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DEGREE DISTANCE INDEX OF GENERALIZED COMPLEMENTS OF GRAPHS

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Abstract. The topological indices are the numerical parameters associated with the graph which are usually graph invariant. The topological indices are classified based on the properties of graphs. The degree distance index is the topological index which is calculated by counting the degrees and distance between the vertices. In this paper, the degree distance indices of generalized complements of graphs are calculated.

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1. Introduction

Let G be a simple connected graph with vertex set $V(G)$. The complement of a graph is obtained by removing all the edges of G and adding the edges which are not in G. Let $P = \{V_1, V_2, \ldots, V_k\}$ be a partition of $V(G)$ of order $k \geq 2$. For all V_r and V_s in P, $r \neq s$, remove the edges between V_r and V_s in G and add the edges between V_r and V_s which are not in G. The resulting graph G_k^P is called $k-$complement of graph G with respect to the partition P. Let $P = \{V_1, V_2, \ldots, V_k\}$ be a partition of $V(G)$ of order $k \geq 2$. For each set V_r in P, remove the edges of G inside V_r and add the edges joining vertices of V_r in $\overline{G}$. The resulting graph $G_{k(i)}^P$ is called $k(i)-$complement of graph G with respect to the partition P [1].

Topological indices are the numerical parameters associated with the graph which are usually graph invariant. From this index it is possible to analyze mathematical values and further investigate some physicochemical properties of a molecule. Therefore, it is also called a molecular descriptor. The Wiener index was introduced by Harry Wiener [2] is the first topological index to be used in Chemistry to determine the boiling point of paraffin. Let $d(v_i, v_j)$ be the distance between the vertices v_i and v_j and $deg(v_i)$ be the degree of the vertex v_i.

The Wiener index of a graph G is,

$$W(G) = \sum_{1 \leq i < j \leq n} d(v_i, v_j).$$

The degree distance index was introduced by Dobrynin and Kochetova [3] which correlates degree distance with Wiener index. The degree distance of a graph G is defined as,

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$$DD(G) = \sum_{1\le i<j\le n} (deg(v_i) + deg(v_j))\, d(v_i, v_j).$$

The various upper bounds for degree distance index of a graph are found in [4]. The degree distance index of few product graphs are calculated in [5]. One can refer [6] for the degree distance index of derived graphs. The generalized degree distance of graphs can be found in [7, 8].

Motivated by the work on distance in complements of graphs [9], in this paper, the concept of degree distance index is extended to generalized complements of a graph. In order to compute degree distance index of G_k^P or $G_{k(i)}^P$, the graph G is partitioned in such a way that the resultant k or $k(i)$ - complement of G is a connected graph. For all the terminologies refer [10, 11].

2. Degree distance index of $k-$complements of graphs

We now find necessary condition for the connectedness of $k-$complement of graph G.

Lemma 2.1. *The $k-$complement of a connected graph G is connected, if $V(G)$ can be partitioned into $\{V_1, V_2, \dots, V_k\}$ such that there exist a path between every pair of vertices in V_i and a vertex v in V_i such that v is non-adjacent to at least one vertex of each partite V_j, $i \ne j$.*

Proof. Suppose $V(G)$ can be partitioned into $\{V_1, V_2, \dots, V_k\}$ such that there exist a path between every pair of vertices in V_i and a vertex v in V_i such that v is non-adjacent to at least one vertex of each partite V_j, $i \ne j$. Then, in G_k^P there exists a path between all the vertices of V_i and at least two vertices of every two different partites are adjacent. Hence G_k^P is connected. □

Proposition 2.2.

1. *Let $P = \{V_1, V_2, \dots, V_n\}$ be a partition of a cycle C_{2n} such that the graph induced by the vertex set V_i denoted as $\langle V_i\rangle$ is K_2. Then, $DD(C_{2n})_n^P = 8n^2(n-1)$.*
2. *Let $P = \{V_1, V_2, \dots, V_{n+1}\}$ be a partition of a cycle C_{2n+1} such that $\langle V_1\rangle = K_1$ and $\langle V_i\rangle = K_2$ for $i = 2, 3, \dots, n$. Then, $DD(C_{2n+1})_n^P = 8n^3 + 4n^2 - 2n - 4$.*

Proof.

