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PSEUDO PARALLEL AND GENERALIZED RICCI PSEUDO PARALLEL INVARIANT SUBMANIFOLDS OF A GENERALIZED SASAKIAN SPACE FORM

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Abstract. The objective of this paper is to study the invariant submanifolds of generalized Sasakian space form. We study pseudo parallel, generalized Ricci-pseudo parallel invariant submanifolds of a generalized Sasakian space form on different curvature tensors and obtain some new results.

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1. Introduction

The concept of generalized Sasakian space forms(GSSF) have been introduced by Alegre and Carriazo (see [1–3]). Subsequently, it has been studied by many authors [7, 12, 15]. An invariant submanifold was studied by Chen and others (see [8–11]). Some related works can be found in [5, 6, 13, 14, 16–21].

In this paper, we acquire needful and competent state for an invariant submanifold of GSSF to be totally geodesic. The pseudo parallel and generalized Ricci-pseudo parallel conditions are verified by using different curvature tensors. An almost contact metric manifold M is called GSSF if there exist three functions f_1, f_2, f_3 on M such that the curvature tensor R is given by

$$\begin{aligned} R(V_1,V_2)V_3 =& f_1\{g(V_2,V_3)V_1 - g(V_1,V_3)V_2\} + f_2\{g(V_1,\phi V_3)\phi V_2 \\ & - g(V_2,\phi V_3)\phi V_1 + 2g(V_1,\phi V_2)\phi V_3\} + f_3\{\eta(V_1)\eta(V_3)V_2 \\ & - \eta(V_2)\eta(V_3)V_1 + g(V_1,V_3)\eta(V_2)\zeta - g(V_2,V_3)\eta(V_1)\zeta\}, \end{aligned} \tag{1}$$

for all vector fields V_1, V_2, V_3 on M. It is denoted by $M(f_1, f_2, f_3)$. If $f_1 = \dfrac{c+3}{4}$, $f_2 = f_3 = \dfrac{c-1}{4}$, then a GSSF with Sasakian structure becomes Sasakian space-from.

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2. Preliminaries

Let $(\overline{M})$ be an odd dimensional manifold provided with almost contact metric structure including ϕ, ζ, η and g are $(1,1)$ tensor field, vector field, 1-form and Riemannian metric respectively satisfying

$$1=\eta(\zeta),\ \phi^2 V_1=-V_1+\eta(V_1)\zeta,\ \phi\zeta=0, \tag{2}$$

$$g(\phi V_1,\phi V_2)=g(V_1,V_2)-\eta(V_1)\eta(V_2), \tag{3}$$

$$g(\phi V_1,V_2)+g(V_1,\phi V_2)=0, \eta(\phi V_1)=0, \tag{4}$$

$$(\overline{\nabla}_{V_1}\eta)(V_2)=g(\overline{\nabla}_{V_1}\zeta,V_2) \tag{5}$$

for all vector fields V_1, V_2. We have in a GSSF $M(f1_1, f_2, f_3)$,

$$(\nabla_{V_1}\phi)V_2=(f_1-f_3)[g(V_1,V_2)\zeta-\eta(V_2)V_1], \tag{6}$$

$$\nabla_{V_1}\zeta=-(f_1-f_3)\phi V_1, \tag{7}$$

$$R(\zeta,V_1)V_2=(f_1-f_3)\{g(V_1,V_2)\zeta-\eta(V_2)V_1\} \tag{8}$$

$$S(V_1,V_2)=(2nf_1+3f_2-f_3)g(V_1,V_2)-\{3f_2+(2n-1)f_3\}\eta(V_1)\eta(V_1), \tag{9}$$

$$S(V_1,\zeta)=2n(f_1-f_3)\eta(V_1), \tag{10}$$

for all $V_1, V_2 \in TM$ and ∇ is Levi-Civita connection M.

