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A NOTE ON GENERALIZED DEGENERATE FROBENIUS-EULER-GENOCCHI POLYNOMIALS

DMITRIY DOLGY, WONJOO KIM, AND HYUNSEOK LEE

ABSTRACT. We consider Frobenius-Genocchi polynomials and introduce their generalization - Frobenius-Euler-Genocchi polynomials, the degenerate and generalized versions of the last and derive some important identities. In addition, we introduce the higher-order Frobenius-Euler-Genocchi polynomials of order α and study their properties involving the generalized falling factorials, the degenerate Euler polynomials of order α and degenerate Stirling numbers of the second kind.

1. INTRODUCTION

Investigations for various degenerate versions of some special numbers and polynomials have drawn the attention of many mathematicians in recent years, which were initiated by Carlitz's work in [4,5]. Many interesting results were obtained by exploiting different tools such as generating functions, combinatorial methods, p-adic analysis, umbral calculus, differential equations, probability theory, operator theory, analytic number theory and quantum physics (see [6, 8–18]). Belbachir et al. introduced the Euler–Genocchi polynomials in [2] and Goubi generalized them to the generalized Euler–Genocchi polynomials of order α in [7]. Here we introduce degenerate versions for both of them. Namely, we introduce the generalized degenerate Frobenius-Euler–Genocchi polynomials as a degenerate version of the Frobenius-Euler-Genocchi polynomials. In addition, we introduce their higher-order version, namely the generalized degenerate Frobenius-Euler–Genocchi polynomials of order α, as a degenerate version of the generalized Frobenius-Euler r–Genocchi polynomials of order α. The aim of this paper is to study certain properties and identities involving these polynomials, the generalized falling factorials, the degenerate Frobenius-Euler polynomials of order α. The novelty of this paper is that it is the first paper that introduces the generalized degenerate Frobenius-Euler–Genocchi polynomials and the generalized degenerate Frobenius Euler–Genocchi polynomials of order α, as degenerate versions of the polynomials introduced earlier in [2,3,8–17].

The outline of this paper is as follows. In Section 1, we recall the degenerate exponentials, the degenerate Euler polynomials, the degenerate Euler polynomials of order α, and the degenerate Genocchi polynomials of order α. Section 2 is the main result of this paper. We introduce Frobenius-Genocchi polynomials, their degenerate version and generalized degenerate Frobenius-Euler-Genocchi polynomials as a generalization of both the degenerate Euler polynomials and the degenerate Genocchi polynomials. Then the generalized degenerate Frobenius-Euler–Genocchi polynomials of order α, are introduced as a higher-order version of degenerate Frobenius-Euler–Genocchi. We deduce some expressions for them with a positive integer *m*. Then we observe certain relations between generalized degenerate Frobenius-Euler–Genocchi polynomials of order α and the degenerate Euler polynomials of order α and on this base degenerate Stirling numbers of the second kind.

In the rest of this section, we recall the facts that we need throughout this paper.

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It is well known that the Euler polynomials are defined by

$$\frac{2}{e^t+1}e^{xt}=\sum_{n=o}^{\infty}E_n(x)\frac{t^n}{n!} \tag{1}$$

When $x=0$, $E_n=E_n(0)$ are called Euler numbers.

The Genocchi polynomials are given by

$$\frac{2t}{e^t+1}e^{xt}=\sum_{n=o}^{\infty}G_n(x)\frac{t^n}{n!} \tag{2}$$

When $x=0$, $G_n=G_n(0)$ are called Euler numbers.

Note that $G_n\in\mathbb{N}$.

For any nonzero $\lambda\in\mathbb{R}$, the degenerate exponentials are defined by

$$e_\lambda^x(t)=(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!},\ \ e_\lambda(t)=e_\lambda^1(t),\ (\text{see [8]}) \tag{3}$$

where the generalized falling factorials are given by

$$(x)_{0,\lambda}=1,\ \ (x)_{n,\lambda}=x(x-\lambda)\cdots(x-(n-1)\lambda),\ \ (n\geq 1). \tag{4}$$

Let $\log_\lambda(t)$ be the compositional inverse function of $e_\lambda(t)$ such that $\log_\lambda(e_\lambda(t))=e_\lambda(\log_\lambda t)=t$. It is called the degenerate logarithm and is given by (see [13])

$$\log_\lambda(1+t)=\sum_{k=1}^{\infty}\frac{\lambda^{k-1}(1)_{k,\frac{1}{\lambda}}}{k!}t^k=\frac{1}{\lambda}\left((1+t)^\lambda-1\right) \tag{5}$$

Note that $\lim_{\lambda\to 0}\log_\lambda(1+t)=\log(1+t)$ and $\lim_{\lambda\to 0}e_\lambda(t)=e^t$.

