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A COMMON FIXED POINT THEOREM FOR A NEW CLASS OF CONTRACTIONS ON b-METRIC SPACES

SAMANEH MOHAMADI, MAHDI IRANMANESH, AND HARIKRISHNAN PANACKAL

Abstract. This paper introduces a new class of generalized contractive mappings to establish a common fixed point theorem for a new class of mappings in complete b-metric spaces. This can be considered as an extension in some of the existing ones. Finally, we provide an example to show that our result is a natural generalization of certain fixed point theorems.

1. Introduction

Let (X, d) be a metric space. A mapping $T : X \to X$ is contraction if there exists a constant $k \in (0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y),$$

for any $x, y \in X$.
If X is complete, then every contraction on X has a unique fixed point, that can be derived as the limit of iteration of the contraction at some point of X which is known as the Banach contraction principle.
In 1997, Alber and Guerre-Delabriere [3] generalized the notion of contraction mapping as follows: A mapping $T : X \to X$ is ϕ-weak contraction if there exists a function $\phi : [0, \infty) \to [0, \infty)$ such that ϕ is positive on $(0, \infty)$, $\phi(0) = 0$, and

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)),$$

for every $x, y \in X$.
Notice that By considering the function $\phi(t) = kt$ with $0 < k < 1$, every contraction is ϕ-weak contraction.
Meanwhile, they have shown that every single-valued ϕ-weak contraction on a Hilbert space has a unique fixed point. Rhoades [27] showed that most parts of the results in [3] are valid

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for any Banach space. He also proved the following generalization of the Banach contraction principle:

Theorem 1.1. *Let* (X, d) *be a complete metric space and let* $T : X \to X$ *be a* ϕ*-weak contraction on* X*, where* $\phi : [0, \infty) \to [0, \infty)$ *is a continuous nondecreasing function with* $\phi(t) > 0$ *for all* $t > 0$ *and* $\phi(0) = 0$*, then* T *has a unique fixed point.*

Dutta and Choudhury [16] proved the following generalization of Theorem 1.1.

Theorem 1.2. *Let* (X, d) *be a complete metric space and let the map* $T : X \to X$ *satisfies in the inequality*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y)) \qquad (x, y \in X),$$

where $\psi, \phi : [0, \infty) \to [0, \infty)$ *are both continuous and monotone nondecreasing functions with* $\psi(t) = \phi(t) = 0$ *if and only if* $t = 0$*. Then* T *has a unique fixed point.*

Doric [15] generalized Theorem 1.2 as follows:

Theorem 1.3. *Let* (X, d) *be a complete metric space and let the map* $T : X \to X$ *satisfies the inequality*

$$\psi(d(Tx, Ty)) \leq \psi(M(x, y)) - \phi(M(x, y)),$$

for any $x, y \in X$*, where* M *is given by*

$$M(x, y) = max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(Tx, y))\},$$

and

(*i*) $\psi : [0, \infty) \to [0, \infty)$ *is a continuous monotone nondecreasing function with* $\psi(t) = 0$ *if and only if* $t = 0$*,*

(*ii*) $\phi : [0, \infty) \to [0, \infty)$ *is a lower semi-continuous function with* $\phi(t) = 0$ *if and only if* $t = 0$*.*

Then T *has a unique fixed point.*

Fixed point theorems for multi-vlaued operators using Hausdorff metric was initiated by Nadler [25] in 1969.

The concept of a b-metric space was introduced by Bakhtin [5], and later used by Czerwik [12]. After that, several interesting results about the existence of fixed points for single-valued and multi-valued operators in b-metric spaces have been obtained (see, e.g, [1, 2, 6, 8, 13, 17, 18, 19, 20, 21, 22, 23, 24, 26, 28]).

In 2012, Bota et al. [4] proved the following theorem in complete b-metric spaces:

Theorem 1.4. *Let (X, d) be a complete b-metric space and $\phi : [0, \infty) \to [0, \infty)$ is a lower semi continuous function with $\phi(t) = 0$ if and only if $t = 0$. Suppose that $T : X \to X$, $S : X \to CB(X)$, where $CB(X)$ denotes the family of all nonempty closed bounded subsets of X, are such that for all $x, y \in X$*

$$H(\{Tx\}, Sy) \leq M(x, y) - \phi(M(x, y))$$

where

$$M(x, y) = max\{d(x, y), D(x, Tx), D(y, Sy), \frac{1}{2s}(D(x, Sy) + D(y, Tx))\},$$

then T and S have a unique common fixed point in X.