Let M be a submanifold submerged in an odd dimensional contact metric manifold $\overline{M}$ convince with metric g. The tangent bundle and the set of vector fields normal to M are TM and $T^{\perp}M$, respectively. Weingarten and Gauss formula are specified by,

$$\overline{\nabla}_{V_1}V_2=\nabla_{V_1}V_2+h(V_1,V_2), \tag{11}$$

$$\overline{\nabla}_{V_1}N=\nabla^{\perp}_{V_1}N-A_N V_1, \tag{12}$$

for any $V_1, V_2 \in TM$, where $\nabla^{\perp}$ is the normal bundle connection and $N \in T^{\perp}M$. The relation between second fundamental form h and A_N are specified by

$$g(A_N V_1,V_2)=g(h(V_1,V_2),N), \tag{13}$$

for any $V_1, V_2 \in \Gamma(TM)$, $N \in T^{\perp}M$.

The submanifold is called totally geodesic when $h = 0$, which suggests that the geodesics in M are also geodesics in $\overline{M}$. We have $(0, K+2)$-type tensor field $Q(F,T)$ interpret as follows

$$\begin{aligned} &Q(F,T)(E_1,E_2,E_3,\ldots,E_K;E,Y)\\ &\quad=-T((E\wedge_F Y)E_1,E_2,E_3,\ldots,E_K)-T(E_1,(E\wedge_F Y)E_2,E_3,\ldots,E_K)\\ &\quad-\cdots-T(E_1,E_2,E_3,\ldots,E_{K-1},(E\wedge_F Y)E_K), \end{aligned} \tag{14}$$

where $(E \wedge_F Y)Z = F(Y,Z)E - F(E,Z)Y$.

A submanifold of a Riemannian manifold $\overline{M}$ is called pseudo-parallel which satisfies the condition

$$\overline{R}(V_1, V_2) \cdot h = f_h Q(g, h), \tag{15}$$

where (15) can be written as

$$\begin{aligned} R^{\perp}(V_1, V_2)h(U, W) - h(R(V_1, V_2)U, W) - h(U, R(V_1, V_2)W) \\ = -f_h[h((V_1 \wedge_g V_2), W) + h(U, (V_1 \wedge_g V_2)W)], \end{aligned} \tag{16}$$

for all vector field V_1, V_2, U, W tangent to M. If $\overline{R}(V_1, V_2) \cdot h = 0$, then submanifold M is called semiparallel. In [4], the authors defined pleasing the condition

$$\overline{R}(V_1, V_2) \cdot h = f_s Q(S, h) \tag{17}$$

This type of submanifold is called Ricci generalized Pseudo-parallel, where (17) given by

$$\begin{aligned} R^{\perp}(V_1, V_2)h(U, W) - h(R(V_1, V_2)U, W) - h(U, R(V_1, V_2)W) \\ = -f_s[h((V_1 \wedge_s V_2), W) + h(U, (V_1 \wedge_s V_2)W)], \end{aligned} \tag{18}$$

for all vector fields V_1, V_2, U, W tangent to M. In an invariant submanifold of GSSF, N is identically zero, i.e.,

$$h(V_1, \zeta) = 0, \tag{19}$$

for any vector field V_1 tangent to M.

In an $(2n+1)$-dimensional Riemannian manifold, the concircular curvature tensor C, M-projective curvature tensor M and conformal curvature tensor V^* are defined as follows:

$$C(V_1, V_2)V_3 = R(V_1, V_2)V_3 - \left(\frac{r}{2n(2n+1)}\right)[g(V_2, V_3)V_1 - g(V_1, V_3)V_2], \tag{20}$$

$$\begin{aligned} M(V_1, V_2)V_3 =& R(V_1, V_2)V_3 - \frac{1}{4n}[S(V_2, V_3)V_1 - S(V_1, V_3)V_2 \\ &+ g(V_2, V_3)QV_1 - g(V_1, V_3)QV_2], \end{aligned} \tag{21}$$

$$\begin{aligned} V^*(V_1, V_2)V_3 =& R(V_1, V_2)V_3 - \frac{1}{2n-1}[S(V_2, V_3)V_1 - S(V_1, V_3)V_2 \\ &+ g(V_2, V_3)QV_1 - g(V_1, V_3)QV_2] \\ &+ \frac{r}{2n(2n-1)}[g(V_2, V_3)V_1 - g(V_1, V_3)V_2], \end{aligned} \tag{22}$$

where a and b are constants.