For non-negative integers $n\geq 0$, the degenerate Stirling numbers of the second kind are introduced by Kim–Kim as

$$(x)_{n,\lambda}=\sum_{k=0}^{n}S_{2,\lambda}(n,k)(x)_k,\ (\text{see [12,14]}). \tag{6}$$

where $(x)_{0,\lambda}=1,\ \ (x)_{n,\lambda}=x(x-\lambda)\cdots(x-n+1)\lambda),\ \ (n\geq 1)$.

For $u\in\mathbb{R}$ with $u\neq 1$ the Frobenius-Euler polynomials are defined by the generating functions to be

$$\frac{1-u}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}H_n(x|u)\frac{t^n}{n!},\ (\text{see [12,14]}). \tag{7}$$

If $x=0$, $H_n(u)=H_n(0|u)$ are called Frobenius-Euler numbers.

If $u=-1$ then

$$\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}H_n(x|-1)\frac{t^n}{n!}. \tag{8}$$

From here and (1) we conclude

$$H_n(x|-1)=E_n(x). \tag{9}$$

If $x=0$ then $H_n(-1)=E_n$.

Recently, the degenerate Frobenius-Euler polynomials were introduced by the generating function as

$$\frac{1-u}{e_\lambda(t)-u}e_\lambda^x(t)=\sum_{n=0}^{\infty}h_{n,\lambda}(x|u)\frac{t^n}{n!},\ (\text{see [1,8,9]}). \tag{10}$$

If $x=0$, $h_{n,\lambda}=h_{n,\lambda}(0|u)$ are called degenerate Frobenius-Euler numbers.

Let $u=-1$ then

$$\frac{2}{e_\lambda(t)+1}e_\lambda^x(t)=\sum_{n=0}^{\infty}h_{n,\lambda}(x|-1)\frac{t^n}{n!}. \tag{11}$$

Carlitz in [4] introduced a degenerate version of Euler polynomials called degenerate Euler polynomials as

$$\frac{2}{e_\lambda(t)+1}e_\lambda^x(t)=\sum_{n=0}^{\infty}\varepsilon_{n,\lambda}(x)\frac{t^n}{n!}. \tag{12}$$

If $x=0$ then $\varepsilon_{n,\lambda}=\varepsilon_{n,\lambda}(0)$ are called degenerate Euler numbers.

From (11) and (12) we get

$$h_{n,\lambda}(x|-1)=\varepsilon_{n,\lambda}(x). \tag{13}$$

If $x=0$ then $h_{n,\lambda}(0|-1)=h_{n,\lambda}(-1)=\varepsilon_{n,\lambda}(0)=\varepsilon_{n,\lambda}$.

The degenerate higher-order Euler polynomials of the order α are given by the generating function as

$$\left(\frac{2}{e_\lambda(t)+1}\right)^{\alpha}e_\lambda^x(t)=\sum_{n=0}^{\infty}\varepsilon_{n,\lambda}^{(\alpha)}(x)\frac{t^n}{n!},\ (\text{see } [8-12]). \tag{14}$$

If $x=0$, $\varepsilon_{\lambda}^{(\alpha)}=\varepsilon_{\lambda}^{(\alpha)}(0)$ are called degenerate Euler numbers of order α.

Degenerate form of Genocchi polynomials is given by the generating function as

$$\frac{2t}{e_\lambda(t)+1}e_\lambda^x(t)=\sum_{n=0}^{\infty}G_{n,\lambda}(x)\frac{t^n}{n!}. \tag{15}$$

If $x=0$ then $G_{n,\lambda}=G_{n,\lambda}(0)$ are called degenerate Genocchi numbers of order α.

The degenerate higher-order Frobenius-Euler polynomials of the order α are given by the generating function as

$$\left(\frac{1-u}{e_\lambda(t)-u}\right)^{\alpha}e_\lambda^x(t)=\sum_{n=0}^{\infty}h_{n,\lambda}^{(\alpha)}(x|u)\frac{t^n}{n!},\ (\text{see } [8-12]). \tag{16}$$

If $x=0$, $h_{n,\lambda}^{(\alpha)}(u)=h_{n,\lambda}^{(\alpha)}(0|u)$ are called degenerate Frobenius-Euler numbers of order α.

