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The Vertex Degree Polynomial of Some Graph Operations

[1] Hanan Ahmed, [2] Anwar Alwardi, [3] Ruby Salestina M. and [4] Ismail Naci Cangul
[1,3] Department of Mathematics,
Yuvaraja's college, University of Mysore, Mysuru, India
e-mail:hananahmed1a@gmail.com & ruby.salestina@gmail.com
[2] Department of Mathematics, Faculty of Science, University of Jeddah, Jeddah, Saudi Arabia
e-mail:asaleh1@uj.edu.sa
[4] Bursa Uludag University, Department of Mathematics,
Gorukle 16059 Bursa, Turkey
e-mail:cangul@uludag.edu.tr

Abstract

Graph polynomials have been developed for measuring structural information of networks using combinatorial graph invariants and for characterizing graphs. Various problems in graph theory and discrete mathematics can be treated and solved in a rather efficient manner by making use of polynomials. Various graph polynomials have been proven useful in discrete mathematics, engineering, information sciences, mathematical chemistry, and related disciplines. The vertex degree polynomial of a graph G is defined as $VD(G,x) = \sum_{uv\in E(G)} d(u)x^{d(v)}$. Graph operations are important tools for constructing new graphs, and they play key roles in the design and analysis of networks. In this study, we give exact values of the vertex degree polynomials of graph operations, including the cartesian product, join, corona product and composition of graphs.

Keywords: Vertex degree polynomial, cartesian product, join, corona product, composition of graphs.
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1 Introduction

In this paper, we considered only finite, undirected, connected graphs without loops and multiple edges. Let $G = (V, E)$ be such a graph with vertex set $V(G)$ and edges set $E(G)$. As usual, we denote by $n = |V|$ and $m = |E|$ to the number of vertices and edges in G, respectively. A vertex u is called a neighbor of v in G if uv is an edge of G. The set $N(v)$ of all neighbors of v is called the open neighborhood of v. Thus $N(v) = \{u \in V : uv \in E\}$. The closed neighborhood of v in G is defined as $N[v] = N(v) \cup \{v\}$. The degree of a vertex v in G is defined to be the number of edges incident to v and is denoted by $d_G(v)$ or $d(v)$. In other words $d(v) = |N(v)|$. The minimum and maximum degrees of vertices of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The distance $d_G(u, v)$ or $d(u, v)$ between two vertices in a graph G is the length of the shortest path joining them, otherwise $d(u, v) = \infty$ [14]. Any undefined term may be found in Chartrand [11], Harary [14] or in Bondy [10].

Numerous graph polynomials were introduced in the literature, several of them also turned out to be applicable in mathematical chemistry. And as we know that some very important and famous topological indices which have a lot of applications are based in the degrees of the vertices. The importance of polynomials to represent a graph in an algebraic form and the application of the topological indices which are based in the degrees of the vertices motivated us to define a new graph polynomial based in the degrees of the vertices which is called as the vertex degree polynomial, [2], defined as $VD(G, x) = \sum_{uv \in E(G)} d(u)x^{d(v)}$. The derivative of the vertex degree polynomial at $x = 1$ will give double of the second Zagreb index for the graph. We foresee that a lot of graph theory problems can be solved by using this polynomial. Also, we can study this graph polynomial in chemical graph theory by studying the molecular graphs and getting the correlation with the physicochemical and biological properties.

The roots of the vertex degree polynomial are called vertex degree roots. The set of vertex degree roots of the graph G is denoted by $Z_{VD}(G)$. The reader is encouraged to refer to the paper [1] for mathematical properties of vertex degree polynomial. For more information about graph operations, see [7, 5, 8, 15].

In this paper, the explicit formulae for the vertex degree polynomials of some graph operations containing the join, cartesian product, corona product and composition of graphs will be presented.

2 Main results

In this section, we introduce the graph operations used for producing the composite graphs that are relevant for our purposes and review their basic properties. Similarly to these results, the multiplicative Zagreb indices were calculated in [12] for some graph operations. Another work is done for lexicographic product in [4]. Also for tensor products, similar results were obtained in [3]. In [9], the Zagreb polynomials are calculated. In [17], some bounds for the sum of cubes of vertex degrees of splice graphs. We consider four operations. Each of them is treated in a separate subsection.

