-1}(x)\, d\mu_{-1}(y)}{\int_{\mathbb{Z}_p}(a+bt)^{\frac{\xi_1\xi_2}{2}x}\, d\mu_{-1}(x)} = \frac{2((a+bt)^{\frac{\xi_1\xi_2}{2}}+1)}{((a+bt)^{\frac{\xi_1}{2}}+1)((a+bt)^{\frac{\xi_2}{2}}+1)}.$$

For $\xi_1,\xi_2 \in \mathbb{N}$ with $\xi_1 \equiv 1 \pmod 2$ and $\xi_2 \equiv 1 \pmod 2$, we consider the following fermionic p-adic invariant integral on $\mathbb{Z}_p$ associated with ξ-generalized Catalan polynomials

(31)
$$\begin{aligned}
J(\xi_1,\xi_2) &= \frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}(a+bt)^{\frac{\xi_1 x_1+\xi_2 x_2+\xi_1\xi_2 x}{2}}\, d\mu_{-1}(x_1)\, d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}(a+bt)^{\frac{\xi_1\xi_2 x}{2}}\, d\mu_{-1}(x)}\\
&= \frac{2(a+bt)^{\frac{\xi_1\xi_2 x}{2}}((a+bt)^{\frac{\xi_1\xi_2}{2}}+1)}{((a+bt)^{\frac{\xi_1}{2}}+1)((a+bt)^{\frac{\xi_2}{2}}+1)}.
\end{aligned}$$

Then we note that $J(\xi_1, \xi_2)$ is symmetric in ξ_1 and ξ_2.

Theorem 4. *For $n \geq 0$, $\xi_1, \xi_2 \in \mathbb{N}$ with $\xi_1 \equiv 1$ (mod 2) and $\xi_2 \equiv 1$ (mod 2), we have*

$$\sum_{j=0}^{n} b^{n-j} H_{n-j,\frac{\xi_2}{2}}(\xi_1, a) W_{j,\xi_1}(a,b|\xi_2 x) = \sum_{j=0}^{n} b^{n-j} H_{n-j,\frac{\xi_1}{2}}(\xi_2, a) W_{j,\xi_2}(a,b|\xi_1 x).$$

In particular, when $\xi_2 = 1$, we have

$$W_n(a,b|\xi_1 x) = \sum_{j=0}^{n} b^{n-j} H_{n-j,\frac{1}{2}}(\xi_1, a) W_{j,\xi_1}(a,b|x).$$

Proof. From (28),(29) and (31), we observe

$$\begin{aligned} J(\xi_1,\xi_2) &= \frac{\int_{\mathbb{Z}_p} (a+bt)^{\frac{\xi_1(x_1+\xi_2 x)}{2}} \, d\mu_{-1}(x_1) \int_{\mathbb{Z}_p} (a+bt)^{\frac{\xi_2 x_2}{2}} \, d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p} (a+bt)^{\frac{\xi_1 \xi_2}{2} x} \, d\mu_{-1}(x)} \\ &= \left(\sum_{j=0}^{\infty} W_{j,\xi_1}(a,b|\xi_2 x) t^j \right) \left(\sum_{k=0}^{\infty} H_{k,\frac{\xi_2}{2}}(\xi_1, a) b^k t^k \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} b^{n-j} W_{j,\xi_1}(a,b|\xi_2 x) H_{n-j,\frac{\xi_2}{2}}(\xi_1, a) \right) t^n. \end{aligned} \tag{32}$$

On the other hand, from (28),(29) and (31), we get

$$\begin{aligned} J(\xi_1,\xi_2) &= \frac{\int_{\mathbb{Z}_p} (a+bt)^{\frac{\xi_2(x_2+\xi_1 x)}{2}} \, d\mu_{-1}(x_2) \int_{\mathbb{Z}_p} (a+bt)^{\frac{\xi_1 x_1}{2}} \, d\mu_{-1}(x_1)}{\int_{\mathbb{Z}_p} (a+bt)^{\frac{\xi_1 \xi_2}{2} x} \, d\mu_{-1}(x)} \\ &= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} b^{n-j} W_{j,\xi_2}(a,b|\xi_1 x) H_{n-j,\frac{\xi_1}{2}}(\xi_2, a) \right) t^n. \end{aligned} \tag{33}$$

