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CONTINUITY OF LOCALLY BOUNDED ENDOMORPHISMS OF CONNECTED LIE GROUPS WITHOUT NONTRIVIAL COMPACT CONNECTED SUBGROUPS

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Abstract. We prove the automatic continuity of locally bounded endomorphisms of connected Lie groups without nontrivial compact connected subgroups.

§ 1. Introduction

As is known [1], every locally bounded automorphism of a connected solvable simply connected Lie group G without nontrivial compact connected subgroups is continuous. In this note, we establish the continuity of every locally bounded endomorphism of a connected solvable simply connected Lie group G without nontrivial compact connected subgroups. In particular, the assertion holds for connected simply connected solvable Lie groups and for the universal covering group of $\mathrm{SL}(2,\mathbb{R})$.

§ 2. Preliminaries

Recall that a (not necessarily continuous) homomorphism π of a topological group G into a topological group H is said to be *relatively compact* if there is a neighborhood $U = U_{e_G}$ of the identity element e_G in G whose image $\pi(U)$ has compact closure in H. Obviously, a homomorphism into a locally compact group is relatively compact if and only if it is *locally bounded*,

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i.e., there is a neighborhood U_{e_G} whose image is contained in some element of the filter $\mathfrak{U}_H$ of neighborhoods of e_H in H having compact closure.

Let us also recall the notion of discontinuity group of a homomorphism π of a topological group G into a topological group H (see [2] and [3–5]). Let $\mathfrak{U} = \mathfrak{U}_G$ be the filter of neighborhoods of e_G in G. For every (not necessarily continuous) locally relatively compact homomorphism π of G into H, the set $\mathrm{DG}(\pi) = \bigcap_{U \in \mathfrak{U}} \overline{\pi(U)}$ is called the discontinuity group of π. Here and below, the bar stands for the closure in the corresponding topology (here the closure is taken in the topology of H). (See Definition 1.1.1 in [2].)

The discontinuity group of a homomorphism has some important properties.

Lemma. *Let π be a homomorphism of a topological group G into a topological group H. The set $\mathrm{DG}(\pi)$ is a compact subgroup of the topological group H and a compact normal subgroup of the closed subgroup $\overline{\pi(G)}$ of H. Moreover, the filter basis $\{\overline{\pi(U)} \mid U \in \mathfrak{U}\}$ converges to $\mathrm{DG}(\pi)$, and the homomorphism π is continuous if and only if $\mathrm{DG}(\pi) = \{e_H\}$. If G is a connected Lie group, then $\mathrm{DG}(\pi)$ is a compact connected subgroup of H. Let G be a connected Lie group, let N be a closed normal subgroup of G, and let π be a locally bounded homomorphism of G into a locally compact group H. Let M be the discontinuity group of the restriction $\mathrm{DG}(\pi|_N)$. Then M is a closed normal subgroup of the compact discontinuity group $\mathrm{DG}(\pi)$, and the corresponding quotient group $\mathrm{DG}(\pi)/M$ is isomorphic to the discontinuity group $\mathrm{DG}(\psi)$ of the homomorphism ψ of G obtained as the composition of the homomorphism π and the canonical homomorphism $\overline{\pi(G)} \to \overline{\pi(G)}/M$.*

Proof. See Theorem 1.1.2, Lemma 1.1.6, and Lemma 1.1.7 of [2].

§ 3. Main Result

Theorem 1. *Let G be a connected Lie group without nontrivial compact connected subgroups, and let π be a locally bounded endomorphism of G. Then π is continuous.*

Proof. Recall (see the lemma) that, since G is a connected Lie group, it follows that the discontinuity group $\mathrm{DG}(\pi)$ is a compact connected subgroup of G. (See Lemma 1.1.6 in [2].) Since G has no nontrivial compact connected groups by assumption, it follows that the discontinuity group $\mathrm{DG}(\pi)$ of π is the identity subgroup of G. Hence π is continuous. (See Theorem 1.1.2 of [2].)

§ 4. Discussion

Some continuity conditions for locally bounded finite-dimensional representations, automorphisms and endomorphisms of Lie groups are related to the equivalence of the continuity of the corresponding mapping to the corresponding property of the restriction of the mapping to the center of the group (see, in particular, [6–9]). Let us use the example given in [11] to show that such an equivalence fails to hold for not simply connected solvable linear Lie groups.

Example. Consider the Heisenberg group

$$H = \left\{ h(a,b,c) = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \qquad a,b,c \in \mathbb{R} \right\}$$

and let G be the direct product $\mathbb{T}^1 \times H$ formed by the elements of the form $dh(a,b,c)$, $d \in \mathbb{C}$, $|d| = 1$, $a,b,c \in \mathbb{R}$. The Lie group G is solvable, linear, and connected. Let ρ be the identical mapping of G onto itself ($\rho(dh) = dh$, $d \in \mathbb{T}^1$, $h \in H$), and let α and β be discontinuous characters of the subgroups $A = \{h(a,b,c) \in H, b = c = 0\}$ and $B = \{h(a,b,c) \in H, a = c = 0\}$, respectively. Let us extend them to characters $\widetilde{\alpha}\colon G \to \mathbb{T}^1$ and $\widetilde{\beta}\colon G \to \mathbb{T}^1$ of G defined by the rule

$$\widetilde{\alpha}(dh(a,b,c)) = \alpha(a), \qquad \widetilde{\beta}(dh(a,b,c)) = \beta(b), \qquad dh(a,b,c) \in G.$$

Then the discontinuous endomorphism $\pi = \widetilde{\alpha}\widetilde{\beta}\rho$ of G is continuous on the center $C = \{dh(a,b,c) \in G, a = b = 0\}$.

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