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A NOTE ON TRANSVERSAL HYPERSURFACES OF PARA-KENMOTSU MANIFOLDS

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Abstract. The purpose of this paper is to study transversal hypersurfaces of para-Kenmotsu manifolds. First, it is proved that every transversal hypersurface of an almost paracontact manifold admits an almost product pseudo-Riemannian structure (J, G). After that, we show that every transversal hypersurface of an almost paracontact manifold also admits a $(f, g, \mu, \nu, \lambda)$-structure and we derive some results allied with relationship between induced almost product pseudo-Riemannian structure (J, G) and induced $(f, g, \mu, \nu, \lambda)$-structure. An example of transversal hypersurface of a para-Kenmotsu manifold admitting $(f, g, \mu, \nu, \lambda)$-structure is also illustrated.

1. Introduction

Para-Kenmotsu manifold known as not only a special case of almost paracontact structure but also an analogous of para-Sasakian manifold and almost product structure. In 1976, the notion of an almost paracontact structure on a differentiable manifold was introduced by I. Sato [23]. After that in 1995, B. B. Sinha and K. L. Sai Prasad [24] defined a class of almost paracontact metric manifolds known as para-Kenmotsu (p-Kenmotsu) and special para Kenmotsu manifolds. Further, such structures were studied by several authors such as [2, 3, 17, 25]. An almost contact manifold is always odd dimensional but an almost paracontact manifold could be even dimensional as well.

The study of hypersurface in pseudo-Riemannian manifold is one of the potent aspects of the theory of pseudo-Riemannian geometry. It has ample significance in general theory of relativity, black holes and quantum mechanics [15, 10, 6]. Therefore, several researchers showed their interest in studying the geometry of hypersurface in different ambient spaces [29, 13, 12, 11].

On the other hand, transversal hypersurface of contact Riemannian manifold is a hypersurface such that the characteristic vector field (or Reeb vector field) ξ of a manifold is never tangent to the hyperplane. The concept of transversal hypersurface is introduced by K.Yano in 1972 [28]. After that transversal hypersurfaces were investigated by several authors in different ambient manifolds such as [1, 21, 22] and many others. For further studies we recommend the papers [4, 5, 8, 9, 18, 19, 20, 26].

A systematic study of transversal hypersurfaces of para Kenmotsu manifold has not been undertaken yet, however para Kenmotsu manifolds have many analogies

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and differences with the Kenmotsu manifolds due to the fact that the geometry of hypersurfaces of pseudo-Riemannian manifold behave differently.

In the present paper, we consider an almost paracontact pseudo-metric manifold $\mathcal{M}$ and we obtain that every transversal hypersurface of $\mathcal{M}$ admits an almost product pseudo-Riemannian structure as well as a $(f, g, \mu, \nu, \lambda)$-structure. After that we find some results related with the relationship between induced almost product pseudo-Riemannian structure (J, G) and induced $(f, g, \mu, \nu, \lambda)$-structure. An example of transversal hypersurfaces is also illustrated.

2. **Preliminaries**

Let $\mathcal{M}$ be a smooth manifold with a tensor field ϕ of $(1,1)$-type, a vector field ξ, a 1-form η and a pseudo-Riemannian metric g, then we say that (ϕ, ξ, η, g) is an almost paracontact metric structure on $\mathcal{M}$ if for all $U, V \in T\mathcal{M}$ [30]:

$$\phi^2 = I - \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad \phi(\xi) = 0, \qquad \eta \circ \phi = 0, \tag{2.1}$$

$$g(\phi U, \phi V) = -g(U, V) + \eta(U)\eta(V), \tag{2.2}$$

$$g(\phi U, V) = -g(U, \phi V), \qquad g(U, \xi) = \eta(U), \qquad g(\xi, \xi) = 1. \tag{2.3}$$

Consequently, we call $(\mathcal{M}, \phi, \xi, \eta, g)$ is an almost paracontact metric manifold, where ϕ is a structural endomorphism, ξ is a characteristic vector field and η is a paracontact form.