For $u=-1$ we get

$$\left(\frac{1-u}{e_\lambda(t)-u}\right)^{\alpha}e_\lambda^x(t)=\sum_{n=0}^{\infty}h_{n,\lambda}^{(\alpha)}(x|-1)\frac{t^n}{n!}\ \text{ and }\ h_{n,\lambda}^{(\alpha)}(x|-1)=\varepsilon_{n,\lambda}^{(\alpha)}(x). \tag{17}$$

2. Generalized degenerate Frobenius-Euler-Genocchi polynomials

We define Frobenius-Genocchi polynomials by generating function to be

$$\frac{t(1-u)}{e^t-u}e^{xt}=\sum_{n=0}^{\infty}FG_n(x|u)\frac{t^n}{n!}. \tag{18}$$

When $x=0$, $FG_n(u)=FG_n(0|u)$ are called Frobenius-Genocchi numbers.

In particular, if $u=-1$, we get

$$\sum_{n=0}^{\infty}FG_n(x|-1)\frac{t^n}{n!}=\frac{2t}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}. \tag{19}$$

From here

$$FG_n(x|-1)=G_n(x).$$

If $x=0$, then

(20) $$FG_n(0|-1) = FG_n(-1) = G_n(0) = G_n.$$

Note that from (18), $FG_0(x|u) = 0$.

Observe that

$$\begin{aligned}\frac{1-u}{e^t-u}e^{xt} &= \frac{1}{t}\sum_{n=1}^{\infty} FG_n(x|u)\frac{t^n}{n!}\\ &= \frac{1}{t}\sum_{n=0}^{\infty} FG_{n+1}(x|u)\frac{t^{n+1}}{(n+1)!}\\ &= \sum_{n=0}^{\infty}\frac{FG_{n+1}(x|u)}{n+1}\frac{t^n}{n!}.\end{aligned}$$

(21)

From (7) and (21) we obtain

(22) $$H_n(x|u) = \frac{FG_{n+1}(x|u)}{n+1}.$$

For integer $r \geq 0$, we consider the generalization of Frobenius- Genocchi polynomials that we shall call Frobenius-Euler-Genicchi polynomials (or Frobenius-Euler r-Genicchi polynomials). They are defined by generating function as

(23) $$\frac{t^r(1-u)}{e^t-u}e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x|u)\frac{t^n}{n!}.$$

When $x = 0$, $B_n^{(r)}(u) = B_n^{(r)}(0|u)$ are called Frobenius-Euler-Genocchi numbers.

In particular, $B_n^{(r)}(x|u) = H_n(x|u)$, $B_n^{(1)}(x|u) = FG_n(x|u)$.

Note that $B_0^{(r)}(x|u) = B_1^{(r)}(x|u) = \cdots = B_{r-1}^{(r)}(x|u) = 0$.

Introduce degenerate Frobenius-Genocchi polynomials by generating function to be

(24) $$\frac{t(1-u)}{e_\lambda(t)-u}e_\lambda^x = \sum_{n=0}^{\infty} FG_{n,\lambda}(x|u)\frac{t^n}{n!}.$$

From (24), it follows $FG_{0,\lambda}(x|u) = 0$.

If $x = 0$, then $FG_{n,\lambda}(u) = FG_{n,\lambda}(0|u)$ are called degenerate Frobenius-Genocchi numbers.

Note that

(25) $$\lim_{\lambda\to 0} h_{n,\lambda}(x|u) = H_n(x|u),\ \lim_{\lambda\to 0} FG_{n,\lambda}(x|u) = FG_n(x|u).$$

Consider the chain of equalities

$$\begin{aligned}\sum_{n=0}^{\infty} h_{n,\lambda}(x|u)\frac{t^n}{n!} &= \frac{1-u}{e_\lambda(t)-u}e_\lambda^x(t) = \frac{1}{t}\sum_{n=0}^{\infty} FG_{n,\lambda}(x|u)\frac{t^n}{n!} = \frac{1}{t}\sum_{n=1}^{\infty} FG_{n,\lambda}(x|u)\frac{t^n}{n!}\\ &= \frac{1}{t}\sum_{n=0}^{\infty} FG_{n+1,\lambda}(x|u)\frac{t^{n+1}}{(n+1)!} = \sum_{n=0}^{\infty}\frac{FG_{n+1,\lambda}(x|u)}{n+1}\frac{t^n}{n!}.\end{aligned}$$