This paper presents a new common fixed point theorem for multi-valued and single-valued operators on complete b-metric spaces. Our results generalize some well-known common fixed point theorems given by Zhang and Song [29], Rhoades [27], Ćirić [9], Daffer and Kaneko [14] and Aydi, Bota, Karapinar and Moradi [4].

2. Preliminaries

Throughout this paper, $\mathbb{R}$ denotes the real line, and $\mathbb{N}$ is the set of all-natural numbers. We recall some definitions and preliminaries that will be needed in this paper.

Definition 2.1. ([11]) Let X be a nonempty set and $s \geq 1$ a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric provided that, for all $x, y, z \in X$, the following conditions hold:

i) $d(x, y) = 0$ iff $x = y$;
ii) $d(x, y) = d(y, x)$;
iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space with the parameter s.

The following example shows that a b-metric need not necessarily be a metric. It should be noted that the class of b-metric spaces is effectively more significant than metric spaces. It should also be remarked that a b-metric is not continuous in general [7].

Example 2.2. Let (X, ρ) be a metric space and $d(x, y) = (\rho(x, y))^p$, where $p > 1$ is a real number. Then d is a b-metric with the parameter $s = 2^{p-1}$.

Remark 2.3. Every metric space is a b-metric space with $s = 1$, however, the converse is generally not valid.

Example 2.4. Let $X = \mathbb{R}$, and let the mapping $d : X \times X \to [0, \infty)$ be defined by $d(x, y) = |x - y|^2$ for all $x, y \in X$. Then (X, d) is a b-metric space with the parameter $s = 2$, but it is not a metric.

Proposition 2.5. *(*[7]*) In a b-metric space the following assertions hold:*

(i) a convergent sequence has a unique limit,

(ii) each convergent sequence is Cauchy,

Definition 2.6. ([10]) Let (X, d) be a b-metric space.

(i) A sequence $\{x_n\}$ in X is called b-convergent if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.

(ii) A sequence $\{x_n\}$ in X is called b-Cauchy if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

(iii) A b-metric space (X, d) is said to be b-complete if every b-Cauchy sequence in X is b-convergent.

(iv) A set $B \subset X$ is said to be b-closed if for any sequence $\{x_n\}$ in B which $\{x_n\}$ is b-convergent to $z \in X$, we have $z \in B$.

Let (X, d) be a b-metric space, and let $CB(X)$ be the family of all nonempty closed bounded subsets of X. For $A, B \in CB(X)$, we define

$$H(A, B) = \max\{\rho(A, B), \rho(B, A)\},$$

where

$$\rho(A, B) = \sup\{D(a, B), a \in A\}, \qquad \rho(B, A) = \sup\{D(b, A), b \in B\}$$

with

$$D(a, C) = \inf\{d(a, x), x \in C\}, \qquad (C \in CB(X)).$$

Lemma 2.7. *([11, 28]) Let (X, d) be a b-metric space. For any $A, B, C \in CB(X)$ and any $x, y \in X$, we have the following assertions:*

(i) $D(x, A) = 0 \Leftrightarrow x \in \bar{A} = A$,

(ii) $D(x, B) \leq d(x, b)$ *for any* $b \in B$,

(iii) $\rho(A, B) \leq H(A, B)$,

(iv) $d(x, B) \leq H(A, B)$ *for all* $x \in A$,

(v) $H(A, A) = 0$,

(vi) $H(A, B) = H(B, A)$,

(vii) $H(A, C) \leq s(H(A, B) + H(B, C))$,

(viii) $D(x, A) \leq s(d(x, y) + D(y, A))$.

(ix) *for every* $\alpha > 0$, $b \in B$, *there exists* $a \in A$ *such that*

$$d(a, b) \leq H(A, B) + \alpha.$$

Definition 2.8. A function $f : X \to \mathbb{R}$ is called lower semi-continuous (upper semi-continuous), if for any $\{x_n\} \subset X$ and $x \in X$

$$x_n \to x \Rightarrow f(x) \leq \lim_{n\to\infty} inf f(x_n),$$

$$(\quad x_n \to x \Rightarrow \lim_{n\to\infty} \sup f(x_n) \leq f(x) \quad)$$

Example 2.9. The indicator function of a closed set is upper semi-continuous, whereas the indicator function of an open set is lower semi-continuous.