3. Concircular pseudo parallel invariant submanifold of GSSF

Definition 3.1. *An invariant submanifold of GSSF $\overline{M}$ is said to be a concircular pseudo parallel invariant submanifold of GSSF if it satisfies the condition*

$$C(V_1, V_2).h = f_h Q(g, h) \tag{23}$$

Theorem 3.1. *Let M be a concircular pseudo parallel invariant submanifold of GSSF $\overline{M}$. Then M is either totally geodesic or $f_h = (f_1 - f_3) - \dfrac{r}{2n(2n+1)}$.*

Proof. By the definition of concircular pseudo parallel invariant submanifold of GSSF and equation (16), we have

$$\begin{aligned} R^{\perp}(V_1, V_2)h(U, W) - h(C(V_1, V_2)U, W) - h(U, C(V_1, V_2)W) \\ = -f[h((V_1 \wedge_g V_2), W) + h(U, (V_1 \wedge_g V_2)W)] \end{aligned} \tag{24}$$

Putting $V_1 = W = \zeta$ and using (14), we have

$$\begin{aligned} & R^{\perp}(\zeta, V_2)h(U, \zeta) - h(C(\zeta, V_2)U, \zeta) - h(U, C(\zeta, V_2)\zeta) \\ & = -f_h[g(V_2, U)h(\zeta, \zeta) - g(\zeta, U)h(V_2, \zeta) + g(V_2, \zeta)h(U, \zeta) - g(\zeta, \zeta)h(U, V_2)] \end{aligned} \tag{25}$$

Substituting (19) and (20) in (25), we get,

$$\left[(f_1 - f_3) - \frac{r}{2n(2n+1)} - f_h\right] h(U, V_2) = 0 \tag{26}$$

This implies that, $f_h = (f_1 - f_3) - \dfrac{r}{2n(2n+1)}$ or $h(U, V_2) = 0$. This proves our assertion. □

From Theorem 3.1, the following corollaries are immediate:

Corollary 3.2. *If M is a concircular pseudo parallel invariant submanifold of GSSF $\overline{M}$ such that $f_h \neq (f_1 - f_3) - \dfrac{r}{2n(2n+1)}$, then M is a totally geodesic.*

Corollary 3.3. *If M is a concircular pseudo parallel invariant submanifold of GSSF $\overline{M}$, then M is not a totally geodesic provided $f_h = (f_1 - f_3) - \dfrac{r}{2n(2n+1)}$.*

4. M-projective pseudo parallel invariant submanifold of GSSF

Definition 4.1. *An invariant submanifold of GSSF $\overline{M}$ is said to be an M-projective pseudo parallel invariant submanifold of GSSF if it satisfies the condition*

$$M(V_1, V_2) \cdot h = f_h Q(g, h) \tag{27}$$

Theorem 4.1. *Let M be an M-projective pseudo parallel invariant submanifold of GSSF $\overline{M}$. Then M is either totally geodesic or*

$$f_h = \frac{1}{4n}\left[(-2n+1)f_3 - 3f_2\right].$$

Proof. By the definition of M-projective pseudo parallel invariant submanifold of GSSF and equation (16), we have

$$R^{\perp}(V_1,V_2)h(U,W) - h(M(V_1,V_2)U,W) - h(U,M(V_1,V_2)W) \\ = -f_h[h((V_1 \wedge_g V_2),W) + h(U,(V_1 \wedge_g V_2)W)] \tag{28}$$

Putting $V_1 = W = \zeta$ and using (14), we have

(29)

$$R^{\perp}(\zeta,V_2)h(U,\zeta) - h(M(\zeta,V_2)U,\zeta) - h(U,M(\zeta,V_2)\zeta) \\ = -f_h[g(V_2,U)h(\zeta,\zeta) - g(\zeta,U)h(V_2,\zeta) + g(V_2,\zeta)h(U,\zeta) - g(\zeta,\zeta)h(U,V_2)]$$

Using (19) and (21) in (29), we come by

$$\left[(f_1 - f_3) - \frac{1}{4n}(4nf_1 - (2n+1)f_3 + 3f_2) - f_h\right] h(U,V_2) = 0 \tag{30}$$