By (32) and (33), we obtain the symmetric identity. □

Corollary 5. *For $\xi_1, \xi_2 \in \mathbb{N}$ with $\xi_1 \equiv 1$ (mod 2) and $\xi_2 \equiv 1$ (mod 2), we have*

$$\sum_{j=0}^{n} b^{n-j} H_{n-j,\frac{\xi_2}{2}}(\xi_1, a) W_{j,\xi_1}(a,b) = \sum_{j=0}^{n} b^{n-j} H_{n-j,\frac{\xi_1}{2}}(\xi_2, a) W_{j,\xi_2}(a,b).$$

Theorem 6. *For $n \geq 0$, $\xi_1, \xi_2 \in \mathbb{N}$ with $\xi_1 \equiv 1$ (mod 2) and $\xi_2 \equiv 1$ (mod 2), we have*

$$\sum_{j=0}^{\xi_1 - 1} (-1)^j W_{n,\xi_1}(a,b|\xi_2 x + \frac{\xi_2}{\xi_1} j) = \sum_{j=0}^{\xi_2 - 1} (-1)^j W_{n,\xi_2}(a,b|\xi_1 x + \frac{\xi_1}{\xi_2} j).$$

In addition, when $\xi_2 = 1$, we have

$$W_n(\xi_1 x) = \sum_{j=0}^{\xi_1 - 1} (-1)^j W_{n,\xi_1}(x + \frac{1}{\xi_1} j).$$

Proof. From (28) and (31), we observe

$$\begin{aligned}
J(\xi_1,\xi_2) &= \left((a+bt)^{\frac{\xi_1\xi_2}{2}x}\int_{\mathbb{Z}_p}(a+bt)^{\frac{\xi_1 x_1}{2}}\,d\mu_{-1}(x_1)\right)\left(\frac{\int_{\mathbb{Z}_p}(a+bt)^{\frac{\xi_2 x_2}{2}}\,d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}(a+bt)^{\frac{\xi_1\xi_2}{2}x}\,d\mu_{-1}(x)}\right)\\
&= \left((a+bt)^{\frac{\xi_1\xi_2}{2}x}\int_{\mathbb{Z}_p}(a+bt)^{\frac{\xi_1 x_1}{2}}\,d\mu_{-1}(x_1)\right)\left(\sum_{j=0}^{\xi_1-1}(-1)^j(a+bt)^{\frac{\xi_2}{2}j}\right)\\
&= \sum_{j=0}^{\xi_1-1}(-1)^j\int_{\mathbb{Z}_p}(a+bt)^{\frac{\xi_1}{2}(x_1+\xi_2 x+\frac{\xi_2}{\xi_1}j)}\,d\mu_{-1}(x_1)\\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{\xi_1-1}(-1)^j W_{n,\xi_1}(a,b|\xi_2 x+\frac{\xi_2}{\xi_1}j)\right)t^n.
\end{aligned} \tag{34}$$