Definition 2.1. [14, 16] *We say that an almost paracontact metric structure (ϕ, ξ, η, g) is a **para-Kenmotsu** structure if the Levi-Civita connection $\overline{\nabla}$ of g satisfies the following conditions:*

$$(\overline{\nabla}_U \phi)V = g(\phi U, V)\xi - \eta(V)\phi U, \tag{2.4}$$

$$\overline{\nabla}_U \xi = \phi^2 U = U - \eta(U)\xi \tag{2.5}$$

for any $U, V \in T\mathcal{M}$.

Note that the para-Kenmotsu structure was introduced by J. Welyczko in [27] for 3-dimensional normal almost paracontact metric structures. A similar notion called P-Kenmotsu structure appears in the paper of B. B. Sinha and K. L. Sai Prasad [24].

3. **Transversal Hypersurfaces**

Let $(\mathcal{M}, \phi, \xi, \eta, \tilde{g})$ be an almost paracontact pseudo-metric manifold, and let M be an *immersed hypersurface* of $\mathcal{M}$ with induced symmetric tensor field g. In view of casual character of the vector fields of the manifold, we have three types of hypersurfaces, namely, pseudo-Riemannian, Riemannian and null (or lightlike) and the metric g is a non-degenerate or a degenerate metric according as M is pseudo-Riemannian, Riemannian hypersurface and lightlike hypersurface, respectively [7].

The hypersurface M with induced metric g is said to be a *transversal hypersurface* of $\mathcal{M}$ if the characteristic vector field ξ is never tangent to the hyperplane. Here, ξ can be considered as affine normal to M. Now, if ξ and $X \in TM$ are linearly independent, therefore we may express ϕX as:

$$\phi X = JX + \omega(X)\xi, \tag{3.1}$$

where J is a tensor field of type (1, 1) and ω is a 1-form on M.
Now, operating ϕ on (3.1) and with the help of equation (2.1), we have

$$J^2 = I, \tag{3.2}$$

and

$$\omega \circ J = -\eta \tag{3.3}$$

from which it follows that

$$\omega = -\eta \circ J. \tag{3.4}$$

In view of (3.1), we have

Theorem 3.1. *Each transversal hypersurface of an almost paracontact manifold admits an almost product structure J and a 1-form ω.*

Now, we consider that M admits an almost paracontact metric structure (ϕ, ξ, η, g). Then for all $X, Y \in TM$, we obtain the following results

$$\begin{aligned} g(\phi X, \phi Y) &= g(JX + \omega(X)\xi, JY + \omega(Y)\xi) \\ &= g(JX, JY) + \omega(Y)(\eta \circ J)(X) + \omega(X)(\eta \circ J)(Y) + \omega(X)\omega(Y). \end{aligned}$$

In the account of equation (3.4) the foregoing equation turns to

$$g(\phi X, \phi Y) = g(JX, JY) + \omega(Y)(\eta \circ J)(X) + \omega(X)(\eta \circ J)(Y) + \omega(X)\omega(Y)$$

which by using equation (2.2) gives

$$g(JX, JY) = -g(X, Y) + \eta(X)\eta(Y) + \omega(X)\omega(Y). \tag{3.5}$$

Now, we define a new metric G on the tansversal hypersurface given by:

$$G(X, Y) = g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y). \tag{3.6}$$

So with the help of the equations (3.4) and (3.5), we have

$$\begin{aligned} G(JX, JY) &= -g(JX, JY) + \eta(JX)\eta(JY) \\ &= g(X, Y) - \eta(X)\eta(Y) - \omega(X)\omega(Y) + (\eta \circ J)(X)(\eta \circ J)(Y) \\ &= g(X, Y) - \eta(X)\eta(Y) - \omega(X)\omega(Y) + \omega(X)\omega(Y) \\ &= g(X, Y) - \eta(X)\eta(Y) \\ &= -G(X, Y). \end{aligned}$$

Hence, G is pseudo-Riemannian metric on M, i.e., (J, G) is an almost product pseudo-Riemannian structure on transversal hypersurface M of $\mathcal{M}$.