Lemma 2.1. *[16] Let G_1, G_2 be graphs. Then we have*
(a) $|V(G_1 \times G_2)| = |V(G_1[G_2])| = |V(G_1)||V(G_2)|$
$|E(G_1 \times G_2)| = |E(G_1)||V(G_2)| + |V(G_1)||E(G_2)|$
$|E(G_1 + G_2)| = |E(G_1)| + |E(G_2)| + |V(G_1)||V(G_2)|$
$|E(G_1[G_2])| = |E(G_1)||V(G_2)|^2 + |E(G_2)||V(G_1)|$

(b) $deg_{G_1 \times G_2}((a, b)) = deg_{G_1}(a) + deg_{G_2}(b)$

(c) $deg_{G_1[G_2]}((a, b)) = |V(G_2)|deg_{G_1}(a) + deg_{G_2}(b)$

(d) $deg_{G_1+G_2}(a) = \begin{cases} deg_{G_1}(a) + |V(G_2)|, & a \in V(G_1); \\ deg_{G_2}(a) + |V(G_1)|, & a \in V(G_2). \end{cases}$

2.1 Join

The join $G_1 + G_2$ of two graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 is the graph on the vertex set $V_1 \cup V_2$ and the edge set $E_1 \cup E_2 \cup \{u_1u_2 : u_1 \in V_1, u_2 \in V_2\}$. Hence, the sum of two graphs is obtained by connecting each vertex of one graph to each vertex of the other graph, while keeping all edges of both graphs.

Theorem 2.2. *Let $G_1 = (n_1, m_1)$ and $G_2 = (n_2, m_2)$ be two graphs. Then*

$$\begin{aligned}VD(G_1+G_2,x) = {} & VD(G_1,x)x^{n_2} + VD(G_2,x)x^{n_1} \\ & + n_2x^{n_2}\sum_{uv\in E(G_1)}(x^{d_{G_1}(u)}+x^{d_{G_1}(v)}) \\ & + n_1x^{n_1}\sum_{uv\in E(G_2)}(x^{d_{G_2}(u)}+x^{d_{G_2}(v)}) \\ & + [2m_1+n_1n_2]x^{n_1}(\sum_{v\in V(G_2)}x^{d_{G_2}(v)}) \\ & + [2m_2+n_1n_2]x^{n_2}(\sum_{u\in V(G_1)}x^{d_{G_1}(u)}).\end{aligned}$$

Proof. By the definition, we have

$$\begin{aligned}VD(G_1+G_2,x) &= \sum_{uv\in E(G_1+G_2)} d_{G_1+G_2}(u)x^{d_{G_1+G_2}(v)} \\ &= \sum_{uv\in E(G_1)} d_{G_1+G_2}(u)x^{d_{G_1+G_2}(v)} + \sum_{uv\in E(G_2)} d_{G_1+G_2}(u)x^{d_{G_1+G_2}(v)} \\ &+ \sum_{u\in V(G_1)v\in V(G_2)} d_{G_1+G_2}(u)x^{d_{G_1+G_2}(v)}.\end{aligned}$$

Now

$$\begin{aligned}\sum_{uv\in E(G_1)} d_{G_1+G_2}(u)x^{d_{G_1+G_2}(v)} &= \sum_{uv\in E(G_1)}(d_{G_1}(u)+n_2)x^{d_{G_1}(v)+n_2} \\ &= \sum_{uv\in E(G_1)} d_{G_1}(u)x^{d_{G_1}(v)}.x^{n_2} \\ &+ n_2x^{n_2}\sum_{uv\in E(G_1)}(x^{d_{G_1}(u)}+x^{d_{G_1}(v)}) \\ &= x^{n_2}VD(G_1,x)+n_2x^{n_2}\sum_{uv\in E(G_1)}(x^{d_{G_1}(u)}+x^{d_{G_1}(v)}) \qquad (1)\end{aligned}$$

and

$$\begin{aligned}\sum_{uv\in E(G_2)} d_{G_1+G_2}(u)x^{d_{G_1+G_2}(v)} &= \sum_{uv\in E(G_2)}(d_{G_2}(u)+n_1)x^{d_{G_2}(v)+n_1} \\ &= \sum_{uv\in E(G_2)} d_{G_2}(u)x^{d_{G_2}(v)}.x^{n_1} \\ &+ n_1x^{n_1}\sum_{uv\in E(G_2)}(x^{d_{G_2}(u)}+x^{d_{G_2}(v)}) \\ &= x^{n_1}VD(G_2,x)+n_1x^{n_1}\sum_{uv\in E(G_2)}(x^{d_{G_2}(u)}+x^{d_{G_2}(v)}) \qquad (2)\end{aligned}$$