Example 2.10. Consider the function f as follows:

$$Tx = \begin{cases} -1 & x \leq 0, \\ 1 & x > 0. \end{cases}$$

This function is lower semi continuous at $x = 0$.
Notice that a function maybe upper or lower semi-continuous without being either left or right continuous. For example,

$$f(x) = \begin{cases} 1 & if x < 1 \\ 2 & if x = 1 \\ \frac{1}{2} & if x > 1 \end{cases}$$

is upper semi-continuous at $x = 1$, but it is neither left nor right continuous at 1.

Definition 2.11. Let (X, d) be a complete metric space, we introduce the following functions:

(*i*) $\psi : [0, \infty) \to [0, \infty)$ is a continuous nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$.

(*ii*) $\phi : [0, \infty) \to [0, \infty)$ is a lower semi continuous function with $\phi(t) = 0$ if and only if $t = 0$.

(*iii*) $\theta : [0, \infty) \to [0, \infty)$ is a continuous function with $\theta(t) = 0$ if and only if $t = 0$.

Throughout the paper, Ψ is the set of all functions ψ satisfying (i), Φ is the set of all functions ϕ satisfying (ii) and Θ is the set of all functions θ satisfying (iii).

3. Main results

We establish a common fixed point theorem for a new class of generalized contractive mappings in the following theorem. Moreover, we will assume that s is the infimum of real numbers such that (iii) in Definition 2.1 is true.

Theorem 3.1. *Let (X, d) be a complete b-metric space and $\psi \in \Psi$, $\phi \in \Phi$ and $\theta \in \Theta$. Consider the maps $T : X \to X$, $S : X \to CB(X)$ where S is a multi-valued map and a constant $L > 0$ be such that the inequality*

$$\psi(s^2 H(\{Tx\}, Sy)) \leq \psi(M_s(x, y)) - \phi(\theta(M_s(x, y))) + L\psi(N(x, y)) \tag{3.1}$$

holds for all $x, y \in X$, where

$$M_s(x, y) = \max\{d(x, y), D(x, Tx), D(y, Sy), \frac{1}{2s}[D(x, Sy) + D(y, Tx)]\},$$
$$N(x, y) = \min\{D(x, Tx), D(y, Ty), D(x, Sy), D(y, Tx)\}.$$

Then S and T have a unique common fixed point in X, that is, there exist $z \in X$ such that $z = Tz$ and $z \in Sz$.

Proof. It is easy to show that $x = y$ is a common fixed point of T and S if and only if $M_s(x, y) = 0$. Thus we suppose that for all $x, y \in X$, we have $M_s(x, y) > 0$.
We complete that proof in the following steps:
Step 1: Let $x_0 \in X$ and $x_1 \in Sx_0$. Set $x_2 = Tx_1$.

By choosing $\alpha = \frac{\phi(\theta(M_s(x_2,x_1)))}{2}$ in Lemma 2.7, there exists $x_3 \in Sx_2$ such that

$$d(x_3, x_2) \leq H(\{Tx_1\}, Sx_2) + \frac{\phi(\theta(M_s(x_2, x_1)))}{2}.$$

We let $x_4 = Tx_3$. In analogous way, one can find $x_5 \in Sx_4$ such that

$$d(x_5, x_4) \leq H(\{Tx_3\}, Sx_4) + \frac{\phi(\theta(M_s(x_4, x_3)))}{2}.$$

Inductively, we let $x_{2n} = Tx_{2n-1}$, and by lemma 2.7, there exists $x_{2n+1} \in Sx_{2n}$ such that

$$\begin{aligned} d(x_{2n+1}, x_{2n}) &\leq H(\{Tx_{2n-1}\}, Sx_{2n}) + \frac{\phi(\theta(M_s(x_{2n}, x_{2n-1})))}{2} \\ &\leq s^2 H(\{Tx_{2n-1}\}, Sx_{2n}) + \frac{\phi(\theta(M_s(x_{2n}, x_{2n-1})))}{2}. \end{aligned}$$

Since ψ is nondecreasing we have

$$\begin{aligned} \psi(d(x_{2n+1}, x_{2n})) &= \psi(D\{Tx_{2n-1}\}, x_{2n+1})) \\ &\leq \psi(H(\{Tx_{2n-1}\}, Sx_{2n})) \\ &\leq \psi(s^2 H(\{Tx_{2n-1}\}, Sx_{2n})). \end{aligned} \tag{3.2}$$