On the other hand, by (28) and (31), we observe that

$$\begin{aligned}
J(\xi_1,\xi_2) &= \left((a+bt)^{\frac{\xi_1\xi_2}{2}x}\int_{\mathbb{Z}_p}(a+bt)^{\frac{\xi_2 x_2}{2}}\,d\mu_{-1}(x_2)\right)\left(\frac{\int_{\mathbb{Z}_p}(a+bt)^{\frac{\xi_1 x_1}{2}}\,d\mu_{-1}(x_1)}{\int_{\mathbb{Z}_p}(a+bt)^{\frac{\xi_1\xi_2}{2}x}\,d\mu_{-1}(x)}\right)\\
&= \left((a+bt)^{\frac{\xi_1\xi_2}{2}x}\int_{\mathbb{Z}_p}(a+bt)^{\frac{\xi_2 x_2}{2}}\,d\mu_{-1}(x_2)\right)\left(\sum_{j=0}^{\xi_2-1}(-1)^j(a+bt)^{\frac{\xi_1}{2}j}\right)\\
&= \sum_{j=0}^{\xi_2-1}(-1)^j\int_{\mathbb{Z}_p}(a+bt)^{\frac{\xi_2}{2}(x_2+\xi_1 x+\frac{\xi_1}{\xi_2}j)}\,d\mu_{-1}(x_2)\\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{\xi_2-1}(-1)^j W_{n,\xi_2}(a,b|\xi_1 x+\frac{\xi_1}{\xi_2}j)\right)t^n.
\end{aligned} \tag{35}$$

By (34) and (35), we get the desired result.

□

3. Symmetric identities of ξ-new numbers and polynomials of order r arising from the Fermionic p-adic integral on $\mathbb{Z}_p$

In this section, we consider ξ-new numbers and polynomials of order r, which derived from p-adic Fermionic integral on $\mathbb{Z}_p$, respectively. We also investigate several symmetric identities of some sums involving the number and polynomials dependent on $\xi \in \mathbb{N}$ for these numbers and polynomials, respectively.

For $r \in \mathbb{N}$, $\xi_1,\xi_2 \in \mathbb{N}$ with $\xi_1 \equiv 1$ (mod2) and $\xi_2 \equiv 1$ (mod2), from (10) and (11), we introduce ξ-generalized Changhee numbers and polynomials of order r, respectively, given by

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(a+bt)^{\xi(x_1+x_2+\cdots+x_r)}\,d\mu_{-1}(x_1)\cdots\,d\mu_{-1}(x_r)\\
&\qquad=\left(\frac{2}{1+(a+bt)^{\xi}}\right)^r=\sum_{n=0}^{\infty}A_{n,\xi}^{(r)}(a,b)t^n.
\end{aligned} \tag{36}$$

and

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(a+bt)^{\xi(x_1+x_2+\cdots+x_r+x)}\,d\mu_{-1}(x_1)\,d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_r) = \left(\frac{2}{1+(a+bt)^{\xi}}\right)^r(a+bt)^{\xi x}=\sum_{n=0}^{\infty}A_{n,\xi}^{(r)}(a,b|x)t^n. \tag{37}$$

For $\xi_1,\xi_2\in\mathbb{N}$, we consider

$$\begin{aligned} I^{(r)}(\xi_1,\xi_2) =& \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r}(a+bt)^{\xi_1(x_1+\cdots+x_r+\xi_2 x)}\,d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\ &\times\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r}(a+bt)^{\xi_2(x_1+\cdots+x_r+\xi_1 x)}\,d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\ &\times\frac{1}{\int_{\mathbb{Z}_p}(a+bt)^{\xi_1\xi_2 x}\,d\mu_{-1}(x)}. \end{aligned}\tag{38}$$

Then we note that $I^{(r)}(\xi_1,\xi_2)$ is symmetric in ξ_1 and ξ_2.

By (37), it is easy to see that

$$\begin{aligned} &(a+bt)^{\xi_1\xi_2 x}\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r}(a+bt)^{\xi_1(x_1+\cdots+x_r)}\,d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\ &=\left(\frac{2}{1+(a+bt)^{\xi_1}}\right)^r(a+bt)^{\xi_1\xi_2 x}=\sum_{n=0}^{\infty}A_{n,\xi_1}^{(r)}(a,b|\xi_2 x)t^n. \end{aligned}\tag{39}$$