(26)

From the latter, comparing coefficients in the first and the last expressions of (26), we get

(27) $$h_{n,\lambda}(x|u) = \frac{FG_{n+1,\lambda}(x|u)}{n+1}.$$

Introduce the generalization of degenerate Frobenius-Genocchi polynomials as

$$\frac{t^r(1-u)}{e_\lambda(t)-u}e_\lambda^x(t)=\sum_{n=0}^{\infty}B_{n,\lambda}^{(r)}(x|u)\frac{t^n}{n!}, \tag{28}$$

where $r\geq 0$, $r\in\mathbb{Z}$.

Polynomials $B_{n,\lambda}^{(r)}(x|u)$ defined by generating function (28) are called generalized degenerate Frobenius-Euler-Genocchi polynomials.

If $x=0$, $B_{n,\lambda}^{(r)}(u)=B_{n,\lambda}^{(r)}(0|u)$ are called generalized degenerate Frobenius-Euler-Genocchi numbers.

If $u=-1$, then

$$\sum_{n=0}^{\infty}B_{n,\lambda}^{(r)}(x|-1)\frac{t^n}{n!}=\frac{2t^r}{e_\lambda(t)+1}e_\lambda^x(t)=\sum_{n=0}^{\infty}A_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}. \tag{29}$$

From here, we get the correspondece $B_{n,\lambda}^{(r)}(x|-1)=A_{n,\lambda}^{(r)}(x)$, where $A_{n,\lambda}^{(r)}(x)$ are Frobenius-Euler polynomials defined in [12].

For generalized degenerate Frobenius-Euler-Genocchi polynomials, we have

$$\begin{aligned}\sum_{n=0}^{\infty}B_{n,\lambda}^{(r)}(x|u)\frac{t^n}{n!}&=\frac{t^r(1-u)}{e_\lambda(t)-u}e_\lambda^x(t)\\&=\sum_{l=0}^{\infty}B_{l,\lambda}^{(r)}(u)\frac{t^l}{l!}e_\lambda^x(t)\\&=\sum_{l=0}^{\infty}B_{l,\lambda}^{(r)}(u)\sum_{k=0}^{\infty}(x)_{k,\lambda}\frac{t^k}{k!}\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}B_{l,\lambda}^{(r)}(u)(x)_{n-l,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{30}$$

Therefore, by comparing coefficients in both sides of (30), we get the following theorem.

Theorem 1. *For $r\geq 0$ with $r\in\mathbb{Z}$, we have*

$$B_{n,\lambda}^{(r)}(x|u)=\sum_{l=0}^{n}\binom{n}{l}B_{l,\lambda}^{(r)}(u)(x)_{n-l,\lambda}.$$

Then, we get

$$\begin{aligned}\sum_{n=0}^{\infty}\varepsilon_{n,\lambda}(x)\frac{t^n}{n!}&=\frac{1}{t^r}\sum_{n=0}^{\infty}B_{n,\lambda}^{(r)}(x|-1)\frac{t^n}{n!}\\&=\sum_{n=r}^{\infty}B_{n,\lambda}^{(r)}(x|-1)\frac{t^{n-r}}{n!}\\&=\sum_{n=0}^{\infty}B_{n+r,\lambda}^{(r)}(x|-1)\frac{t^n}{(n+r)!}\\&=\sum_{n=0}^{\infty}B_{n+r,\lambda}^{(r)}(x|-1)(n+r)_r\frac{t^n}{n!}.\end{aligned} \tag{31}$$

Comparing coefficient from the left and right sides of the last equality, we obtain

Theorem 2. *For $r \geq 0$ with $r \in \mathbb{Z}$, we have*

$$B_{n+r,\lambda}^{(r)}(x|-1) = \frac{1}{(n+r)_r}\varepsilon_{n,\lambda}(x).$$

Observe that

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n,\lambda}^{(r)}(x+1|u)\frac{t^n}{n!} &= \frac{t^r(1-u)}{e_\lambda(t)-u}e_\lambda^{x+1}(t) \\ &= \frac{t^r(1-u)}{e_\lambda(t)-u}e_\lambda^{x}(t)e_\lambda(t) \\ &= \sum_{l=0}^{\infty} B_{l,\lambda}^{(r)}(x|u)\frac{t^l}{l!}e_\lambda(t) \\ &= \sum_{l=0}^{\infty} B_{l,\lambda}^{(r)}(x|u)\frac{t^l}{l!}\sum_{k=0}^{\infty}(1)_{k,\lambda}\frac{t^k}{k!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}B_{l,\lambda}^{(r)}(x|u)(1)_{n-l,\lambda}\right)\frac{t^n}{n!}.\end{aligned}$$