Thus

$$\psi(d(x_{2n+1}, x_{2n})) \leq \psi(s^2 H(\{Tx_{2n-1}\}, Sx_{2n})) + \frac{\phi(\theta(M_s(x_{2n}, x_{2n-1})))}{2}$$

From (3.1) we get that

$$\begin{aligned} \psi(d(x_{2n+1}, x_{2n})) \leq &\psi(M_s(x_{2n}, x_{2n-1})) - \frac{\phi(\theta(M_s(x_{2n}, x_{2n-1})))}{2} + \\ &L\psi(N(x_{2n}, x_{2n-1})). \end{aligned} \tag{3.3}$$

Step 2: We show that $\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0$.
For any $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_{2n-1}, x_{2n}) \leq & M_s(x_{2n-1}, x_{2n}) \\ = & \max\{d(x_{2n-1}, x_{2n}), D(x_{2n-1}, Tx_{2n-1}), D(x_{2n}, Sx_{2n}), \\ & \frac{1}{2s}[D(x_{2n-1}, Sx_{2n}) + D(x_{2n}, Tx_{2n-1})]\} \\ \leq & \max\{d(x_{2n-1}, x_{2n}), d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1}), \\ & \frac{1}{2s} d(x_{2n-1}, x_{2n+1})\} \\ \leq & \max\{d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1}), \\ & \frac{1}{2s}[s(d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1}))]\} \\ = & \max\{d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1})\}. \end{aligned}$$

If $\max\{d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1})\} = d(x_{2n}, x_{2n+1})$, by (3.3) and the fact that $N(x_{2n}, x_{2n-1}) = 0$ we have

$$\begin{aligned} \psi(d(x_{2n+1}, x_{2n})) \leq & \psi(M_s(x_{2n}, x_{2n-1})) - \frac{\phi(\theta(M_s(x_{2n}, x_{2n-1})))}{2} \\ & + L\psi(N(x_{2n}, x_{2n-1}))) \\ \leq & \psi(d(x_{2n}, x_{2n+1})) - \frac{\phi(\theta(M_s(x_{2n}, x_{2n-1})))}{2}. \end{aligned}$$

So $\frac{\varphi(\theta(M_s(x_{2n}, x_{2n-1})))}{2} = 0$, that is, $M_s(x_{2n}, x_{2n-1}) = 0$, which is a contradiction. Hence $\max\{d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1})\} = d(x_{2n-1}, x_{2n})$. Then $M_s(x_{2n-1}, x_{2n}) = d(x_{2n-1}, x_{2n})$ for each $n \geq 1$. We have

$$d(x_{2n}, x_{2n+1}) \leq d(x_{2n-1}, x_{2n}). \tag{3.4}$$

Similar to the process of (3.2) we get also

$$\psi(d(x_{2n+1}, x_{2n+2})) \leq \psi(s^2 H(\{Tx_{2n+1}\}, Sx_{2n})).$$

By using (3.1) we have

$$\begin{aligned} \psi(d(x_{2n+1}, x_{2n+2})) \leq & \psi(M_s(x_{2n+1}, x_{2n})) - \phi(\theta(M_s(x_{2n+1}, x_{2n})))+ \\ & L\psi(N(x_{2n+1}, x_{2n})), \end{aligned} \tag{3.5}$$

where

$$\begin{aligned}
M_s(x_{2n+1}, x_{2n}) =& \max\{d(x_{2n+1}, x_{2n}), D(x_{2n+1}, Tx_{2n+1}), D(x_{2n}, Sx_{2n}),\\
& \frac{1}{2s}[D(x_{2n+1}, Sx_{2n}) + D(x_{2n}, Tx_{2n+1})]\}\\
\leq& \max\{d(x_{2n+1}, x_{2n}), d(x_{2n+1}x_{2n+2}), d(x_{2n}, x_{2n+1})\\
& \frac{1}{2s}[d(x_{2n+1}, x_{2n+1}) + d(x_{2n}, x_{2n+2})]\}\\
\leq& \max\{dx_{2n+1}, x_{2n}), d(x_{2n+1}, x_{2n+2}),\\
& \frac{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})}{2}\}\\
=& \max\{d(x_{2n+1}, x_{2n}), d(x_{2n+1}, x_{2n+2})\}\\
=& d(x_{2n+1}, x_{2n}).
\end{aligned}$$