This implies that, $f_h = \dfrac{1}{4n}\left((-2n+1)f_3 - 3f_2\right)$ or $h(U,V_2) = 0$. □

From Theorem 4.1, the following corollaries are immediate:

Corollary 4.2. *If M is a M-projective pseudo parallel invariant submanifold of GSSF $\overline{M}$ such that $f_h \neq \dfrac{1}{4n}\left((-2n+1)f_3 - 3f_2\right)$, then M is a totally geodesic.*

Corollary 4.3. *If M is a M-projective pseudo parallel invariant submanifold of GSSF $\overline{M}$, then M is not a totally geodesic provided*

$$f_h = \frac{1}{4n}\left((-2n+1)f_3 - 3f_2\right).$$

5. Conformal pseudo parallel invariant submanifold of GSSF

Definition 5.1. *An invariant submanifold of GSSF $\overline{M}$ is said to be a conformal pseudo parallel invariant submanifold of GSSF if it satisfies the condition*

$$V^*(V_1,V_2).h = f_h Q(g,h) \tag{31}$$

Theorem 5.1. *Let M be a conformal pseudo parallel invariant submanifold of GSSF $\overline{M}$. Then M is either totally geodesic or*

$$f_h = \frac{1}{2n-1}\left((-2n+1)f_1 - 3f_2 + 2f_3 - \frac{r}{2n}\right).$$

Proof. By the definition of conformal pseudo parallel invariant submanifold of GSSF and equation (16), then we have

$$R^{\perp}(V_1,V_2)h(U,W) - h(V^*(V_1,V_2)U,W) - h(U,V^*(V_1,V_2)W) \\ = -f_h[h((V_1 \wedge_g V_2),W) + h(U,(V_1 \wedge_g V_2)W)] \tag{32}$$

Also, by putting $V_1 = W = \zeta$ and using (14), we infer that

(33)

$$R^{\perp}(\zeta,V_2)h(U,\zeta) - h(V^*(\zeta,V_2)U,\zeta) - h(U,V^*(\zeta,V_2)\zeta) \\ = -f_h[g(V_2,U)h(\zeta,\zeta) - g(\zeta,U)h(V_2,\zeta) + g(V_2,\zeta)h(U,\zeta) - g(\zeta,\zeta)h(U,V_2)]$$

Substituting (19) and (22) in (33), we have

$$\Big[(f_1 - f_3) - \frac{1}{2n-1}[4nf_1 - (2n+1)f_3 + 3f_2]v - \frac{r}{2n(2n-1)} - f_h\Big]h(U, V_2) = 0 \tag{34}$$

This implies that $f_h = \frac{1}{2n-1}\left((-2n+1)f_1 - 3f_2 + 2f_3 - \frac{r}{2n}\right)$ or $h(U, V_2) = 0$. This completes the proof. □

From Theorem 5.1, the following corollaries are immediate:

Corollary 5.2. *If M is a conformal pseudo parallel invariant submanifold of GSSF $\overline{M}$ such that $f_h \neq \frac{1}{2n-1}\left[(-2n+1)f_1 - 3f_2 + 2f_3 - \frac{r}{2n}\right]$, then M is a totally geodesic.*

Corollary 5.3. *If M is a conformal pseudo parallel invariant submanifold of GSSF $\overline{M}$, then M is not a totally geodesic provided*

$$f_h = \frac{1}{2n-1}\left[(-2n+1)f_1 - 3f_2 + 2f_3 - \frac{r}{2n}\right].$$

6. Concircular Ricci generalized pseudo parallel invariant submanifold of GSSF

Definition 6.1. *An invariant submanifold of GSSF $\overline{M}$ is called concircular Ricci generalized pseudo parallel invariant submanifold of GSSF if it satisfies the condition*

$$C(V_1, V_2).h = f_s Q(S, h) \tag{35}$$

Theorem 6.1. *Let M be a concircular Ricci generalized pseudo parallel invariant submanifold of GSSF $\overline{M}$. Then M is either totally geodesic or*

$$f_s = \frac{1}{2n} - \frac{r}{4n^2(2n+1)(f_1 - f_3)}.$$

Proof. By the definition of concircular Ricci generalized pseudo parallel invariant submanifold of GSSF and equation (18), we have

$$\begin{aligned} &R^{\perp}(V_1, V_2)h(U, W) - h(C(V_1, V_2)U, W) - h(U, C(V_1, V_2)W) \\ &= -f_s[h((X \wedge_S V_2), W) + h(U, (V_1 \wedge_S V_2)W)] \end{aligned} \tag{36}$$