(32)

Comparing coefficients in both sides of this expression, we get the following

Theorem 3. *For $r \geq 0$ with $r \in \mathbb{Z}$, we have*

$$B_{n,\lambda}^{(r)}(x+1|u) = \sum_{l=0}^{n}\binom{n}{l}B_{l,\lambda}^{(r)}(x|u)(1)_{n-l,\lambda}.$$

Observe that

$$\begin{aligned}\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!} &= \frac{1}{t^r}B_{n,\lambda}^{(r)}(x|u)\frac{t^n}{n!}\left(\frac{e_\lambda(t)-u}{1-u}\right) \\ &= \frac{1}{1-u}\frac{1}{t^r}\sum_{n=0}^{\infty}B_{n,\lambda}^{(r)}(x|u)\frac{t^n}{n!}e_\lambda(t) - \frac{u}{1-u}\frac{1}{t^r}\sum_{n=0}^{\infty}B_{n,\lambda}^{(r)}(x|u)\frac{t^n}{n!} \\ &= \frac{1}{1-u}\sum_{l=0}^{\infty}B_{l+r,\lambda}^{(r)}(x|u)\frac{t^l}{(l+r)!}\sum_{k=o}^{\infty}(1)_{k,\lambda}\frac{t^k}{k!} - \frac{u}{1-u}\sum_{n=0}^{\infty}B_{n+r,\lambda}^{(r)}(x|u)\frac{t^n}{(n+r)!} \\ &= \frac{1}{1-u}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\binom{n+r}{l+r}B_{l+r,\lambda}^{(r)}(x|u)\frac{n!}{(n+r)!}\right)\frac{t^n}{n!} - \frac{u}{1-u}\sum_{n=0}^{\infty}B_{n+r,\lambda}^{(r)}(x|u)\frac{n!}{(n+r)!}\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{1-u}\sum_{l=0}^{n}\binom{n+r}{l+r}B_{l+r,\lambda}^{(r)}(x|u)(1)_{n-l,\lambda}\frac{1}{(n+r)_r} - \frac{u}{1-u}B_{n+r,\lambda}^{(r)}(x|u)\frac{1}{(n+r)_r}\right)\frac{t^n}{n!}.\end{aligned}$$

(33)

Therefore, we arrive at the following output.

Theorem 4. *For $r \geq 0$ with $r \in \mathbb{Z}$, we have*

$$(x)_{n,\lambda} = \frac{1}{1-u}\frac{1}{(n+r)_r}\left(\sum_{l=0}^{n}\binom{n+r}{l+r}B_{l+r,\lambda}^{(r)}(x|u)(1)_{n-l,\lambda} - uB_{n+r,\lambda}^{(r)}(x|u)\right).$$

Observe now

$$\begin{aligned}
\sum_{n=0}^{\infty} B_{n,\lambda}^{(r)}(x|u)\frac{t^n}{n!} &= \frac{t^r(1-u)}{e_\lambda(t)-u}e_\lambda^x(t) = \frac{t^r(1-u)}{e_\lambda^m(t)-u^m}\sum_{k=0}^{m-1} e_\lambda^{x+k}(t)u^{m-1-k} \\
&= \frac{(mt)^r(1-u)}{e_{\frac{\lambda}{m}}(mt)-u^m}\frac{1}{m^r}\sum_{k=0}^{m-1} e_{\frac{\lambda}{m}}^{\frac{x+k}{m}}(mt)u^{m-1-k} \\
&= \frac{1}{m^r}\frac{1}{1+u+u^2+\cdots+u^{m-1}}\sum_{k=0}^{m-1}\frac{(mt)^r(1-u^m)}{e_{\frac{\lambda}{m}}(mt)-u^m}e_{\frac{\lambda}{m}}^{\frac{x+k}{m}}(mt)u^{m-1-k} \\
&= \frac{1}{m^r}\frac{1}{1+u+u^2+\cdots+u^{m-1}}\sum_{k=0}^{m-1}\sum_{n=0}^{\infty} B_{n,\frac{\lambda}{m}}^{(r)}\left(\frac{x+k}{m}|u^m\right)m^n\frac{t^n}{n!}u^{m-1-k} \\
&= \sum_{n=0}^{\infty}\left(m^{n-r}\frac{1}{1+u+u^2+\cdots+u^{m-1}}\sum_{k=0}^{m-1} B_{n,\frac{\lambda}{m}}^{(r)}\left(\frac{x+k}{m}|u^m\right)u^{m-1-k}\right)\frac{t^n}{n!}.
\end{aligned} \tag{34}$$