If $\max\{d(x_{2n+1}, x_{2n}), d(x_{2n+1}, x_{2n+2})\} = d(x_{2n+1}, x_{2n+2})$, by (3.5) and the fact $N(x_{2n+1}, x_{2n}) = 0$ we have

$$\begin{aligned}
\psi(d(x_{2n+1}, x_{2n+2})) \leq& \psi(M_s(x_{2n+1}, x_{2n})) - \phi(\theta(M_s(x_{2n+1}, x_{2n})))\\
<& \psi(M_s(x_{2n+1}, x_{2n}))\\
=& \psi(d(x_{2n+1}, x_{2n+2})).
\end{aligned}$$

Thus

$$\psi(d(x_{2n+1}, x_{2n+2})) < \psi(d(x_{2n+1}, x_{2n+2})),$$

which is a contradiction, therefore

$$M_s(x_{2n+1}, x_{2n}) = d(x_{2n+1}, x_{2n})$$

and

$$d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n+1}, x_{2n}). \tag{3.6}$$

From (3.4) and (3.6), we get that

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n), \qquad \forall n \geq 0.$$

Thus $\{d(x_n, x_{n+1});\ n \in \mathbb{N}\}$ is a non-increasing sequence of positive numbers. Hence, there is $l \geq 0$ such that

$$\lim_{n\to\infty} M_s(x_n, x_{n+1}) = \lim_{n\to\infty} d(x_n, x_{n+1}) = l \geq 0.$$

We show that $l = 0$. On the contrary suppose $l > 0$. We know $\phi(\theta(l)) > 0$ from (3.3) and taking limits as $n \to \infty$, since ϕ is lower semi-continuous, we get

$$\begin{aligned} \psi(l) \leq & \psi(l) - \frac{\phi(\theta(l))}{2} + L\psi(0) \\ < & \psi(l) \end{aligned}$$

this is contradiction, thus $l = 0$. So we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.7}$$

Step 3: We will prove that $\{x_n\}$ is a b-Cauchy sequence. Because of (3.7), it is sufficient to show that $\{x_{2n}\}$ is a b-Cauchy sequence.

Suppose $\{x_{2n}\}$ is not a b-Cauchy sequence. Then there exists $\varepsilon > 0$, for which we can find two subsequences $\{x_{2mi}\}$, $\{x_{2ni}\}$ of $\{x_{2n}\}$ such that ni is the smallest index, for which

$$ni > mi > i, \qquad d(x_{2mi}, x_{2ni}) \geq \varepsilon.$$

This means that

$$d(x_{2mi}, x_{2ni-2}) < \varepsilon. \tag{3.8}$$

By using the triangular inequality, we get

$$\varepsilon \leq d(x_{2mi}, x_{2ni}) \leq sd(x_{2mi}, x_{2mi+1}) + sd(x_{2mi+1}, x_{2n_i}).$$

By taking the upper limits as $i \to \infty$, we get

$$\frac{\varepsilon}{s} \leq \lim_{i\to\infty} \sup d(x_{2mi+1}, x_{2ni}). \tag{3.9}$$

On the other hand, we have

$$d(x_{2mi}, x_{2ni-1}) \leq sd(x_{2mi}, x_{2ni-2}) + sd(x_{2ni-2}, x_{2ni-1}).$$

Using (3.8) and taking the upper limit as $i \to \infty$, we get

$$\lim_{i\to\infty} \sup d(x_{2mi}, x_{2ni-1}) \leq \varepsilon s. \tag{3.10}$$

Using the triangular inequality, we have

$$\varepsilon \leq d(x_{2mi}, x_{2ni}) \leq sd(x_{2mi}, x_{2ni-1}) + sd(x_{2ni-1}, x_{2ni}).$$

By taking the upper limit as $i \to \infty$, we get

$$\frac{\varepsilon}{s} \leq \lim_{i\to\infty} \sup d(x_{2mi}, x_{2ni-1}). \tag{3.11}$$

From (3.10) and (3.11), we have

$$\frac{\varepsilon}{s} \leq \limsup_{i\to\infty} d(x_{2mi}, x_{2ni-1}) \leq \varepsilon s. \tag{3.12}$$

Using the triangular inequality, we have

$$\begin{aligned}\varepsilon \leq & d(x_{2mi}, x_{2ni}) \leq sd(x_{2mi}, x_{2ni-2}) + sd(x_{2ni-2}, x_{2ni}) \\ \leq & sd(x_{2mi}, x_{2ni-2}) + s[sd(x_{2ni-2}, x_{2ni-1}) + sd(x_{2ni-1}, x_{2ni})].\end{aligned}$$