Consequently, we can state the following theorem:

Theorem 3.2. *Each transversal hypersurface of an almost paracontact manifold admits an almost product pseudo-Riemannian structure.*

We now assume that M is orientable and choose a unit vector field N of $\mathcal{M}$ which is normal to M. Then, the Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y)N, \tag{3.7}$$

and

$$\overline{\nabla}_X N = -HX \tag{3.8}$$

for any $X, Y \in TM$, respectively. Here ∇ represents the Levi-Civita connection with respect to pseudo-Riemannian metric g induced by $\tilde{g}$ on $\mathcal{M}$ and h is a second fundamental form related to H by

$$h(X, Y) = g(HX, Y). \tag{3.9}$$

The hypersurface M is totally geodesic in $\mathcal{M}$ if second fundamental form vanishes identically. Next, for any $X \in TM$, we define

$$\phi X = fX + \mu(X)N, \tag{3.10}$$

$$\phi N = -U, \tag{3.11}$$

$$\xi = V + \lambda N, \tag{3.12}$$

$$\eta(X) = \nu(X), \tag{3.13}$$

where $\eta(N) = \lambda$ (a smooth function on M), f is (1,1)-type tensor field and μ represents a non-zero 1-form. Hence, we can observe that $\lambda \neq 0$, because if $\lambda = 0$, then $\eta(N) = g(N, \xi) = 0$, which implies that ξ is perpendicular to N, so we have $\xi \in TM$, which contradicts the fact that M is a transversal hypersurface of $\mathcal{M}$.

We get an induced structure $(f, g, \mu, \nu, \lambda)$-structure [29] on transversal hypersurface such that

$$f^2 = I + \mu \otimes U - \nu \otimes V, \tag{3.14}$$

$$fU = \lambda V \quad \textit{and} \quad fV = \lambda U, \tag{3.15}$$

$$\mu \circ f = -\lambda\nu \quad \textit{and} \quad \nu \circ f = -\lambda\mu, \tag{3.16}$$

$$\mu(U) = \lambda^2 - 1, \quad \nu(V) = 1 - \lambda^2 \quad \textit{and} \quad \nu(U) = 0 = \mu(V), \tag{3.17}$$

$$g(fX, fY) = -g(X, Y) + \nu(X)\nu(Y) - \mu(X)\mu(Y), \tag{3.18}$$

$$g(fX, Y) = -g(X, fY), \quad g(X, U) = \mu(X) \quad \textit{and} \quad g(X, V) = \nu(X), \tag{3.19}$$

for all $X, Y \in TM$, where $\lambda = \eta(N)$.
Thus we see that the following result:

Theorem 3.3. *Every transversal hypersurface of an almost paracontact manifold also admits a $(f, g, \mu, \nu, \lambda)$-structure.*

Now, we find a relationship between the induced almost product pseudo-Riemannian structure (J, G) and induced $(f, g, \mu, \nu, \lambda)$-structure on transversal hypersurface of an almost paracontact pseudo-metric manifold.

Theorem 3.4. *If M be a transversal hypersurface of an almost paracontact pseudo-metric manifold $\mathcal{M}$ equipped with an almost paracontact metric structure (ϕ, η, ξ, g), then we have*

$$\omega = \frac{1}{\lambda}\mu, \tag{3.20}$$

$$J = f - \frac{1}{\lambda}\mu \otimes V, \tag{3.21}$$

$$JU = \frac{1}{\lambda}V, \tag{3.22}$$

$$\mu \circ J = \mu \circ f = -\lambda\nu, \tag{3.23}$$

$$JV = fV = \lambda U, \tag{3.24}$$

$$\nu \circ J = -\frac{1}{\lambda}\mu, \tag{3.25}$$

and

$$G(X, JY) = -g(X, fY) \tag{3.26}$$

for all $X, Y \in T(M)$.

Proof. By using (3.12) in the relation $\phi X = JX + \omega(X)\xi$, we have

$$\phi X = JX + \omega(X)V + \omega(X)\lambda N.$$

In account of (3.10), we find

$$fX + \mu(X)N = JX + \omega(X)V + \omega(X)\lambda N.$$

On comparing normal and tangential parts it follows that (i) $\mu(X) = \lambda\omega(X)$ and (ii) $fX = JX + \omega(X)V$, respectively. From the first case (3.20) follows. To find (3.21), we use (3.20) in the second case.