Comparing coefficients in both sides of this equality, we get the following result.

Theorem 5. *For $r \geq 0$ with $r \in \mathbb{Z}$, we have*

$$B_{n,\lambda}^{(r)}(x|u) = m^{n-r}\frac{1}{1+u+u^2+\cdots+u^{m-1}}\sum_{k=0}^{m-1} B_{n,\frac{\lambda}{m}}^{(r)}\left(\frac{x+k}{m}|u^m\right)u^{m-1-k}.$$

For nonzero $\alpha \in \mathbb{C}$, and $r \in \mathbb{Z}$ with $r \geq 0$, we consider the generalized degenerate Frobenius-Euler–Genocchi polynomials of order α which are given by

$$t^r\left(\frac{1-u}{e_\lambda(t)-u}\right)^\alpha e_\lambda^x(t) = \sum_{n=0}^{\infty} B_{n,\lambda}^{(r,\alpha)}(x|u)\frac{t^n}{n!}. \tag{35}$$

Note that $B_{0,\lambda}^{(r,\alpha)}(x|u) = B_{1,\lambda}^{(r,\alpha)}(x|u) = \cdots = B_{n,\lambda}^{(r-1,\alpha)}(x|u) = 0$.

When $x=0$, $B_{n,\lambda}^{(r,\alpha)}(u) = B_{n,\lambda}^{(r,\alpha)}(0|u)$ are called the generalized degenerate Frobenius-Euler–Genocchi numbers of order α.

We mention here that these polynomials can be viewed as a special case of polynomials $L_n^{(f,g,h)}$ defined by the generating function

$$f(t)g\circ h(t) = \sum_{n\geq 0} L_n^{(f,g,h)}\frac{t^n}{n!}. \tag{36}$$

which are recently studied by Goubi [7].

For this, one can take

$$f(t) = t^r e_\lambda^x(t), g(t) = t^a h(t) = \frac{1-u}{e_\lambda(t)-u}. \tag{37}$$

From (35), we get

$$B_{n,\lambda}^{(r,\alpha)}(x|u) = \sum_{k=0}^{n}\binom{n}{k}B_{n,\lambda}^{(r,\alpha)}(u)(x)_{n-k,\lambda} = \sum_{k=0}^{n}\binom{n}{k}B_{n-k,\lambda}^{(r,\alpha)}(u)(x)_{k,\lambda}. \tag{38}$$

Let $\alpha=-m(m\in\mathbb{N})$. Then, by (28)

$$\begin{aligned}\sum_{n=0}^{\infty}B_{n,\lambda}^{(r,-m)}(x|u)\frac{t^n}{n!}&=\frac{t^r}{(1-u)^m}(e_\lambda(t)-u)^m e_\lambda^x(t)\\&=\frac{t^r}{(1-u)^m}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}e_\lambda^{x+k}(t)u^{m-k}\\&=\frac{t^r}{(1-u)^m}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\sum_{n=0}^{\infty}(x+k)_{n,\lambda}\frac{t^n}{n!}u^{m-k}\\&=\sum_{n=0}^{\infty}\frac{1}{(1-u)^m}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}u^{m-k}(x+k)_{n,\lambda}\frac{t^{n+r}}{n!}\\&=\sum_{n=0}^{\infty}\frac{1}{(1-u)^m}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}u^{m-k}(x+k)_{n-r,\lambda}(n)_r\frac{t^n}{n!}.\end{aligned}\tag{39}$$

Therefore, by comparing cofficients in both sides of (39), we obtain the following theorem.