By taking the upper limit as $i \to \infty$ and using (3.8) we have

$$\varepsilon \leq \limsup_{i\to\infty} d(x_{2mi}, x_{2ni}) \leq \varepsilon s. \tag{3.13}$$

Again, using the triangular inequality, we have

$$d(x_{2mi+1}, x_{2ni-1}) \leq sd(x_{2mi+1}, x_{2mi}) + sd(x_{2mi}, x_{2ni-1}).$$

By using (3.10) and taking the upper limit as $i \to \infty$, we have

$$\limsup_{i\to\infty} d(x_{2mi+1}, x_{2ni-1}) \leq s^2\varepsilon. \tag{3.14}$$

Again, using the triangular inequality, we have

$$\begin{aligned}\varepsilon \leq & d(x_{2mi}, x_{2ni}) \leq sd(x_{2mi}, x_{2mi+1}) + sd(x_{2mi+1}, x_{2ni}) \\ \leq & sd(x_{2mi}, x_{2mi+1}) + s^2 d(x_{2mi+1}, x_{2ni-1}) + s^2 d(x_{2ni-1}, x_{2ni}).\end{aligned}$$

By taking the upper limit as $i \to \infty$, and using (3.14) we have

$$\frac{\varepsilon}{s^2} \leq \limsup_{i\to\infty} d(x_{2mi+1}, x_{2ni-1}) \leq s^2\varepsilon. \tag{3.15}$$

From (3.1) and similar to the process (3.2) we get

$$\begin{aligned}\psi(sd(x_{2mi+1}, x_{2ni})) \leq & \psi(s^2 H(Sx_{2mi}, \{Tx_{2ni-1}\})) \\ \leq & \psi(M_s(x_{2mi}, x_{2ni-1})) - \phi(\theta(M_s(x_{2mi}, x_{2ni-1})) \\ & + L\psi(N(x_{2mi}, x_{2ni-1})),\end{aligned} \tag{3.16}$$

where

$$
\begin{aligned}
M_s(x_{2mi}, x_{2ni-1}) =& \max\{d(x_{2mi}, x_{2ni-1}), D(x_{2mi}, Tx_{2mi}), D(x_{2ni-1}, Sx_{2ni-1}),\\
&\frac{1}{2s}[D(x_{2mi}, Sx_{2ni-1}) + D(x_{2ni-1}, Tx_{2mi})]\}\\
\leq& \max\{d(x_{2mi}, x_{2ni-1}), d(x_{2mi}, x_{2mi+1}), d(x_{2ni}, x_{2ni-1}),\\
&\frac{1}{2s}[d(x_{2mi}, x_{2ni}) + d(x_{2mi+1}, x_{2ni-1})]\}.
\end{aligned}
$$

Taking the upper limit and using (3.10), (3.13) and (3.14), we have

$$
\begin{aligned}
\limsup_{i\to\infty} M_s(x_{2mi}, x_{2ni-1}) \leq& \max\{\limsup_{i\to\infty} d(x_{2mi}, x_{2ni-1}),\\
&\limsup_{i\to\infty} d(x_{2mi}, x_{2mi+1}), \limsup_{i\to\infty} d(x_{2ni}, x_{2ni-1}),\\
&\frac{\limsup_{i\to\infty} d(x_{2mi}, x_{2ni}) + \limsup_{i\to\infty} d(x_{2mi+1}, x_{2ni-1})}{2s}\}\\
\leq& \max\{\varepsilon s, 0, 0, \frac{\varepsilon s + \varepsilon s^2}{2s}\} = \varepsilon s,
\end{aligned}
$$

and using (3.12), (3.13), and (3.15) we have

$$
\min\left\{\frac{\varepsilon}{s}, \frac{\varepsilon + \frac{\varepsilon}{s^2}}{2s}\right\} = \frac{\varepsilon + \frac{\varepsilon}{s^2}}{2s}
$$

we get

$$
\frac{\varepsilon + \frac{\varepsilon}{s^2}}{2s} \leq \limsup_{i\to\infty} M_s(x_{2mi}, x_{2ni-1}) \leq \varepsilon s
$$

and

$$
\frac{\varepsilon + \frac{\varepsilon}{s^2}}{2s} \leq \liminf_{i\to\infty} M_s(x_{2mi}, x_{2ni-1}) \leq \varepsilon s \tag{3.17}
$$

and

$$
\begin{aligned}
N(x_{2mi}, x_{2ni-1}) =& \min\{D(x_{2mi}, Tx_{2mi}), D(x_{2ni-1}, Tx_{2ni-1}),\\
&D(x_{2mi}, Sx_{2ni-1}), D(x_{2ni-1}, Tx_{2mi})\}.
\end{aligned} \tag{3.18}
$$

From (3.18), $\limsup_{i\to\infty} N(x_{2mi}, x_{2ni-1}) = 0$.