(1) Let $P = \{V_1, V_2, \dots, V_n\}$ be a partition of C_{2n} such that each $\langle V_i\rangle$ is K_2. Note that $(C_{2n})_n^P$ is a $(2n-2)-$regular graph with n pairs of vertices of $(C_{2n})_n^P$ are at distance 2 and remaining $\binom{2n}{2} - n$ pairs of vertices are at distance 1.

Therefore,

$$\begin{aligned} DD(C_{2n})_n^P &= n(2n-2+2n-2)(2) + 2n(n-1)(2n-2+2n-2)(1) \\ &= 8n^2(n-1). \end{aligned}$$

(2) Let $P = \{V_1, V_2, \dots, V_{n+1}\}$ be a partition of C_{2n+1} such that $\langle V_1\rangle = K_1$ and $\langle V_i\rangle = K_2$ for $i = 2, 3, \dots, n$. Note that one vertex of degree $2n-2$ and remaining $2n$ vertices of degree $2n-1$. The $n+1$ pairs of vertices of $(C_{2n+1})_{(n+1)}^P$ are at distance 2 and remaining $\binom{2n+1}{2} - (n+1)$ pairs of vertices are at distance 1.

Therefore,

$$
\begin{aligned}
DD(C_{2n+1})^P_{n+1} &= 2(2n-2+2n-1)(2) + (2n-2)(2n-2+2n-1)(1) \\
&+ (n-1)(2n-1+2n-1)(2) + (2n-1)(2n-1+2n-1)(1) \\
&+ 2(2n-1+2n-1)((2n-3)+(2n-5)+\cdots+1) \\
&= 8n^3+4n^2-2n-4.
\end{aligned}
$$

□

Proposition 2.3. *(1) Let $P = \{V_1, V_2, \ldots, V_n\}$ be a partition of the path P_{2n} such that $\langle V_i\rangle = P_2$ for all $i = 1, 2, \ldots, n$. Then, $DD(P_{2n})^P_n = 8n^3 - 8n^2 + 2$.*

(2) Let $P = \{V_1, V_2, \ldots, V_{n+1}\}$ be a partition of the path $v_1, v_2, \ldots, v_{2n+1}$ such that $\langle V_i\rangle = P_2$, $i = 1, 2, \ldots, n$ and $\langle V_{n+1}\rangle = \{v_{2n+1}\}$. Then, $DD(P_{2n+1})^P_{n+1} = 8n^3 + 4n^2 - 2n$.

(3) Let $P = \{V_1, V_2, \ldots, V_{n+1}\}$ be a partition of the path $v_1, v_2, \ldots, v_{2n+1}$ such that $\langle V_i\rangle = P_2$ except one $\langle V_i\rangle = \{v_i\}$, $i \neq 1, 2n+1$. Then, $DD(P_{2n+1})^P_{n+1} = 8n^3 + 4n^2 - 2n - 2$.

Proof. (1) Let $P = \{V_1, V_2, \ldots, V_n\}$ be a partition of P_{2n} such that $\langle V_i\rangle = P_2$ for all $i = 1, 2, \ldots, n$. Observe that two vertices have degree $(2n-1)$ and remaining $(2n-2)$ vertices have degree $(2n-2)$. Now, $(n-1)$ pairs of vertices of $(P_{2n})^P_n$ are at distance 2 and remaining $\binom{2n}{2} - (n-1)$ pairs of vertices are at distance 1. Therefore,

$$
\begin{aligned}
DD(P_{2n})^P_n &= (1)(2n-1+2n)(1) + (2n-2)(2n-1+2n-2)(1) \\
&+ (2n-2)(2n-2+2n-1)(1) + (n-1)(2n-2+2n-2)(2) \\
&+ (2)(2n-2+2n-2)((2n-4)+(2n-6)+\cdots+2)(1) \\
&= 8n^3 - 8n^2 + 2.
\end{aligned}
$$

(2) Let $P = \{V_1, V_2, \ldots, V_{n+1}\}$ be a partition of P_{2n+1} such that $\langle V_i\rangle = P_2$, $i = 1, 2, \ldots, n$ and $\langle V_{n+1}\rangle = \{v_{2n+1}\}$. Then, $(P_{2n+1})^P_{n+1}$ has one vertex of degree $2n$ and remaining vertices of degree $2n-1$. Also n pairs of vertices of $(P_{2n+1})^P_{(n+1)}$ are at distance 2 and remaining $\binom{2n+1}{2} - n$ pairs of vertices are at distance 1. Therefore,

$$
\begin{aligned}
DD(P_{2n+1})^P_{n+1} &= (1)(2n-1+2n)(1) + (2n-1)(2n-1+2n)(1) \\
&+ (2n-2)(2n-1+2n-1)(1) + (n)(2n-1+2n-1)(2) \\
&+ (2n-2)(2n-1+2n-1) + (2)(2n-1+2n-1)((2n-4) \\
&+ (2n-6)+\cdots+2) \\
&= 8n^3 + 4n^2 - 2n.
\end{aligned}
$$