Theorem 6. *For $r\in\mathbb{Z}$ with $r\geq 1$ and $n\in\mathbb{N}\cup\{0\}$, we have*

$$B_{n,\lambda}^{(r,-m)}(x|u)=\frac{(n)_r}{(1-u)^m}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}u^{m-k}(x+k)_{n-r,\lambda}.$$

When $x=0$, we have

$$B_{n,\lambda}^{(r,-m)}(u)=\frac{(n)_r}{(1-u)^m}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}u^{m-k}(k)_{n-r,\lambda}.\tag{40}$$

From (38) and (40), we have

$$\begin{aligned}B_{n,\lambda}^{(r,-m)}(x|u)&=\sum_{k=0}^{n}\binom{n}{k}B_{n-k,\lambda}^{(r,-m)}(u)(x)_{k,\lambda}\\&=\sum_{k=0}^{n}\binom{n}{k}\frac{(n-k)_r}{(1-u)^m}\sum_{k=0}^{n}\binom{m}{j}(-1)^{m-j}u^{m-j}(j)_{n-k-r,\lambda}(x)_{k,\lambda}\\&=\frac{1}{(1-u)^m}\sum_{k=0}^{n}\sum_{j=0}^{m}\binom{n}{k}\binom{m}{j}(-1)^{m-j}u^{m-j}(j)_{n-k-r,\lambda}(n-k)_r(x)_{k,\lambda}.\end{aligned}\tag{41}$$

By (35), we get

$$\begin{aligned}\sum_{n=0}^{\infty}B_{n,\lambda}^{(r,\alpha)}(x|u)\frac{t^n}{n!}&=t^r\left(\frac{1-u}{e_\lambda(t)-u}\right)^{\alpha}e_\lambda^x(t)=t^r\sum_{n=0}^{\infty}h_{n,\lambda}^{(\alpha)}(x|u)\frac{t^n}{n!}\\&=\sum_{n=r}^{\infty}h_{n-r,\lambda}^{(\alpha)}(x|u)\frac{n!}{(n-r)!}\frac{t^n}{n!}=\sum_{n=r}^{\infty}(n)_r h_{n-r,\lambda}^{(\alpha)}(x|u)\frac{t^n}{n!}.\end{aligned}\tag{42}$$

By comparing cofficients in both sides of (42), we arrive at the following result.

Theorem 7. *For $n,r\geq 0$ with $n\geq r$, we have*

$$B_{n,\lambda}^{(r,\alpha)}(x|u)=(n)_r h_{n-r,\lambda}^{(\alpha)}(x|u).$$

In particular, for $x = 0$, we have

$$B^{(r,\alpha)}_{n,\lambda}(u) = (n)_r h^{(\alpha)}_{n-r,\lambda}(u). \tag{43}$$

By (38) and (43), we get

$$\begin{aligned} B^{(r,\alpha)}_{n,\lambda}(x|u) &= \sum_{k=o}^{n}\binom{n}{k}B^{(r,\alpha)}_{n-k,\lambda}(u)\cdot(x)_{k,\lambda} = \sum_{k=o}^{n-r}\binom{n}{k}B^{(r,\alpha)}_{n-k,\lambda}(u)\cdot(x)_{k,\lambda} \\ &= \sum_{k=o}^{n-r}\binom{n}{k}(n-k)_r h^{(\alpha)}_{n-k-r,\lambda}(u)\cdot(x)_{k,\lambda} = (n)_r\sum_{k=o}^{n-r}\binom{n-r}{k}^{n-r} h^{(\alpha)}_{n-k-r,\lambda}(u)\cdot(x)_{k,\lambda} \\ &= (n)_r(x)_{n-r,\lambda} + (n)_r\sum_{k=o}^{n-r-1}\binom{n-r}{k}h^{(\alpha)}_{n-k-r,\lambda}(u)\cdot(x)_{k,\lambda}. \end{aligned} \tag{44}$$

Therefore, by (44), we obtain the following theorem.

Theorem 8. *For $\alpha(\neq 0) \in \mathbb{C}$ and $n, r \geq 0$ with $n \geq r$, we have*

$$B^{(r,\alpha)}_{n,\lambda}(x|u) = (n)_r(x)_{n-r,\lambda} + (n)_r\sum_{k=o}^{n-r-1}\binom{n-r}{k}h^{(\alpha)}_{n-k-r,\lambda}(u)\cdot(x)_{k,\lambda}.$$

Observe that from (35),

$$t^r\left(\frac{1-u}{e_\lambda(t)-u}\right)^{\alpha} e^x_\lambda(t) = \sum_{n=0}^{\infty} B^{(r,\alpha)}_{n,\lambda}(x|u)\frac{t^n}{n!}.$$