Now taking the upper limit as $i \to \infty$ in (3.16) and using (3.9) and (3.18) we have

$$\begin{aligned}\psi(\varepsilon s) = \psi(s^2 \cdot \frac{\varepsilon}{s}) \leq & \psi(s^2 \limsup_{i\to\infty} d(x_{2mi+1}, x_{2ni})) \\ \leq & \psi(\limsup_{i\to\infty} M_s(x_{2mi}, x_{2ni-1})) - \\ & \phi(\theta(\liminf_{i\to\infty} M_s(x_{2mi}, x_{2ni-1}))) \\ \leq & \psi(\varepsilon s) - \phi(\theta(\liminf_{i\to\infty} M_s(x_{2mi}, x_{2ni-1})))\end{aligned}$$

which implies that

$$\phi(\theta(\liminf_{i\to\infty} M_s(x_{2mi}, x_{2ni-1}))) = 0.$$

Thus

$$\liminf_{i\to\infty} M_s(x_{2mi}, x_{2ni-1}) = 0$$

which is in contradiction with (3.17).

Hence $\{x_n\}$ is a b-Cauchy sequence in X.

Step 4: As $\{x_n\}$ is a b-Cauchy sequence and X is a complete b-metric space, there exists $u \in X$ such that

$$\lim_{n\to\infty} d(x_n, u) = 0.$$

We show that $u = Tu$ and $u \in Su$. Similar to the process (3.2)

$$\begin{aligned}\psi(D(x_{2n+2}, Su) \leq & \psi(s^2 H(\{Tx_{2n+1}\}, Su)) \\ \leq & \psi(M_s(x_{2n+1}, u)) - \phi(\theta(M_s(x_{2n+1}, u))) \\ & + L\psi(N(x_{2n+1}, u)),\end{aligned} \tag{3.19}$$

$$\begin{aligned}D(u, Su) \leq & M_s(x_{2n+1}, u) \\ = & \max\{d(x_{2n+1}, u), d(x_{2n+1}, x_{2n+2}), D(u, Su), \\ & \frac{1}{2s}[D(x_{2n+1}, Su) + d(u, x_{2n+2})]\}.\end{aligned} \tag{3.20}$$

By using the triangular inequality, we have

$$d(x_{2n+1}, u) \leq sd(x_{2n+1}, x_{2n}) + sd(x_{2n}, u).$$

As

$$\lim_{n\to\infty} d(x_{2n+1}, u) = 0 \;\; and \;\; \lim_{n\to\infty} d(x_{2n+2}, u) = 0. \tag{3.21}$$

Again, using the triangular inequality, we get

$$D(x_{2n+1}, Su) \leq s(d(x_{2n+1}, u) + D(u, Su)).$$

By (3.21)

$$\lim_{n\to\infty} D(x_{2n+1}, Su) \leq sD(u, Su).$$

By taking limit from (3.20)

$$\begin{aligned} D(u, Su) \leq & \lim_{n\to\infty} M_s(x_{2n+1}, u) \\ \leq & \max\{D(u, Su), \frac{sD(u, Su)}{2s}\} = D(u, Su), \\ & \lim_{n\to\infty} M_s(x_{2n+1}, u) = D(u, Su). \end{aligned} \tag{3.22}$$

As

$$D(u, Su) \leq s[d(u, x_{2n+2}) + D(x_{2n+2}, Su)],$$

by taking the upper limit as $n \to \infty$, we have

$$\frac{D(u, Su)}{s} \leq \limsup_{n\to\infty} D(x_{2n+2}, Su) \tag{3.23}$$

and $\lim_{n\to\infty} N(x_{2n+1}, u) = 0$. Since ψ is continuous, by using (3.19), (3.22), (3.23) we have

$$\psi(D(u, Su)) \leq \psi(D(u, Su)) - \phi(\theta(D(u, Su))).$$

Then $\phi(\theta(D(u, Su))) = 0$. Thus $D(u, Su) = 0$.