Theorem 7. *For $r,\xi_1,\xi_2\in\mathbb{N}$ with $\xi_1\equiv 1\ (\mathrm{mod}2)$ and $\xi_2\equiv 1\ (\mathrm{mod}2)$, $n\geq 0$, we have*

$$\begin{aligned} &\sum_{i=0}^{n}\sum_{k=0}^{i}b^kH_{k,\xi_1}(\xi_2,a)A_{n-i,\xi_1}^{(r)}(a,b|\xi_2 x)A_{i-k,\xi_2}^{(r-1)}(a,b|\xi_1 x)\\ &=\sum_{i=0}^{n}\sum_{k=0}^{i}b^kH_{k,\xi_2}(\xi_1,a)A_{n-j,\xi_2}^{(r)}(a,b|\xi_1 x)A_{i-k,\xi_1}^{(r-1)}(a,b|\xi_2 x). \end{aligned}$$

Proof. From (38) and (39), we have

$$\begin{aligned} I^{(r)}(\xi_1,\xi_2) =&(a+bt)^{\xi_1\xi_2 x}\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r}(a+bt)^{\xi_1(x_1+\cdots+x_r)}\,d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\ &\times\left(\frac{\int_{\mathbb{Z}_p}(a+bt)^{\xi_2 x_r}\,d\mu_{-1}(x_r)}{\int_{\mathbb{Z}_p}(a+bt)^{\xi_1\xi_2 x}\,d\mu_{-1}(x)}\right)\\ &\times(a+bt)^{\xi_1\xi_2 x}\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r-1}(a+bt)^{\xi_2(x_1+\cdots+x_{r-1})}\,d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_{r-1}) \end{aligned}\tag{40}$$

$$
\begin{aligned}
&= \left(\sum_{j=0}^{\infty} A_{j,\xi_1}^{(r)}(a,b|\xi_2 x)t^j\right)\left(\sum_{k=0}^{\infty} H_{k,\xi_1}(\xi_2,a)b^k t^k\right)\left(\sum_{l=0}^{\infty} A_{l,\xi_2}^{(r-1)}(a,b|\xi_1 x)t^l\right)\\
&= \left(\sum_{j=0}^{\infty} A_{j,\xi_1}^{(r)}(a,b|\xi_2 x)t^j\right)\left(\sum_{i=0}^{\infty}\left(\sum_{k=0}^{i} H_{k,\xi_1}(\xi_2,a)b^k A_{i-k,\xi_2}^{(r-1)}(a,b|\xi_1 x)t^i\right)\right)\\
&= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\sum_{k=0}^{i} b^k H_{k,\xi_1}(\xi_2,a)A_{n-i,\xi_1}^{(r)}(a,b|\xi_2 x)A_{i-k,\xi_2}^{(r-1)}(a,b|\xi_1 x)\right)t^n.
\end{aligned}
$$

On the other hand, from (38) and (39), we have

$$
\begin{aligned}
I^{(r)}(\xi_1,\xi_2) =&(a+bt)^{\xi_1\xi_2 x}\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r}(a+bt)^{\xi_2(x_1+\cdots+x_r)}\,d\mu_{-1}(x_1)\cdots\,d\mu_{-1}(x_r)\\
&\times\left(\frac{\int_{\mathbb{Z}_p}(a+bt)^{\xi_1 x_r}\,d\mu_{-1}(x_r)}{\int_{\mathbb{Z}_p}(a+bt)^{\xi_1\xi_2 x}\,d\mu_{-1}(x)}\right)\\
&\times(a+bt)^{\xi_1\xi_2 x}\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r-1}(a+bt)^{\xi_1(x_1+\cdots+x_{r-1})}\,d\mu_{-1}(x_1)\cdots\,d\mu_{-1}(x_{r-1})\\
&= \left(\sum_{j=0}^{\infty} A_{j,\xi_2}^{(r)}(a,b|\xi_1 x)t^j\right)\left(\sum_{k=0}^{\infty} b^k H_{k,\xi_2}(\xi_1,a)t^k\right)\left(\sum_{l=0}^{\infty} A_{l,\xi_1}^{(r-1)}(a,b|\xi_2 x)t^l\right)\\
&= \left(\sum_{j=0}^{\infty} A_{j,\xi_2}^{(r)}(a,b|\xi_1 z_r)\frac{t^j}{j!}\right)\left(\sum_{i=0}^{\infty}\left(\sum_{k=0}^{i} b^k H_{k,\xi_2}(\xi_1,a)A_{i-k,\xi_1}^{(r-1)}(a,b|\xi_2 x)t^i\right)\right)\\
&= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\sum_{k=0}^{i} b^k H_{k,\xi_2}(\xi_1,a)A_{n-j,\xi_2}^{(r)}(a,b|\xi_1 x)A_{i-k,\xi_1}^{(r-1)}(a,b|\xi_2 x)\right)t^n.
\end{aligned}
\tag{41}
$$