and

$$\sum_{u\in V(G_1)v\in V(G_2)} d_{G_1+G_2}(u)x^{d_{G_1+G_2}(v)} = (d(u_1)+n_2)x^{d(v_1)+n_1} + (d(v_1)+n_1)x^{d(u_1)+n_2}$$
$$+ (d(u_1)+n_2)x^{d(v_2)+n_1} + (d(v_2)+n_1)x^{d(u_1)+n_2}$$
$$+ ... + (d(u_1)+n_2)x^{d(v_{n_2})+n_1} + (d(v_{n_2})+n_1)x^{d(u_1)+n_2}$$
$$+ ... + (d(u_{n_1})+n_2)x^{d(v_1)+n_1} + (d(v_1)+n_1)x^{d(u_{n_1})+n_2}$$
$$+ (d(u_{n_1})+n_2)x^{d(v_2)+n_1} + (d(v_2)+n_1)x^{d(u_{n_1})+n_2}$$
$$+ ... + (d(u_{n_1})+n_2)x^{d(v_{n_2})+n_1} + (d(v_{n_2})+n_1)x^{d(u_{n_1})+n_2}$$
$$= [(d(u_1)+n_2) + ... + (d(u_{n_1})+n_2)][x^{d(v_1)+n_1} + ... + x^{d(v_{n_2})+n_1}]$$
$$+ [(d(v_1)+n_1) + ... + (d(v_{n_2})+n_1)][x^{d(u_1)+n_2} + ... + x_{x^{d(u_{n_1})+n_1}}]$$
$$= x^{n_1}[2m_1 + n_1n_1] \sum_{v\in V(G_2)} x^{d_{G_2}(v)}$$
$$+ x^{n_2}[2m_2 + n_1n_1] \sum_{u\in V(G_1)} x^{d_{G_1}(u)}. \qquad (3)$$

Hence,

$$VD(G_1+G_2,x) = VD(G_1,x)x^{n_2} + VD(G_2,x)x^{n_1}$$
$$+ n_2x^{n_2} \sum_{uv\in E(G_1)} (x^{d_{G_1}(u)} + x^{d_{G_1}(v)})$$
$$+ n_1x^{n_1} \sum_{uv\in E(G_2)} (x^{d_{G_2}(u)} + x^{d_{G_2}(v)})$$
$$+ [2m_1 + n_1n_2]x^{n_1}(\sum_{v\in V(G_2)} x^{d_{G_2}(v)})$$
$$+ [2m_2 + n_1n_2]x^{n_2}(\sum_{u\in V(G_1)} x^{d_{G_1}(u)}).$$

□

2.2 Cartesian product

For given graphs G_1 and G_2, their cartesian product $G_1 \times G_2$ is defined as the graph on the vertex set $V(G_1)\times V(G_2)$ with vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ connected by an edge if and only if either ($[u_1 = v_1$ and $\{u_2, v_2\} \in E(G_2)]$) or ($[u_2 = v_2$ and $\{u_1, v_1\} \in E(G_1)]$), [13]. The distance between two vertices in $G_1 \times G_2$ is given by $d_{G_1\times G_2}(u, v) = d_{G_1}(u_1, v_1) + d_{G_2}(u_2, v_2)$. For the proof, see [18].

Theorem 2.3. *The vertex degree polynomial of the cartesian product of* $G_1 = (n_1, m_1)$ *and* $G_2 = (n_2, m_2)$ *is given by*

$$\begin{aligned}
VD(G_1 \times G_2, x) &= VD(G_1, x) \sum_{v \in G_2} x^{d_{G_2}(v)} + VD(G_2, x) \sum_{u \in G_1} x^{d_{G_1}(u)} \\
&+ \sum_{u \in V(G_1)} d_{G_1}(u) x^{d_{G_1}(u)} \sum_{v_i v_j \in E(G_2)} (x^{d_{G_2}(v_i)} + x^{d_{G_2}(v_j)}) \\
&+ \sum_{v \in V(G_2)} d_{G_2}(v) x^{d_{G_2}(v)} \sum_{u_i u_j \in E(G_1)} (x^{d_{G_1}(u_i)} + x^{d_{G_1}(u_j)}).
\end{aligned}$$

Proof.