Replacing $V_1 = W = \zeta$ and making use of (14), we get

$$\begin{aligned} &R^{\perp}(\zeta, V_2)h(U, \zeta) - h(C(\zeta, V_2)U, \zeta) - h(U, C(\zeta, V_2)\zeta) = \\ &- f_s[S(V_2, U)h(\zeta, \zeta) - S(\zeta, U)h(V_2, \zeta) + S(V_2, \zeta)h(U, \zeta) - S(\zeta, \zeta)h(U, V_2)] \end{aligned} \tag{37}$$

Using (19), (10), (20) in (37), we obtain

$$\left[(1 - 2nf_s)(f_1 - f_3) - \frac{r}{2n(2n+1)}\right]h(U, V_2) = 0. \tag{38}$$

Equation (38) implies that $f_s = \frac{1}{2n} - \frac{r}{4n^2(2n+1)(f_1 - f_3)}$ or $h(U, V_2) = 0$. Hence the proof. □

From Theorem 6.1, the following corollaries are immediate:

Corollary 6.2. *If M is concircular Ricci generalized pseudo parallel invariant submanifold of GSSF $\overline{M}$ such that $f_s \neq \frac{1}{2n} - \frac{r}{4n^2(2n+1)(f_1-f_3)}$, then M is a totally geodesic.*

Corollary 6.3. *If M is a concircular Ricci generalized pseudo parallel invariant submanifold of GSSF $\overline{M}$, then M is not a totally geodesic provided*

$$f_s = \frac{1}{2n} - \frac{r}{4n^2(2n+1)(f_1-f_3)}.$$

7. M-projective Ricci generalized pseudo parallel invariant submanifold of GSSF

Definition 7.1. *An invariant submanifold of GSSF $\overline{M}$ is called M-projective Ricci generalized pseudo parallel invariant submanifold of GSSF if it satisfies the condition*

$$M(V_1, V_2) \cdot h = f_s Q(S, h) \tag{39}$$

Theorem 7.1. *Let M be an M-projective Ricci generalized pseudo parallel invariant submanifold of GSSF $\overline{M}$. Then M is either totally geodesic or*

$$f_s = \frac{1}{8n^2(f_1-f_3)}[(-2n+1)f_3 - 3f_2].$$

Proof. By the definition of M-projective Ricci generalized pseudo parallel invariant submanifold of GSSF and equation (18), we have

$$\begin{aligned} &R^{\perp}(V_1, V_2)h(U, W) - h(M(V_1, V_2)U, W) - h(U, M(V_1, V_2)W) \\ &\quad = -f_s[h((V_1 \wedge_S V_2), W) + h(U, (V_1 \wedge_S V_2)W)] \end{aligned} \tag{40}$$

Also, putting $V_1 = W = \zeta$ and using (14), we have

(41)

$$\begin{aligned} &R^{\perp}(\zeta, V_2)h(U, \zeta) - h(M(\zeta, V_2)U, \zeta) - h(U, M(\zeta, V_2)\zeta) = \\ &- f_s[S(V_2, U)h(\zeta, \zeta) - S(\zeta, U)h(V_2, \zeta) + S(V_2, \zeta)h(U, \zeta) - S(\zeta, \zeta)h(U, V_2)] \end{aligned}$$

By virtue of (19), (10), (21) and (41), we obtain

$$\left[(1-2nf_s)(f_1-f_3) - \frac{1}{4n}(4nf_1 - (2n+1)f_3 + 3f_2)\right] h(U, V_2) = 0 \tag{42}$$

This implies that $f_s = \frac{1}{8n^2(f_1-f_3)}[(-2n+1)f_3 - 3f_2]$ or $h(U, V_2) = 0$. This completes the proof. □