By comparing the coefficients of (40) with (41), we have the desired identity.

□

Theorem 8. *For r, $\xi_1,\xi_2\in\mathbb{N}$ with $\xi_1\equiv 1$ (mod2) and $\xi_2\equiv 1$ (mod2), $n\geq 0$, we have*

$$
\begin{aligned}
&\sum_{j=0}^{n}\sum_{k=0}^{\xi_1-1}(-1)^k A_{j,\xi_1}^{(r)}(a,b|\xi_2 x+\frac{\xi_2}{\xi_1}k)A_{n-j,\xi_2}^{(r-1)}(a,b|\xi_1 x)\\
&\qquad\qquad = \sum_{j=0}^{n}\sum_{k=0}^{\xi_2-1}(-1)^k A_{j,\xi_2}^{(r)}(a,b|\xi_1 x+\frac{\xi_1}{\xi_2}k)A_{n-j,\xi_1}^{(r-1)}(a,b|\xi_2 x).
\end{aligned}
$$

Proof. From (37) and (38), we observe that

$$
\begin{aligned}
I^{(r)}(\xi_1,\xi_2) &= (a+bt)^{\xi_1\xi_2 x}\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r}(a+bt)^{\xi_1(x_1+\cdots+x_r)}\, d\mu_{-1}(x_1)\cdots\, d\mu_{-1}(x_r)\\
&\quad\times\left(\frac{\int_{\mathbb{Z}_p}(a+bt)^{\xi_2 x_r}\, d\mu_{-1}(x_r)}{\int_{\mathbb{Z}_p}(a+bt)^{\xi_1\xi_2 x}\, d\mu_{-1}(x)}\right)\\
&\quad\times(a+bt)^{\xi_1\xi_2 x}\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r-1}(a+bt)^{\xi_2(x_1+\cdots+x_{r-1})}\, d\mu_{-1}(x_1)\cdots\, d\mu_{-1}(x_{r-1})\\
&= (a+bt)^{\xi_1\xi_2 x}\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r}(a+bt)^{\xi_1(x_1+\cdots+x_r)}\, d\mu_{-1}(x_1)\cdots\, d\mu_{-1}(x_r)\\
&\quad\times\left(\sum_{k=0}^{\xi_1-1}(-1)^k(a+bt)^{\xi_2 k}\right)\\
&\quad\times(a+bt)^{\xi_1\xi_2 x}\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r-1}(a+bt)^{\xi_2(x_1+\cdots+x_{r-1})}\, d\mu_{-1}(x_1)\cdots\, d\mu_{-1}(x_{r-1})\\
&= \sum_{k=0}^{\xi_1-1}(-1)^k\left(\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r}(a+bt)^{\xi_1(x_1+\cdots+x_r+\frac{\xi_2}{\xi_1}k+\zeta_2 x)}\, d\mu_{-1}(x_1)\cdots\, d\mu_{-1}(x_r)\right)\\
&\quad\times\left(\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r-1}(a+bt)^{\xi_2(x_1+\cdots+x_{r-1}+\xi_1 x)}\, d\mu_{-1}(x_1)\cdots\, d\mu_{-1}(x_{r-1})\right)\\
&= \left(\sum_{k=0}^{\xi_1-1}(-1)^k\sum_{j=0}^{\infty}A^{(r)}_{j,\xi_1}(a,b|\xi_2 x+\frac{\xi_2}{\xi_1}k)\frac{t^j}{j!}\right)\left(\sum_{l=0}^{\infty}A^{(r-1)}_{l,\xi_2}(a,b|\xi_1 x)t^l\right)\\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\sum_{k=0}^{\xi_1-1}(-1)^k A^{(r)}_{j,\xi_1}(a,b|\xi_2 x+\frac{\xi_2}{\xi_1}k)A^{(r-1)}_{n-j,\xi_2}(a,b|\xi_1 x)\right)t^n.
\end{aligned}
\tag{42}
$$