$$\begin{aligned}
VD(G_1 \times G_2, x) &= \sum_{(u_i,v_i)(u_j,v_j) \in E(G_1 \times G_2)} d(u_i, v_i) x^{d(u_j,v_j)} \\
&= \sum_{(v_i,v_j) \in E(G_2), u_i = u_j} (d_{G_1}(u_i) + d_{G_2}(v_i)) x^{d_{G_1}(u_j) + d_{G_2}(v_j)} \\
&+ \sum_{u_i u_j \in E(G_1), v_i = v_j} (d_{G_1}(u_i) + d_{G_2}(v_i)) x^{d_{G_1}(u_j) + d_{G_2}(v_j)} \\
&= VD(G_1, x) \sum_{v \in G_2} x^{d_{G_2}(v)} + VD(G_2, x) \sum_{u \in G_1} x^{d_{G_1}(u)} \\
&+ \sum_{u \in V(G_1)} d_{G_1}(u) x^{d_{G_1}(u)} \sum_{v_i v_j \in E(G_2)} (x^{d_{G_2}(v_i)} + x^{d_{G_2}(v_j)}) \\
&+ \sum_{v \in V(G_2)} d_{G_2}(v) x^{d_{G_2}(v)} \sum_{u_i u_j \in E(G_1)} (x^{d_{G_1}(u_i)} + x^{d_{G_1}(u_j)}).
\end{aligned}$$

□

2.3 Corona product

The corona product $G_1 \circ G_2$ is defined as a graph obtained from G_1 and G_2 by taking one copy of G_1 and $|V(G_1)|$ copies of G_2, and then joining each vertex of the i^{th} copy of G_2 named (G_2, i), with the i^{th} vertex of G_1 by an edge [**?**].

If u is a vertex of $G_1 \circ G_2$, then

$$d_{G_1 \circ G_2}(u) = \begin{cases} d_{G_1}(u) + |V(G_2)|, & u \in V(G_1); \\ d_{G_2}(u) + 1, & u \in V(G_2, i). \end{cases}$$

Theorem 2.4.

Let $G_1 = (n_1, m_1)$, $G_2(n_2, m_2)$. *Then*

$$VD(G_1 \circ G_2, x) = x^{n_2} VD(G_1, x) + n_2 x^{n_2} \sum_{uv \in E(G_1)} (x^{d_{G_1}(u)} + x^{d_{G_1}(v)}) + n_1[xVD(G_2, x) + x \sum_{uv \in E(G_2)} (x^{d_{G_2}(u)} + x^{d_{G_2}(v)})] + [x(2m_1 + n_1 n_2) \sum_{v \in V(G_2)} x^{d_{G_2}(v)}] + [x^{n_2}(2m_2 + n_2) \sum_{u \in V(G_1)} x^{d_{G_1}(u)}].$$

Proof.

$$\begin{aligned} VD(G_1 \circ G_2, x) &= \sum_{uv \in E(G_1 \circ G_2)} d_{G_1 \circ G_2}(u) x^{d_{G_1 \circ G_2}(v)} \\ &= \sum_{uv \in E(G_1)} d_{G_1 \circ G_2}(u) x^{d_{G_1 \circ G_2}(v)} + \sum_{uv \in E(G_2)} d_{G_1 \circ G_2}(u) x^{d_{G_1 \circ G_2}(v)} \\ &+ \sum_{u \in V(G_1), v \in V(G_2)} d_{G_1 \circ G_2}(u) x^{d_{G_1 \circ G_2}(v)} \end{aligned}$$

Now,

$$\begin{aligned} \sum_{uv \in E(G_1)} d_{G_1 \circ G_2}(u) x^{d_{G_1 \circ G_2}(v)} &= \sum_{uv \in E(G_1)} (d_{G_1}(u) + n_2) x^{d_{G_1}(v) + n_2} \\ &= x^{n_2} VD(G_1, x) + n_2 x^{n_2} \sum_{uv \in E(G_1)} (x^{d_{G_1}(u)} + x^{d_{G_1}(v)}) \qquad (1) \end{aligned}$$

and,

$$\begin{aligned} \sum_{uv \in E(G_2)} d_{G_1 \circ G_2}(u) x^{d_{G_1 \circ G_2}(v)} &= n_1[\sum_{uv \in E(G_2)} (d_{G_2}(u) + 1) x^{d_{G_2}(v) + 1}] \\ &= n_1[xVD(G_2, x) + x \sum_{uv \in E(G_2)} (x^{d_{G_2}(u)} + x^{d_{G_2}(v)})] \qquad (2) \end{aligned}$$