We conclude that $u \in Su$. From (3.1) we have

$$\begin{aligned} \psi(D(Tu, u)) \leq & \psi(s^2 D(Tu, u)) \\ \leq & \psi(s^2 H(\{Tu\}, Su)) \\ \leq & \psi(M_s(u, u) - \phi(\theta(M_s(u, u))) \\ & + L\psi(N(u, u)). \end{aligned} \tag{3.24}$$

As

$$\begin{aligned} M_s(u, u) &= max\{d(u, u), D(u, Tu), D(u, Su), \frac{1}{2s}[D(u, Su) + D(u, Tu)]\} \\ &= max\{D(u, Tu), \frac{D(u, Tu)}{2s}\} = D(u, Tu). \end{aligned}$$

Moreover as $N(u,u)=0$ from (3.24) we have

$$\psi(D(Tu,u)) \leq \psi(D(Tu,u)) - \phi(\theta(D(Tu,u))) + 0.$$

Then $\phi(\theta(D(Tu,u)))=0$. Thus $D(u,Tu)=0 \Rightarrow u=Tu$.

Step 5: Now, we show that the fixed point is unique. Suppose that z is another fixed point of S and T, i.e., $z=Tz$ and $z\in Sz$, then

$$\begin{aligned}\psi(d(u,z)) =& \psi(D(Tu,z)) \leq \psi(s^2D(Tu,z)) \leq \psi(s^2H(\{Tu\},Sz))\\ \leq& \psi(M_s(u,z)) - \phi(\theta(M_s(u,z)) + L\psi(N(u,z))\end{aligned} \tag{3.25}$$

$$\begin{aligned} d(u,z) \leq M_s(u,z) =& \max\{d(u,z), D(u,Tu), D(z,Sz)\\ &, \frac{1}{2s}[D(u,Sz)+D(z,Tu)]\}\\ d(u,z) \leq& \max\{d(u,z), \frac{1}{s}d(u,z)\}\\ =& d(u,z).\end{aligned}$$

Hence

$$M_s(u,z) = d(u,z).$$

Moreover, $N(u,z)=0$, from (3.25) we have

$$\psi(d(u,z)) \leq \psi(d(u,z)) - \phi(\theta(d(u,z))).$$

Then $\phi(\theta(d(u,z)))=0$. Thus $d(u,z)=0$, i.e. $u=z$.

□

Remark 3.2. If we consider S as a single-valued operator in Theorem 3.1, then T and S have a unique common fixed point in X. Further, T has a unique fixed point in X whenever $S=T$.

Example 3.3. Let $X=[0,1]$ be equipped with the b-mertic $d(x,y)=|x-y|^2$ for all $x,y\in X$, (s=2) and let $T:X\to X$, $S:X\to CB(X)$ defined by

$$Tx = 0\ ,\ Sx = [0, \frac{x}{6}]$$

for all $x \in X$. Let $\theta(t) = t$, $\psi(t) = t$ and $\phi(t) = \frac{1}{2}t$ for all $t \geq 0$. Then

$$H(\{Tx\}, Sy) = H(\{0\}, [0, \frac{y}{6}]) = \frac{y^2}{36}$$

$$\psi(s^2 H(\{Tx\}, Sy)) = \frac{1}{9}y^2 \leq \psi(M_s(x,y)) - \phi(\theta(M_s(x,y))) + L\psi(N(x,y))$$

where

$$M_s(x,y) = max\{|x-y|^2, x^2, |y - \frac{y}{6}|^2, \frac{1}{4}[|x - \frac{y}{6}|^2 + y^2]\}.$$

Then $u = 0$ is the unique common fixed point of T and S.

Corollary 3.4. *If $s = 1$ and ψ, θ are the same functions in Theorem 3.1, then we obtain the main result in* [4].

Corollary 3.5. *If $s = 1$ and S is a single-valued mapping and θ is the same function in Theorem 3.1, then we have the main result in* [15].

Corollary 3.6. *Taking $S = T$ (single-valued) and set $s = 1$ and suppose that θ is the same function in Theorem 3.1, so we get the main result in* [26].

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(S. Mohamadi) Department of mathematical sciences, Shahrood University of Technology

E-mail address: s.mohamadi16@yahoo.com

(M. Iranmanesh) Department of mathematical sciences, Shahrood University of Technology

E-mail address: m.iranmanesh2012@gmail.com

(H. Panackal) Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India

E-mail address: pk.harikrishnan@manipal.edu