From Theorem 7.1, the following corollaries are immediate:

Corollary 7.2. *If M is a M-projective Ricci generalized pseudo parallel invariant submanifold of GSSF $\overline{M}$ such that $f_s \neq \frac{1}{8n^2(f_1-f_3)}[(-2n+1)f_3 - 3f_2]$, then M is a totally geodesic.*

Corollary 7.3. *If M is a M-projective Ricci generalized pseudo parallel invariant submanifold of GSSF $\overline{M}$, then M is not a totally geodesic provided*

$$f_s = \frac{1}{8n^2(f_1 - f_3)}[(-2n+1)f_3 - 3f_2].$$

8. Conformal Ricci generalized pseudo parallel invariant submanifold of GSSF

Definition 8.1. *An invariant submanifold of GSSF $\overline{M}$ is called conformal Ricci generalized pseudo parallel invariant submanifold of GSSF if it satisfies the condition*

$$V^*(V_1, V_2) \cdot h = f_s Q(S, h). \tag{43}$$

Theorem 8.1. *Let M be a conformal Ricci generalized pseudo parallel invariant submanifold of GSSF $\overline{M}$. Then M is either totally geodesic or*

$$\begin{aligned} f_s = \frac{1}{2n} - \frac{1}{2n(2n-1)(f_1 - f_3)}[4nf_1 - (2n+1)f_3 + 3f_2] \\ - \frac{r}{4n^2(2n-1)(f_1 - f_3)} \end{aligned} \tag{44}$$

Proof. By the definition of conformal Ricci generalized pseudo parallel invariant submanifold of GSSF and equation (18), we have

$$\begin{aligned} R^{\perp}(V_1, V_2)h(U, W) - h(V^*(V_1, V_2)U, W) - h(U, V^*(V_1, V_2)W) \\ = -f_s[h((V_1 \wedge_S V_2), W) + h(U, (V_1 \wedge_S V_2)W)] \end{aligned} \tag{45}$$

By putting $V_1 = W = \zeta$ and make use of (14) we get,

$$\begin{aligned} & R^{\perp}(\zeta, V_2)h(U, \zeta) - h(V^*(\zeta, V_2)U, \zeta) - h(U, V^*(\zeta, V_2)\zeta) = \\ & - f_s[S(V_2, U)h(\zeta, \zeta) - S(\zeta, U)h(V_2, \zeta) + S(V_2, \zeta)h(U, \zeta) - S(\zeta, \zeta)h(U, V_2)] \end{aligned} \tag{46}$$

Making use of (19), (10), (22) in (46), we have

$$\begin{aligned} \Big[(1 - 2nf_s)(f_1 - f_3) - \frac{1}{2n-1}(4nf_1 - (2n+1)f_3 + 3f_2) \\ - \frac{r}{2n(2n-1)}\Big]h(U, V_2) = 0 \end{aligned} \tag{47}$$

which implies that

$$\begin{aligned} f_s = \frac{1}{2n} - \frac{1}{2n(2n-1)(f_1 - f_3)}[4nf_1 - (2n+1)f_3 + 3f_2] \\ - \frac{r}{4n^2(2n-1)(f_1 - f_3)} \end{aligned}$$

or $h(U, V_2) = 0$. This completes the proof. □

From Theorem 8.1, the following corollaries are immediate:

Corollary 8.2. *If M is a conformal Ricci generalized pseudo parallel invariant submanifold of GSSF $\overline{M}$ such that*

$$f_s \neq \frac{1}{2n} - \frac{1}{2n(2n-1)(f_1 - f_3)}[4nf_1 - (2n+1)f_3 + 3f_2] - \frac{r}{4n^2(2n-1)(f_1 - f_3)},$$

then M is a totally geodesic.

Corollary 8.3. *If M is a conformal Ricci generalized pseudo parallel invariant submanifold of GSSF $\overline{M}$, then M is not a totally geodesic provided*

$$f_s = \frac{1}{2n} - \frac{1}{2n(2n-1)(f_1 - f_3)}[4nf_1 - (2n+1)f_3 + 3f_2] - \frac{r}{4n^2(2n-1)(f_1 - f_3)}.$$

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