and

$$\begin{aligned}\sum_{u\in V(G_1),v\in V(G_2)} d_{G_1\circ G_2}(u)x^{d_{G_1\circ G_2}(v)} &= (d_{G_1}(u_1)+n_2)x^{d_{G_2}(v_1)+1} + (d_{G_2}(v_1)+1)x^{d_{G_1}(u_1)+n_2}\\ &+ ... + (d_{G_1}(u_1)+n_2)x^{d_{G_2}(v_{n_2})+1} + (d_{G_2}(v_{n_2})+1)x^{d_{G_1}(u_1)+n_2}\\ &+ ... + (d_{G_1}(u_{n_1})+n_2)x^{d_{G_2}(v_1)+1} + (d_{G_2}(v_1)+1)x^{d_{G_1}(u_{n_1})+n_2}\\ &+ ... + (d_{G_1}(u_{n_1})+n_2)x^{d_{G_2}(v_{n_2})+1} + (d_{G_2}(v_{n_2})+1)x^{d_{G_1}(u_{n_1})+n_2}\\ &= [(d_{G_1}(u_1)+n_2)+...+(d_{G_1}(u_{n_1})+n_2)][x^{d_{G_2}(v_1)+1}+...+x^{d_{G_2}(v_{n_2})+1}]\\ &+ [(d_{G_2}(v_1)+1)+...+(d_{G_2}(v_{n_2})+1)][x^{d_{G_1}(u_1)+n_2}+...+x^{d_{G_1}(u_{n_1})+n_2}]\\ &= x\sum_{u\in V(G_1)}(d_{G_1}(u)+n_2)\sum_{v\in V(G_2)}x^{d_{G_2}(v)}\\ &+ x^{n_2}\sum_{v\in V(G_2)}(d_{G_2}(v)+1)\sum_{u\in V(G_1)}x^{d_{G_1}(u)}\\ &= [x(2m_1+n_1n_2)\sum_{v\in V(G_2)}x^{d_{G_2}(v)}] + [x^{n_2}(2m_2+n_2)\sum_{u\in V(G_1)}x^{d_{G_1}(u)}].\end{aligned}$$

Hence

$$\begin{aligned}VD(G_1\circ G_2,x) &= x^{n_2}VD(G_1,x) + n_2x^{n_2}\sum_{uv\in E(G_1)}(x^{d_{G_1}(u)}+x^{d_{G_1}(v)})\\ &+ n_1[xVD(G_2,x) + x\sum_{uv\in E(G_2)}(x^{d_{G_2}(u)}+x^{d_{G_2}(v)})]\\ &+ [x(2m_1+n_1n_2)\sum_{v\in V(G_2)}x^{d_{G_2}(v)}] + [x^{n_2}(2m_2+n_2)\sum_{u\in V(G_1)}x^{d_{G_1}(u)}].\end{aligned}$$

□

Corollary 2.5. *Let* $G = C_noK_1$. *Then* $VD(G,x) = (6n_1+n_2)x^3 + 3n_2x$.

2.4 Composition

The co-vertex degree polynomial of a graph $G = (V,E)$ is define as follws:

$$\overline{VD}(G,x) = \sum_{uv\notin E(G)} d(u)x^{d(v)}.$$

The composition $G_1[G_2]$ of graphs G_1 and G_2 with disjoint vertex sets and edge sets is again a graph on vertex set $V(G_1)\times V(G_2)$ in which $u = (u_1,u_2)$ is adjacent with $v = (v_1,v_2)$ whenever u_1 *is adjacent with* v_1 or, $u_1 = v_1$ *and* u_2 *is adjacent with* v_2. The composition is not commutative. The easiest way to visualize the composition $G_1[G_2]$ is [6] to expand each vertex of G_1 into a copy of G_2, with each edge of G_1 replaced by the set of all possible edges between the corresponding copies of G_2.