Now, by using the equations (3.15) and (3.17) in (3.21), we have

$$JU = \lambda V - \frac{1}{\lambda}(\lambda^2 - 1)V$$

which gives equation (3.22).
Next, from the equation (3.21), we have

$$(\mu \circ J)(X) = (\mu \circ f)(X) - \frac{1}{\lambda}\mu(X)\mu(V),$$

which by using (3.16) takes the form

$$\begin{aligned} &(\mu \circ J)(X) = (\mu \circ f)(X) = -\lambda\nu(X) \\ \Longrightarrow\ &(\mu \circ J) = (\mu \circ f) = -\lambda\nu, \qquad \forall X \in TM, \end{aligned}$$

which is equation (3.23).
Similarly, with the help of the equations (3.16) and (3.17) and (3.21) we get (3.25).

Now, from the equations (3.15), (3.17) and (3.21), we get

$$\begin{aligned} &JV = fV - \frac{1}{\lambda}\mu(V)V \\ \Longrightarrow\ &JV = fV = \lambda U \end{aligned}$$

which is equation (3.24).
Further, by using the equations (3.4), (3.13), (3.19) and (3.20) in (3.6), we have

$$\begin{aligned} G(X,JY) &= -g(X,JY) + \eta(X)\eta(JY) \\ &= -g(X,JY) - \eta(X)\omega(Y) \\ &= -g(X, fY - \frac{1}{\lambda}\mu(Y)V) - \eta(X)\omega(Y) \\ &= -g(X,fY) + \frac{1}{\lambda}\mu(Y)\nu(X) - \frac{1}{\lambda}\mu(Y)\nu(X) \\ &= -g(X,fY) \end{aligned}$$

which is equation (3.26). □

4. **Some Properties of Transversal Hypersurfaces**

Firstly, we state the following lemma:

Lemma 4.1. *Let M be a transversal hypersurface with the structure (f,g,μ,ν,λ) of an almost paracontact manifold $\mathcal{M}$, then*

$$(\overline{\nabla}_X\phi)Y = ((\nabla_X f)Y - \mu(Y)HX + h(X,Y)U) + ((\nabla_X\mu)Y + h(X,fY))N, \tag{4.1}$$

$$\overline{\nabla}_X\xi = (\nabla_X V - \lambda HX) + (h(X,V) + X\lambda)N, \tag{4.2}$$

$$(\overline{\nabla}_X\phi)N = (-\nabla_X U + f(HX)) + (-h(X,U) + \mu(HX))N, \tag{4.3}$$

since $(-h(X,U) + \mu(HX))N = 0$, then we have $(\overline{\nabla}_X\phi)N = -\nabla_X U + f(HX)$, and

$$(\overline{\nabla}_X\eta)Y = (\nabla_X\nu)Y - \lambda h(X,Y) \qquad \forall X,Y \in TM. \tag{4.4}$$

Proof. First, covariant differentiation of (3.10) along X and making use of (3.7), (3.8) and (3.11) gives (4.1). Next, by differentiating (3.12) covariantly along X, then using (3.7) and (3.8), we obtain (4.2). Further, the covariant differentiation of (3.11) along X, and using the equations (3.7)-(3.10), (3.19) leads to (4.3). At last, (4.4) follows by the covariant differentiation of (3.13) along X and utilization of (3.7). □

Theorem 4.2. *Let M be a transversal hypersurface with the structure (f,g,μ,ν,λ) of an para-Kenmotsu manifold $\mathcal{M}$, then we have*

$$(\nabla_X f)Y - \mu(Y)HX + h(X,Y)U = g(fX,Y)V - \nu(Y)fX, \tag{4.5}$$

$$(\nabla_X\mu)Y + h(X,fY) = -\nu(Y)\mu(X) + \lambda g(fX,Y), \tag{4.6}$$

$$\nabla_X V - \lambda HX = X - \nu(X)V, \tag{4.7}$$

$$h(X,V) + X\lambda = -\lambda\nu(X), \tag{4.8}$$

$$\nabla_X U = fHX + \lambda fX - \mu(X)V, \tag{4.9}$$

and

$$(\nabla_X\nu)Y = \lambda h(X,Y) + g(X,Y) - \nu(X)\nu(Y) \qquad \forall X,Y \in TM. \tag{4.10}$$