On the other hand, by (37) and (38), we observe that

$$
\begin{aligned}
I^{(r)}(\xi_2,\xi_1) &= (a+bt)^{\xi_1\xi_2 x}\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r}(a+bt)^{\xi_2(x_1+\cdots+x_r)}\, d\mu_{-1}(x_1)\cdots\, d\mu_{-1}(x_r)\\
&\quad\times\left(\frac{\int_{\mathbb{Z}_p}(a+bt)^{\xi_1 x_r}\, d\mu_{-1}(x_r)}{\int_{\mathbb{Z}_p}(a+bt)^{\xi_1\xi_2 x}\, d\mu_{-1}(x)}\right)\\
&\quad\times(a+bt)^{\xi_1\xi_2 x}\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r-1}(a+bt)^{\xi_1(x_1+\cdots+x_{r-1}+\xi_2 x)}\, d\mu_{-1}(x_1)\cdots\, d\mu_{-1}(x_{r-1})
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
&= (a+bt)^{\xi_1\xi_2 x}\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r}(a+bt)^{\xi_2(x_1+\cdots+x_r)}\,d\mu_{-1}(x_1)\cdots\,d\mu_{-1}(x_r)\\
&\qquad\qquad \times\left(\sum_{k=0}^{\xi_2-1}(-1)^k(a+bt)^{\xi_1 k}\right)\\
&\times (a+bt)^{\xi_1\xi_2 x}\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r-1}(a+bt)^{\xi_1(x_1+\cdots+x_{r-1})}\,d\mu_{-1}(x_1)\cdots\,d\mu_{-1}(x_{r-1})\\
&= \sum_{k=0}^{\xi_2-1}(-1)^k\left(\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r}(a+bt)^{\xi_2(x_1+\cdots+x_r+\frac{\xi_1}{\xi_2}k+\xi_1 x)}\,d\mu_{-1}(x_1)\cdots\,d\mu_{-1}(x_r)\right)\\
&\times\left(\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r-1}(a+bt)^{\xi_1(x_1+\cdots+x_{r-1}+\xi_2 x)}\,d\mu_{-1}(x_1)\cdots\,d\mu_{-1}(x_{r-1})\right)\\
&= \left(\sum_{k=0}^{\xi_2-1}(-1)^k\sum_{j=0}^{\infty}A_{j,\xi_2}^{(r)}(a,b|\xi_1 x+\frac{\xi_1}{\xi_2}k)\frac{t^j}{j!}\right)\left(\sum_{l=0}^{\infty}A_{l,\xi_1}^{(r-1)}(a,b|\xi_2 x)t^l\right)\\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\sum_{k=0}^{\xi_2-1}(-1)^k A_{j,\xi_2}^{(r)}(a,b|\xi_1 x+\frac{\xi_1}{\xi_2}k)A_{n-j,\xi_1}^{(r-1)}(a,b|\xi_2 x)\right)t^n.
\end{aligned}
$$