Theorem 2.6. *Let $G_1 = (n_1, m_1)$, $G_2 = (n_2, m_2)$ be two graphs. Then the vertex degree polynomial of the composition G_1 and G_2 is given by*

$$\begin{aligned} VD(G_1[G_2], x) &= n_2 \sum_{u_iu_j\in E(G_1)} d_{G_1}(u_i)x^{n_2d_{G_1}(u_j)}\big[\sum_{v_iv_j\in E(G_2)} (x^{d_{G_2}(v_i)} + x^{d_{G_2}(v_j)}) \\ &+ \sum_{v\in V(G_2)} x^{d_{G_2}(v)} + \sum_{v_iv_j\notin E(G_2)} (x^{d_{G_2}(v_i)} + x^{d_{G_2}(v_j)})\big] \\ &+ \sum_{u_iu_j\in E(G_1)} (x^{n_2d_{G_1}(u_i)} + x^{n_2d_{G_1}(u_j)})\big[VD(G_2, x) + \overline{VD}(G_2, x) + \sum_{v\in V(G_2)} d_{G_2}(v)x^{d_{G_2}(v)}\big] \\ &+ [n_2 \sum_{u\in V(G_1)} d_{G_1}(u)x^{n_2d_{G_1}(u)} \sum_{v_iv_j\in E(G_2)} (x^{d_{G_2}(v_i)} + x^{d_{G_2}(v_j)})] + VD(G_2, x) \sum_{u\in V(G_1)} x^{n_2d_{G_1}(u)} \end{aligned}$$

Proof. Let $V(G_1) = \{u_1, u_2, ..., u_{n_1}\}$, and $V(G_2) = \{v_1, v_2, ..., v_{n_2}\}$ We have

$$\begin{aligned} VD(G_1[G_2], x) = &\sum_{u_iu_j\in E(G_1), v_iv_j\in E(G_2)} d(u_i, v_i)x^{d(u_j,v_j)} + \sum_{u_iu_j\in E(G_1), v_i=v_j} d(u_i, v_i)x^{d(u_j,v_j)} \\ &+ \sum_{u_iu_j\in E(G_1), v_iv_j\notin E(G_2)} d(u_i, v_i)x^{d(u_j,v_j)} + \sum_{u_i=u_j, v_iv_j\in E(G_2)} d(u_i, v_i)x^{d(u_j,v_j)} \end{aligned}$$

Now

$$\begin{aligned} \sum_{u_iu_j\in E(G_1), v_iv_j\in E(G_2)} d(u_i, v_i)x^{d(u_j,v_j)} &= \sum_{u_iu_j\in E(G_1), v_iv_j\in E(G_2)} (|V(G_2)|d_{G_1}(u_i) + d_{G_2}(v_i))x^{|V(G_2)|d_{G_1}(u_j)+d_{G_2}(v_j)} \\ &= n_2 \sum_{u_iu_j\in E(G_1)} d_{G_1}(u_i)x^{n_2d_{G_1}(u_j)} \sum_{v_iv_j\in E(G_2)} (x^{d_{G_2}(v_i)} + x^{d_{G_2}(v_j)}) \\ &+ \sum_{v_iv_j\in E(G_2)} d_{G_2}(v_i)x^{d_{G_2}(v_j)} \sum_{u_iu_j\in E(G_1)} (x^{n_2d_{G_1}(u_i)} + x^{n_2d_{G_1}(u_j)}) \\ &= n_2 \sum_{u_iu_j\in E(G_1)} d_{G_1}(u_i)x^{n_2d_{G_1}(u_j)} \sum_{v_iv_j\in E(G_2)} (x^{d_{G_2}(v_i)} + x^{d_{G_2}(v_j)}) \\ &+ VD(G_2, x) \sum_{u_iu_j\in E(G_1)} (x^{n_2d_{G_1}(u_i)} + x^{n_2d_{G_1}(u_j)}) \end{aligned}$$

$$\begin{aligned} \sum_{u_iu_j\in E(G_1), v_i=v_j} d(u_i, v_i)x^{d(u_j,v_j)} &= \sum_{u_iu_j\in E(G_1), v_i=v_j} (|V(G_2)|d_{G_1}(u_i) + d_{G_2}(v_i))x^{|V(G_2)|d_{G_1}(u_j)+d_{G_2}(v_j)} \\ &= n_2 \sum_{u_iu_j\in E(G_1)} d_{G_1}(u_i)x^{n_2d_{G_1}(u_j)} \sum_{v\in V(G_2)} x^{d_{G_2}(v)} \\ &+ \sum_{v\in V(G_2)} d_{G_2}(v)x^{d_{G_2}(v)} \sum_{u_iu_j\in E(G_1)} (x^{n_2d_{G_1}(u_i)} + x^{n_2d_{G_1}(u_j)}) \end{aligned}$$