For $u = -1$, we get

$$t^r\left(\frac{1-u}{e_\lambda(t)-u}\right)^{\alpha} e^x_\lambda(t) = \sum_{n=0}^{\infty} B^{(r,\alpha)}_{n,\lambda}(x|-1)\frac{t^n}{n!}. \tag{45}$$

And

$$\begin{aligned} \sum_{n=0}^{\infty}\varepsilon^{(\alpha)}_{n,\lambda}(x)\frac{t^n}{n!} &= \left(\frac{2}{e_\lambda(t)+1}\right)^{\alpha} e^x_\lambda(t) = \frac{1}{t^r}\sum_{n=0}^{\infty}B^{(r,\alpha)}_{n,\lambda}(x|-1)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}B^{(r,\alpha)}_{n,\lambda}(x|-1)\frac{t^{n-r}}{n!} = \sum_{n=0}^{\infty}B^{(r,\alpha)}_{n+r,\lambda}(x|-1)\frac{t^n}{(n+r)!} \\ &= \sum_{n=0}^{\infty}B^{(r,\alpha)}_{n+r}(x|-1)(n+r)_r\frac{t^n}{n!}. \end{aligned} \tag{46}$$

Comparing coefficient from the left and right sides of the last equality, we obtain

Theorem 9. *For $r \geq 0$, $r \in 1$, $\alpha \in \mathbb{C}$, we have*

$$B^{(r,\alpha)}_{n+r}(x|-1) = \frac{1}{(n+r)_r}\varepsilon^{(\alpha)}_{n,\lambda}(x).$$

In particular, for $x = 0$, we enclose

$$B^{(r,\alpha)}_{n+r,\alpha}(0|-1) = \frac{1}{(n+r)_r}\varepsilon^{(\alpha)}_{n,\lambda}. \tag{47}$$

In [12, (Theorem 2.6)], it was obtained the equality

$$\varepsilon_{n,\lambda}^{(\alpha)} = \sum_{k=1}^{n} (-\alpha)_k \left(\frac{1}{2}\right)^k S_{2,\lambda}(n,k) \tag{48}$$

Therefore,

$$\begin{aligned} B_{n+r,\lambda}^{(r,\alpha)}(0|-1) &= \frac{1}{(n+r)_r}\varepsilon_{n,\lambda}^{(\alpha)} = \frac{1}{(n+r)_r}\sum_{k=1}^{n} (-\alpha)_k \left(\frac{1}{2}\right)^k S_{2,\lambda}(n,k) \\ &= \sum_{k=1}^{n} (-\alpha)_k \left(\frac{1}{2}\right)^k S_{2,\lambda}(n,k)\frac{1}{(n+r)_r}. \end{aligned} \tag{49}$$

3. Conclusion

In recent years, various degenerate versions of many special numbers and polynomials have been explored by using different methods as aforementioned in the introduction. In this paper, we studied the generalized degenerate Frobenius-Euler–Genocchi polynomials as a degenerate version of the introduced Frobenius–Genocchi polynomials. In addition, we introduced their higher-order version, namely the generalized degenerate Frobenius-Euler–Genocchi polynomials of order α, as a degenerate version of Frobenius-Euler r–Genocchi polynomials in order α. Then we studied certain properties and identities involving these polynomials, the generalized falling factorials, the degenerate Euler polynomials of order α and degenerate Stirling numbers of the second kind. It is one of our future projects to continue to study various degenerate versions of some special numbers and polynomials and to find their applications to physics, science and engineering as well as to mathematics.

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GLOCAL EDUCATION CENTER, KWANGWOON UNIVERSITY, SEOUL 139-701, REPUBLIC OF KOREA; SCHOOL OF MATHEMATICS AND COMPUTER TECHNOLOGIES, FAR EASTERN FEDERAL UNIVERSITY, VLADIVOSTOK, RUSSIA.
Email address: `dolgii.dmitrii1964@gmail.com (corresponding author)`

DEPARTMENT OF APPLIED MATHEMATICS, KYUNG HEE UNIVERSITY, SEOUL 02447, REPUBLIC OF KOREA
Email address: `wjookim@khu.ac.kr`

DEPARTMENT OF APPLIED MATHEMATICS, KWANGWOON UNIVERSITY, SEOUL 139-701, REPUBLIC OF KOREA
Email address: `luciasconstant@kw.ac.kr`