By comparing with the coefficients of (42) and (43), we have the desired identity. □

Finally, from (14) and (15), we consider ξ-generalized Catalan numbers and polynomials, respectively, given by

(44)
$$
\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(a+bt)^{\frac{\xi(x_1+x_2+\cdots+x_r)}{2}}\,d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_r)\\
&\qquad\qquad = \left(\frac{2}{\sqrt{a+bt}+1}\right)^r = \sum_{n=0}^{\infty}W_{n,\xi}^{(r)}(a,b)t^n,
\end{aligned}
$$

and

(45)
$$
\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(a+bt)^{\frac{\xi(x_1+x_2+\cdots+x_r+x)}{2}}\,d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_r)\\
&\qquad\qquad = \left(\frac{2}{\sqrt{a+bt}+1}\right)^r(a+bt)^{\frac{\xi x}{2}} = \sum_{n=0}^{\infty}W_{n,\xi}^{(r)}(a,b|x)t^n.
\end{aligned}
$$

For $\xi_1, \xi_2 \in \mathbb{N}$, we consider

$$\begin{aligned} J^{(r)}(\xi_1,\xi_2) = &\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r}(a+bt)^{\frac{\xi_1(x_1+\cdots+x_r+\xi_2 x)}{2}}\,d\mu_{-1}(x_1)\cdots\,d\mu_{-1}(x_r) \\ &\times\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r}(a+bt)^{\frac{\xi_2(x_1+\cdots+x_r+\xi_1 x)}{2}}\,d\mu_{-1}(x_1)\cdots\,d\mu_{-1}(x_r) \\ &\times\frac{1}{\int_{\mathbb{Z}_p}(a+bt)^{\frac{\xi_1\xi_2 x}{2}}\,d\mu_{-1}(x)}. \end{aligned} \tag{46}$$

Then we note that $J^{(r)}(\xi_1,\xi_2)$ is symmetric in ξ_1 and ξ_2.
In the same way as above, the following theorems can be obtained.

Theorem 9. *For $r, \xi_1, \xi_2 \in \mathbb{N}$ with $\xi_1 \equiv 1 \pmod 2$ and $\xi_2 \equiv 1 \pmod 2$, $n \geq 0$, we have*

$$\begin{aligned} &\sum_{i=0}^{n}\sum_{k=0}^{i} b^k H_{k,\frac{\xi_1}{2}}(\xi_2,a)W^{(r)}_{n-i,\xi_1}(a,b|\xi_2 x)W^{(r-1)}_{i-k,\xi_2}(a,b|\xi_1 x) \\ &\qquad = \sum_{i=0}^{n}\sum_{k=0}^{i} b^k H_{k,\frac{\xi_2}{2}}(\xi_1,a)W^{(r)}_{n-j,\xi_2}(a,b|\xi_1 x)W^{(r-1)}_{i-k,\xi_1}(a,b|\xi_2 x). \end{aligned}$$

Theorem 10. *For $r,\ \xi_1, \xi_2 \in \mathbb{N}$ with $\xi_1 \equiv 1 \pmod 2$ and $\xi_2 \equiv 1 \pmod 2$, $n \geq 0$, we have*

$$\begin{aligned} &\sum_{j=0}^{n}\sum_{k=0}^{\xi_1-1}(-1)^k W^{(r)}_{j,\xi_1}(a,b|\xi_2 x+\frac{\xi_2}{\xi_1}k)W^{(r-1)}_{n-j,\xi_2}(a,b|\xi_1 x) \\ &\qquad = \sum_{j=0}^{n}\sum_{k=0}^{\xi_2-1}(-1)^k W^{(r)}_{j,\xi_2}(a,b|\xi_1 x+\frac{\xi_1}{\xi_2}k)W^{(r-1)}_{n-j,\xi_1}(a,b|\xi_2 x). \end{aligned}$$

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