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A ONE-DIMENSIONAL PSEUDOREPRESENTATION OF AN ALMOST CONNECTED LOCALLY COMPACT GROUP IS AN EXPONENTIAL OF A PSEUDOCHARACTER ON THE GROUP IF AND ONLY IF IT IS TRIVIAL ON A LEE SUPPLEMENTARY SUBGROUP OF THE GROUP

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Abstract. We prove here that a one-dimensional pseudorepresentation of an almost connected locally compact group is an exponential of a pseudocharacter on the group if and only if it is trivial on a Lee supplementary subgroup of the group.

§ 1. Introduction

For the definitions, notation, and generalities concerning pseudocharacters, quasicharacters, pseudorepresentations, and quasirepresentations, see [1]–[3]. In particular, recall that a mapping π of a given group G into the field $\mathbb{C}$ of complex numbers is said to be a *one-dimensional quasirepresentation* of G on E if $\pi(e_G) = 1 \in \mathbb{C}$, where e_G stands for the identity element of G and if

$$|\pi(g_1g_2) - \pi(g_1)\pi(g_2)| \leq \varepsilon, \qquad g_1, g_2 \in G,$$

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for some $\varepsilon \geq 0$, which is usually assumed to be sufficiently small, and the least upper bound of $|\pi(g_1 g_2) - \pi(g_1)\pi(g_2)|$ for a one-dimensional quasirepresentation π is referred to as the *defect* of π; a one-dimensional quasirepresentation π of G is said to be a one-dimensional *pseudorepresentation* of G if $\pi(g^n) = \pi(g)^n$ for any $n \in \mathbb{Z}$ and $g \in G$.

We prove here that a one-dimensional pseudorepresentation of an almost connected locally compact group is an exponential of a pseudocharacter on the group if and only if it is trivial on a Lee supplementary subgroup of the group.

§ 2. Preliminaries

We need a well-known lemma. For the convenience of the reader, we present it with a proof.

Lemma 1. *Let φ and ψ be unitary characters of the group $\mathbb{Z}$ of integers. If*

$$|\varphi(n) - \psi(n)| \leq q < 1 \tag{1}$$

for all $n \in \mathbb{Z}$, then $\varphi = \psi$.

Proof. It follows from (1) that $|\varphi \circ \psi^{-1}(n) - 1| \leq q < 1$ for all $n \in \mathbb{N}$. Applying an invariant mean I on $\mathbb{N}$, we see that $|S - 1| < q$ for $S = I_n(\varphi \circ \psi^{-1}(n))$. It follows from the invariance of I that $S\varphi \circ \psi^{-1}(1) = S$, where $S \neq 0$. Thus, $\varphi \circ \psi^{-1}(1) = 1$, $\varphi(1) = \psi(1)$, and $\varphi = \psi$, as was to be proved.

Recall Dong Hoon Lee's supplement theorem (Theorem 2.13 of [5]): every almost connected locally compact group G with the connected component G_0 (i.e., a locally compact group G for which the quotient group G/G_0 is compact) admits a totally disconnected compact subgroup D such that $G = G_0 D$.

§ 3. Main result

Recall a known result (Theorem 1 of [6]).

Lemma 2. *Let G be a group and let π and ρ be one-dimensional pseudorepresentations of G. If $|\pi(g) - \rho(g)| \leq q < \sqrt{3}$ for all $g \in G$, then $\pi = \rho$.*

Theorem 1. *A one-dimensional pseudorepresentation π with a defect $\varepsilon < q_0 = \sqrt{3}/5$ of an almost connected locally compact group G with the connected component G_0 is equal to an exponential of a pseudocharacter on G coinciding* *with π on G_0 and on a Lee's supplementary subgroup D of the group G if and only if π is trivial on D.*

Proof. A pseudocharacter on an almost connected locally compact group vanishes on any Lee's supplementary subgroup D since D is compact (see [1]–[3]). This proves the "only if" part.

Let a one-dimensional pseudorepresentation π with a defect $\varepsilon < q_0$ of an almost connected locally compact group G be trivial on a Lee's supplementary subgroup of G. The restriction of π to the connected component G_0 is a one-dimensional pseudorepresentation of a connected locally compact group G_0, and hence this restriction is an imaginary exponential of a real pseudocharacter φ on G_0 [4] whose defect is less than $\log(1+\varepsilon)$. Hence $\pi(g) = 1 \in \mathbb{C}$ for all $g \in G_0 \cap D$, which agrees with the assumption that $\pi(g) = 1 \in \mathbb{C}$ for all $g \in D$.

Consider a section $s\colon D/G_0 \cap D \to D$ taking e_{D/G_0} to e_D and such that the canonical homomorphism ρ of D onto $D/G_0 \cap D$ takes $s(x)$ to x for all $x \in D/G_0 \cap D$. Define a mapping $f\colon G \to \mathbb{C}$ by the formula $f(g_0 d) = \varphi(g_0')$ for all $g_0 \in G_0$ and $d \in D$, where g_0' is defined by the condition that $g_0' s(\rho(d)) = g_0 d$, i.e., $g_0^{-1} g_0' = ds(\rho(d))^{-1}$, which means that $ds(\rho(d))^{-1} \in G_0 \cap D$. If $g_0 d = g_0^{(1)} d_1$, then $dd_1^{-1} = g_0^{-1} g_0^{(1)}$, and hence d and d_1 belong to the same coset by G_0, and therefore $\rho(d) = \rho(d_1)$ and $s(\rho(d)) = s(\rho(d_1))$ and $g_0' = {g'}_0^{(1)}$, and thus the mapping f is well defined. Then $f(g_0^{(1)} d_1 g_0^{(2)} d_2) = f(g_0^{(1)} d_1 g_0^{(2)} d_1^{-1} d_1 d_2) = \varphi((g_0^{(1)} d_1 g_0^{(2)} d_1^{-1})') = \varphi(g_0^{(1)} d_1 g_0^{(2)} d_1^{-1} d_0)$ for some $d \in D$. Therefore, $f(g_0^{(1)} d_1 g_0^{(2)} d_1^{-1})$ differs from $f(g_0^{(1)}) + f(d_1 g_0^{(2)} d_1^{-1}) = f(g_0^{(1)}) + f(g_0^{(2)})$ by the defect of the pseudocharacter φ (since every pseudocharacter is invariant with respect to the inner automorphisms). Thus, f is a quasicharacter on G whose restriction to G_0 obviously coincides with φ. Then the pseudocharacter σ on G related to f (see [1]–[3]) coincides with φ on G_0, vanishes on D (since D is finite), and has the defect not exceeding $4\log(1+\varepsilon)$. Moreover, it is unique with these properties [7]. The appropriate imaginary exponential of σ, which is a unitary one-dimensional pseudorepresentation of G, coincides with π on G_0 and on D, and the defect of σ does not exceed $4\log(1+\varepsilon)$ [4].

Since $|\pi(g_0 d) - \pi(g_0)| le \varepsilon$, $|\exp(i\sigma(g_0 d)) - \exp(i\sigma(g_0))| = |\exp(i(\sigma(g_0 d) - \sigma(g_0)) - 1| \le |\sigma(g_0 d) - \sigma(g_0)| \le 4\log(1+\varepsilon) \le 4\varepsilon$, and $\pi(g_0) = \exp(i\sigma(g_0))$, it follows that $|\pi(g_0 d) - \exp(i\sigma(g_0 d))| \le 5\varepsilon < \sqrt{3}$. By [6], $\pi(g_0 d) = \exp(i\sigma(g_0 d)$ for all $g = g_0 d \in G$, as was to be proved.

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