(3) Let $P = \{V_1, V_2, \ldots, V_{n+1}\}$ be a partition of P_{2n+1} such that $\langle V_i\rangle = P_2$ except one $\langle V_i\rangle = \{v_i\}$, $i \neq 1, 2n+1$. Then, $(P_{2n+1})^P_{n+1}$ has two vertices of degree $2n$, one vertex of degree $2n-2$ and remaining $2n-2$ vertices of degree $2n-1$. So, n pairs of vertices of $(P_{2n+1})^P_{(n+1)}$ are at distance 2 and remaining $\binom{2n+1}{2} - n$ pairs of vertices are at

distance 1. Therefore,

$$\begin{aligned}DD(P_{2n+1})^P_{n+1} &= 1(2n+2n)(1)+4(2n+2n-1)(1)\\&+2(2n+2n-2)(1)+2(2n-4)(2n+2n-1)(1)\\&+2(2n-1+2n-2)(2)+(2n-4)(2n-1+2n-2)(1)\\&+(n-2)(2n-1+2n-1)(2)\\&+(2n^2-6n+5)(2n-1+2n-1)(1)\\&=8n^3+4n^2-2n-2.\end{aligned}$$

□

Proposition 2.4. *Let $P=\{V_1,V_2,\ldots,V_m\}$ be a partition of complete bipartite graph $K_{m,n}$ such that $\langle V_1\rangle = K_{1,n-m+1}$ and $\langle V_i\rangle = K_2$ for $i = 2,3,\ldots,m$. Then, $DD(K_{m,n})^P_m$=$mn(3m+3n-4)$.*

Proof. Let $P=\{V_1,V_2,\ldots,V_m\}$ be a partition $K_{m,n}$ such that $\langle V_1\rangle = K_{1,n-m+1}$ and $\langle V_i\rangle = K_2$ for $i=2,3,\ldots,m$. Let u, v be two vertices of $K_{m,n}$.

Table 1. Degree distance of $k-$complement of $K_{m,n}$

Number of pairs of vertices	**deg(u)+ deg(v)**	**d(u,v)**	**DD(u,v)**
$\binom{m-1}{2}$	$2m$	1	m^3-3m^2+2m
$m-1$	$m+n$	1	$m^2+mn-m-n$
$\binom{n-1}{2}$	$2n$	1	$m^2n-3mn+2n$
$(m-1)(n-m+1)$	$m+n$	1	$mn^2-n^2-m^3+m^2$
$m-1$	$m+n$	1	$m^2+mn-m-n$
$n-m+1$	$m+n$	1	$n^2-m^2+m+n+2m^3-6m^2$
$\binom{n-m+1}{2}$	$2m$	2	$2m^3+2mn^2-4m^2n+3mn-m^2-m-n$
$m-1$	$2n$	2	$4mn-4n$
$(m-1)(m-2)$	$m+n$	2	$2m^2n+4m+4n-6mn$

Thus, $DD(K_{m,n})^P_m = mn(3m+3n-4)$.

□

3. Degree distance index of $k(i)-$complements of graphs

We shall now find the necessary condition for $G^P_{k(i)}$ to be connected whenever G is connected graph.

Lemma 3.1. *The $k(i)-$complement of a connected graph G is connected, if $V(G)$ can be partitioned into $\{V_1,V_2,\ldots,V_k\}$ such that no two vertices of V_i are adjacent.*

Proof. Suppose $\{V_1, V_2, \dots, V_k\}$ is a partition of a graph G such that no two vertices of V_i are adjacent. Then, the vertices of V_i form a complete subgraph in $G^P_{k(i)}$. Since G is connected, there exists a path between the vertices of different partites. Hence $G^P_{k(i)}$ is connected. □

Proposition 3.2. *Let $P = \{V_1, V_2, \dots, V_n\}$ be a partition of a cycle C_{2n} such that each $\langle V_i\rangle = \langle u, v\rangle$ in which $d(u, v) = n$. Then,*

(1) $DD(C_{4l})^P_{2l(i)} = 12l(2l^2 + 2l - 1)$, *if* $n = 2l$
(2) $DD(C_{4l+2})^P_{(2l+1)(i)} = 6(4l^3 + 10l^2 + 6l + 1)$, *if* $n = 2l + 1$.