Proof. Using equations (4.1), (2.4), (3.10), (3.12) and (3.13), we have

$$g(fX,Y)(V+\lambda N)-\nu(Y)f(X)-\nu(Y)\mu(X)N=((\nabla_X f)Y-\mu(Y)HX+h(X,Y)U) + ((\nabla_X\mu)Y+h(X,fY))N.$$

On comparing tangential and normal parts in the foregoing equation, we get the results (4.5) and (4.6), respectively.
Now, from the equations (4.2), (2.5), (3.12) and (3.13), we get

$$(X-\nu(X))(V+\lambda N)=(\nabla_X V-\lambda HX)+(h(X,V)+X\lambda)N,$$

from which on comparing tangential and normal parts, the repective results (4.7) and (4.8) follows.
Similarly, by using the equations (2.4), (3.10) and (3.12) in (4.3) we get (4.9).

Further, since we have

$$\begin{aligned}(\overline{\nabla}_X\eta)Y &= \overline{\nabla}_X\{\eta(Y)\}-\eta(\overline{\nabla}_X Y)\\ &= \overline{\nabla}_X\{g(Y,\xi)\}-\eta(\overline{\nabla}_X Y)\\ &= g(\overline{\nabla}_X Y,\xi)+g(Y,\overline{\nabla}_X\xi)-\eta(\overline{\nabla}_X Y)\\ &= g(Y,\overline{\nabla}_X\xi)\\ &= g(Y,X-\eta(X)\xi)\\ &= g(Y,X)-\eta(X)\eta(Y),\end{aligned}$$

which along with (4.4) gives (4.10). □

Theorem 4.3. *If M is a transversal hypersurface with the structure(f,g,μ,ν,λ) of a para-Kenmotsu manifold $\mathcal{M}$, then 2-form F on M is given by $F(X,Y)=g(X,fY)$ holds the following condition:*

$$\begin{aligned}(\nabla_X F)(Y,Z)+(\nabla_Y F)(Z,X)+(\nabla_Z F)(X,Y)\\ =2\{g(fY,X)\nu(Z)+g(fZ,Y)\nu(X)+g(fX,Z)\nu(Y)\}\end{aligned} \tag{4.11}$$

and consequently, 2-form F is not closed on M.

Proof. In the view of the equations (3.9), (3.19) and (4.5), we get

$$\begin{aligned}(\nabla_X F)(Y,Z) &= g(Y,(\nabla_X f)Z)\\ &= g(Y,g(fX,Z)V-\nu(Z)fX+\mu(Z)HX-h(X,Z)U)\\ &= g(fX,Z)\nu(Y)-g(Y,fX)\nu(Z)+\mu(Z)h(X,Y)-\mu(Y)h(X,Z),\end{aligned}$$

which gives (4.11).
Now, since $dF(X,Y)Z=(\nabla_X F)(Y,Z)+(\nabla_Y F)(Z,X)+(\nabla_Z F)(X,Y)\neq 0$ for any $X,Y,Z\in TM$. Hence, 2-form F is not closed on M. □

5. Example of transversal hypersurface of para-Kenmotsu manifold admitting a (f,g,μ,ν,λ)-structure

Let us consider an 11-dimensional manifold

Let us assume a 5 dimensional manifold $\mathcal{M}^5=\{(x_1,x_2,x_3,x_4,t)\in\mathbb{R}^5 : t>0, x_1>0\}$, where (x_1,x_2,x_3,x_4,t) are the standard coordinates in $\mathbb{R}^5$. Then the vector fields

$$e_1=t\frac{\partial}{\partial x_1},\quad e_2=t\frac{\partial}{\partial x_2},\quad e_3=t\frac{\partial}{\partial x_3},\quad e_4=t\frac{\partial}{\partial x_4},\quad e_5=-t\frac{\partial}{\partial t}=\xi$$

are linearly independent at each point of $\mathcal{M}^5$. Let the semi-Riemannian metric tensor, $\tilde{g}$ is defined as

$$\tilde{g}(e_i, e_i) = -1, \text{ if } i = 2, 4,$$
$$\tilde{g}(e_i, e_i) = 1, \text{ if } i = 1, 3, 5,$$
$$\tilde{g}(e_i, e_j) = 0, \text{ if } \quad i \neq j, \ where\ 1 \leq i, j \leq 5.$$