$$\begin{aligned}\sum_{u_iu_j\in E(G_1),v_iv_j\notin E(G_2)} d(u_i,v_i)x^{d(u_j,v_j)} &= \sum_{u_iu_j\in E(G_1),v_iv_j\notin E(G_2)} (|V(G_2)|d_{G_1}(u_i)+d_{G_2}(v_i))x^{|V(G_2)|d_{G_1}(u_j)+d_{G_2}(v_j)}\\ &= n_2\sum_{u_iu_j\in E(G_1)} d_{G_1}(u_i)x^{n_2d_{G_1}(u_j)}\sum_{v_iv_j\notin E(G_2)}(x^{d_{G_2}(v_i)}+x^{d_{G_2}(v_j)})\\ &+\sum_{v_iv_j\notin E(G_2)} d_{G_2}(v_i)x^{d_{G_2}(v_i)}\sum_{u_iu_j\in E(G_1)}(x^{n_2d_{G_1}(u_i)}+x^{n_2d_{G_1}(u_j)})\\ &= n_2\sum_{u_iu_j\in E(G_1)} d_{G_1}(u_i)x^{n_2d_{G_1}(u_j)}\sum_{v_iv_j\notin E(G_2)}(x^{d_{G_2}(v_i)}+x^{d_{G_2}(v_j)})\\ &+\overline{VD}(G_2,x)\sum_{u_iu_j\in E(G_1)}(x^{n_2d_{G_1}(u_i)}+x^{n_2d_{G_1}(u_j)})\end{aligned}$$

$$\begin{aligned}\sum_{u_i=u_j,v_iv_j\in E(G_2)} d(u_i,v_i)x^{d(u_j,v_j)} &= \sum_{u_i=u_j,v_iv_j\in E(G_2)} (|V(G_2)|d_{G_1}(u_i)+d_{G_2}(v_i))x^{|V(G_2)|d_{G_1}(u_j)+d_{G_2}(v_j)}\\ &= n_2\sum_{u\in V(G_1)} d_{G_1}(u)x^{n_2d_{G_1}(u)}\sum_{v_iv_j\in E(G_2)}(x^{d_{G_2}(v_i)}+x^{d_{G_2}(v_j)})\\ &+\sum_{v_iv_j\in E(G_2)} d_{G_2}(v_i)x^{d_{G_2}(v_j)}\sum_{u\in V(G_1)}x^{n_2d_{G_1}(u)}\\ &= n_2\sum_{u\in V(G_1)} d_{G_1}(u)x^{n_2d_{G_1}(u)}\sum_{v_iv_j\in E(G_2)}(x^{d_{G_2}(v_i)}+x^{d_{G_2}(v_j)})\\ &+VD(G_2,x)\sum_{u\in V(G_1)}x^{n_2d_{G_1}(u)}\end{aligned}$$

Hence

$$\begin{aligned}VD(G_1[G_2],x) &= n_2\sum_{u_iu_j\in E(G_1)} d_{G_1}(u_i)x^{n_2d_{G_1}(u_j)}\Big[\sum_{v_iv_j\in E(G_2)}(x^{d_{G_2}(v_i)}+x^{d_{G_2}(v_j)})\\ &+\sum_{v\in V(G_2)}x^{d_{G_2}(v)}+\sum_{v_iv_j\notin E(G_2)}(x^{d_{G_2}(v_i)}+x^{d_{G_2}(v_j)})\Big]\\ &+\sum_{u_iu_j\in E(G_1)}(x^{n_2d_{G_1}(u_i)}+x^{n_2d_{G_1}(u_j)})\Big[VD(G_2,x)+\overline{VD}(G_2,x)+\sum_{v\in V(G_2)}d_{G_2}(v)x^{d_{G_2}(v)}\Big]\\ &+[n_2\sum_{u\in V(G_1)} d_{G_1}(u)x^{n_2d_{G_1}(u)}\sum_{v_iv_j\in E(G_2)}(x^{d_{G_2}(v_i)}+x^{d_{G_2}(v_j)})]+VD(G_2,x)\sum_{u\in V(G_1)}x^{n_2d_{G_1}(u)}\end{aligned}$$

□

3 Applications

In this section, we apply the derived results of the vertex degree polynomial of some graph operations on some families of graphs of chemical and general interest.

Corollary 3.1. *Suppose G be a grid graph $(P_n \times P_m)$ such that $n, m \geq 3$. Then the vertex degree polynomial of G is* $VD(G, x) = 24x^2 + [20m + 20n - 88]x^3 + [16nm - 34m - 34n + 72]x^4$.

Corollary 3.2. *If $G \cong (P_n \times P_2)$ is a ladder graph with $n \geq 3$, then* $VD(G, x) = (18n - 40)x^3 + 20x^2$.