Proof. If $P = \{V_1, V_2, \dots, V_n\}$ is a partition of C_{2n} such that each partite has two non adjacent vertices with distance between them is n. Then, $(C_{2n})^P_{n(i)}$ is a $3-$regular graph.

(1) Suppose $n = 2l$, $6l$ pairs of vertices are at distance 1, $8l$ pairs of vertices are at distance 2, $8l$ pairs of vertices are at distance $3, 4, \dots, 8l$ pairs of vertices are at distance l. Thus,

$$\begin{aligned} DD(C_{4l})^P_{2l(i)} &= 6l(6)(1) + 8l(6)(2) + \cdots + 8l(6)(l) \\ &= 12l(2l^2 + 2l - 1). \end{aligned}$$

(2) Suppose $n = 2l + 1$, $3(2l + 1)$ pairs of vertices are at distance 1, $4(2l+1)$ pairs of vertices are at distance 2, $4(2l+1)$ pairs of vertices are at distance $3, 4, \dots, 4(2l+1)$ pairs of vertices are at distance $l-1$ and $2(2l + 1)$ pairs of vertices are at distance l. Thus

$$\begin{aligned} DD(C_{4l+2})^P_{(2l+1)(i)} &= 3(2l + 1)(6)(1) + 4(2l + 1)(6)(2) + \cdots \\ &\quad + 4(2l + 1)(6)(l - 1) + 2(2l + 1)(6)(l) \\ &= 6(4l^3 + 10l^2 + 6l + 1). \end{aligned}$$

□

Proposition 3.3. *Let $P = \{V_1, V_2, \dots, V_{n+1}\}$ be a partition of cycle C_{2n+1} such that $\langle V_1\rangle = K_1$ and $\langle V_i\rangle = \langle v_i, v_j\rangle$, $i = 2, 3, \dots, 2n + 1$, in which $d(v_i, v_j) = n$. Then,*

(1) $DD(C_{4l+1})^P_{(2l+1)(i)} = 24l^3 + 37l^2 + 11l$, *if* $n = 2l$
(2) $DD(C_{4l+3})^P_{(2l+2)(i)} = 24l^3 + 73l^2 + 69l + 16$, *if* $n = 2l + 1$.

Proof. (1) Suppose $n = 2l$. Let $v_1, v_2, v_3, \dots, v_{4l}, v_{4l+1}$ be the vertices of C_{4l+1}. Let $P = \{V_1, V_2, \dots, V_{2l+1}\}$ be a partition of C_{4l+1} such that $\langle V_1\rangle = \{v_1\}$, $\langle V_i\rangle = \langle v_i, v_{i+l}\rangle$, $2 \le i \le \frac{2l+1}{2}$. In this partition one partite consists of single vertex and remaining partites consist two vertices which are at distance $2l$.