Let η be the 1-form such that $\eta(X) = \tilde{g}(X, e_5), \forall X \in \Gamma(T\mathcal{M}^5)$. Now, we define the tensor field ϕ of (1,1) type such that
$\phi e_1 = e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = e_4, \quad \phi e_4 = e_3, \quad \phi e_5 = 0.$
Then, we can easily see that

$$\eta(e_5) = 1, \text{ for } e_5 = \xi,$$
$$\phi^2 X = X - \eta(X)\xi,$$
$$and \quad \tilde{g}(\phi X, \phi Y) = -\tilde{g}(X, Y) + \eta(X)\eta(Y)$$

$\forall X, Y \in \Gamma(T\mathcal{M}^5)$.
Thus, $\mathcal{M}^5(\phi, \xi, \eta, \tilde{g})$ defines an almost paracontact metric manifold. Let by ∇, we denote the Levi-civita connection on $\mathcal{M}^5$, then by direct computations, we get

$$[e_i, e_5] = e_i, \quad \nabla_{e_i} e_5 = e_i, \quad 1 \leq i \leq 4,$$

$$\nabla_{e_i} e_j = 0, \quad \text{otherwise}.$$

Thus, we have $\nabla_X e_5 = \phi^2 X - \eta(X) e_5$. Hence, $\mathcal{M}^5(\phi, \xi, \eta, \tilde{g})$ is a para Kenmotsu manifold of dimension 5.

Now, let (P, g) be a pseudo-Riemannian hypersurface of $\mathcal{M}^5$, which is given by $\chi : \mathcal{M}^4 \to \mathcal{M}^5$ such that $\chi(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4, -x_1)$. Then, the local basis of tangent hyperplane of P is given by

$$X_1 = \frac{\partial}{\partial x_1} - \frac{1}{x_1}\frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x_2}, \quad X_3 = \frac{\partial}{\partial x_3}, \quad X_4 = \frac{\partial}{\partial x_4}$$

and unit normal vector field N of the hypersurface (P, g) is given by

$$N = \frac{t}{\sqrt{1+x_1^2}}\left(\frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial t}\right).$$

Here, it is clear that $\xi_p, p \in P$ is not tangent to the hypersurface. Therefore, P is a transversal hypersurface of $\mathcal{M}^5$. Also, we have

$$\eta(N) = -\frac{x_1}{\sqrt{1+x_1^2}} = \lambda, \quad V = \frac{t x_1}{1+x_1^2}\frac{\partial}{\partial x_1} - \frac{t}{1+x_1^2}\frac{\partial}{\partial t} \quad and \quad U = -\frac{t}{\sqrt{1+x_1^2}}\frac{\partial}{\partial x_2}.$$

Further, any tangent vector field of the transversal hypersurface P can be expressed as $X = \sum_{i=1}^{4} c_i X_i$, where $c_i, 1 \leq i \leq 4$ are smooth functions. Operating ϕ on both sides, we obtain

$$\begin{aligned}\phi X &= c_2\left(1 + \frac{t x_1}{\sqrt{1+x_1^2}}\right)\frac{\partial}{\partial x_1} + c_1 \frac{\partial}{\partial x_2} + c_4 \frac{\partial}{\partial x_3} + c_3 \frac{\partial}{\partial x_4} + c_2 \frac{t {x_1}^2}{\sqrt{1+x_1^2}}\frac{\partial}{\partial t} - c_2 x_1 N \\ &= fX + \mu(X)N,\end{aligned}$$

where $\mu(X) = -c_2 x_1$ and f is given by

$$f = \begin{bmatrix} 0 & 1 + \frac{tx_1}{\sqrt{1+x_1^2}} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{t{x_1}^2}{\sqrt{1+x_1^2}} & 0 & 0 & 0 \end{bmatrix}.$$

Hence, P is a transversal hypersurface of $\mathcal{M}$ which admits $(f, g, \mu, \nu, \lambda)$-structure.

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