Corollary 3.3. *For $G \cong C_4-$nanotorus $TC_4(n, m) = C_n \times C_m$, the vertex degree polynomial of G is* $VD(G, x) = 16nmx^4$.

Corollary 3.4. *For $G \cong C_4-$nanotube $TUC_4(n, m) = P_n \times P_m$ with $n, m \geq 3$, the vertex degree polynomial of G is given by* $VD(G, x) = (16m + 4n)x^3[12nm - 26m + 4n2 - 8n]x^4$.

Corollary 3.5. *Let G be the n-prism $K_2 \times C_n$, $n \geq 3$. Then* $VD(G, x) = 18nx^3$.

Corollary 3.6. *The vertex degree polynomials of the fence graph $P_n[P_2]$ and closed fence graph $C_n[P_2]$ are defined as follows:*

$$DV(P_n[P_2], x) = (50n - 116)x^5 + 52x^3,$$

$$VD(C_n[P_2], x) = 50nx^5.$$

References

[1] H. Ahmed, A. Alwardi, M. R. Salestina, *An atlas of vertex degree polynomials of graphs of order at most six*, Research and Reviews: Discrete Mathematical Structures, 7, 7-14 (2020)

[2] H. Ahmed, A. Alwardi, M. R. Salestina, *On The Vertex Degree Polynomial of Graphs*, preprint

[3] N. Akgunes, K. Ch. Das, A. S. Cevik, Some properties on the tensor product of graphs obtained by monogenic semigroups, Applied Mathematics and Computation, 235, 352-357 (2014).

[4] N. Akgunes, K. Ch. Das, A. S. Cevik, I. N. Cangul, Some properties on the lexicographic product of graphs obtained by monogenic semigroups, Journal of Inequalities and Applications 2013, 2013:238.

[5] M. Ascioglu, M. Demirci, I. N. Cangul, *Omega invariant of union, join and corona product of two graphs*, Advanced Studies in Contemporary Mathematics, 30 (3), 297-306 (2020)

[6] A. R. Ashrafi, T. Doslic, A. Hamzeh, *The Zagreb coindices of graph operations*, Discrete Applied Mathematics, 158, 1571-1578 (2010)

[7] M. Azari, *Eccentric connectivity coindex under graph operations*, J. Appl. Math. Comput. 62, 23–35 (2020)

[8] B. Basavanagoud, V. R. Desai, Keerthi G. Mirajkar, B. Pooja, I. N. Cangul, *Four new tensor products of graphs and their Zagreb indices and coindices*, Electronic Journal of Mathematical Analysis and Applications, 8 (1), 209-219 (2020)

[9] A. R. Bindusree, I. N. Cangul, V. Lokesha, A. S. Cevik, Zagreb polynomials of three graph operators, Filomat, 30 (7), 1979-1986, (2016).

[10] J. A. Bondy, U. S. R. Murthy, *Graph Theory*, Springer, Berlin, (2008)

[11] G. Chartrand, L. Lesniak, *Graph and Digraph*, Chap. Hall/CRC, (1996)

[12] K. Ch. Das, A. Yurttas, M. Togan, A. S. Cevik, I. N. Cangul, The multiplicative Zagreb indices of graph operations, Journal of Inequalities and Applications, 2013, 2013:90.

[13] T. Doslic, M. Ghorbani, M. A. Hosseinzadeh, *Eccentric connectivity polynomial of some graph operations*, Serdica J. Computing, 5, 101-116 (2011)

[14] F. Harary, *Graph Theory*, Narosa Publishing House, New Delhi, (2001)

[15] S. M. Hosamani, S. H. Malghan, I. N. Cangul, *The first geometric-arithmetic indices of graph operations*, Advances and Applications in Mathematical Sciences, 6, 155-163 (2015)

[16] M. H. Khalifeh, H. Yousefi-Azari, A. R. Ashrafi, *The first and second Zagreb indices of some graph operations*, Discrete Applied Mathematics 157, 804-811 (2009)

[17] V. Lokesha, S. Jain, M. Muddalapuram, A. S. Cevik, I. N. Cangul, Bounds for the sum of cubes of vertex degrees of splice graphs, TWMS J. App. Eng. Math. 10 (3), 741-750, (2020).

[18] Y. N. Yeh, I. Gutman, *On the sum of all distances in composite graphs*, Discrete Math. 135, 359–365 (1994)