TABLE 2. Distance between pair of vertices of $k-$complement of C_{4l+1}

deg(v_i) + deg(v_j)	**Number of pairs**	**Pair of vertices**	$\begin{bmatrix} \textbf{distance} \\ \textbf{Number of pairs} \end{bmatrix}$
5	$4l$	(v_1, v_i), $2 \leq i \leq 4l+1$	$\begin{bmatrix} 1 & 2 & 3 & 4 & \dots & l & l+1 \\ 2 & 4 & 4 & 4 & \dots & 4 & 2 \end{bmatrix}$
6	$3l^2$	(v_i, v_{i+j}), $2 \leq i \leq 3l+1$, $1 \leq j \leq l$	$\begin{bmatrix} 1 & 2 & 3 & 4 & \dots & l \\ 3l & 3l & 3l & 3l & \dots & 3l \end{bmatrix}$
6	$2l^2$	(v_i, v_{i+j}), $2 \leq i \leq 2l+1$, $l+1 \leq j \leq 2l$	$\begin{bmatrix} l & l-1 & \dots & 3 & 2 & 1 \\ 2l & 2l & \dots & 2l & 2l & 2l \end{bmatrix}$
6	$\frac{l^2-l}{2}$	(v_i, v_{i+j}), $3l+2 \leq i \leq 4l$, $1 \leq j \leq l-1$ such that $i+j \leq 4l+1$	$\begin{bmatrix} 1 & 2 & 3 & \dots & l-2 & l-1 \\ l-1 & l-2 & l-3 & \dots & 2 & 1 \end{bmatrix}$
6	$\frac{l^2-l}{2}$	(v_i, v_{i+j}), $l+2 \leq i \leq 2l$, $2l+1 \leq j \leq \frac{3l-1}{2}$ such that $i+j \leq 4l+1$	$\begin{bmatrix} 2 & 3 & 4 & \dots & l-1 & l \\ l-1 & l-2 & l-3 & \dots & 2 & 1 \end{bmatrix}$
6	$2(\frac{l^2-l}{2})$	(v_i, v_{i+j}), (v_x, v_{x+y}), $2 \leq i \leq l$, $2l+2 \leq j \leq 3l$, $2l+2 \leq x \leq 3l$, $l+1 \leq y \leq 2l-1$ such that $i+j \leq 4l+1$	$\begin{bmatrix} 2 & 3 & 4 & \dots & l-1 & l \\ 1 & 2 & 3 & \dots & l-2 & l-1 \end{bmatrix}$
6	l^2	(v_i, v_{i+j}), $2 \leq i \leq l+1$, $2l+1 \leq j \leq 3l$ such that $i+j \leq 4l+1$	$\begin{bmatrix} 2 & 3 & 4 & \dots & l & l+1 \\ l & l & l & \dots & l & l \end{bmatrix}$

The degree distance between the vertices corresponding to first row of the table is $10l^2 + 20l$. The degree distance between the vertices of next rows are $9l^3 + 9l^2$, $6l^3 + 6l^2$, $l^3 - l$, $l^3 + 3l^2 - 4l$, $4l^3 - 4l$ and $3l^3 + 9l^2$ respectively. Hence $DD(C_{4l+1})^P_{(2l+1)(i)} = 24l^3 + 37l^2 + 11l$.

(2) Suppose $n = 2l+1$. Let $v_1, v_2, v_3, \dots, v_{4l+2}, v_{4l+3}$ be the vertices of C_{4l+3}. Let $P = \{V_1, V_2, \dots, V_{2l+2}\}$ be a partition of cycle such that $\langle V_1 \rangle = \{v_1\}$, $\langle V_i \rangle = \langle v_i, v_{i+l} \rangle$, $2 \leq i \leq l+1$. In this partition one

partite consists of single vertex and remaining partites consist of two vertices which are at distance $2l+1$.

TABLE 3. Distance between pair of vertices of $k-$complement of C_{4l+3}

$\mathbf{deg}(v_i)$ + $\mathbf{deg}(v_j)$	**Number of pairs**	**Pair of vertices**	$\begin{bmatrix}\textbf{distance}\\ \textbf{Number of pairs}\end{bmatrix}$
5	$4l+2$	(v_1, v_i), $2 \le i \le 4l+3$	$\begin{bmatrix} 1 & 2 & 3 & 4 & \dots & l & l+1 \\ 2 & 4 & 4 & 4 & \dots & 4 & 4 \end{bmatrix}$
6	$3l^2+4l$ $+1$	(v_i, v_{i+j}), $2 \le i \le 3l+2$, $1 \le j \le l+1$	$\begin{bmatrix} 1 & 2 & \dots & l & l+1 \\ 3l+1 & 3l+1 & \dots & 3l+1 & 3l+1 \end{bmatrix}$
6	$\frac{l^2+l}{2}$	(v_i, v_{i+j}), $3l+3 \le i \le 4l+2$, $1 \le j \le l$ such that $i+j \le 4l+3$	$\begin{bmatrix} 1 & 2 & 3 & \dots & l-2 & l-1 & l \\ l & l-1 & l-2 & \dots & 3 & 2 & 1 \end{bmatrix}$
6	$2l^2+l$	(v_i, v_{i+j}), $2 \le i \le 2l+2$, $\frac{2l+3}{2} \le j \le 2l+1$	$\begin{bmatrix} l & l-1 & \dots & 2 & 1 \\ 2l+1 & 2l+1 & \dots & 2l+1 & 2l+1 \end{bmatrix}$
6	$\frac{l^2-l}{2}$	(v_i, v_{i+j}), $2l+3 \le i \le 3l+1$, $l+2 \le j \le 2l$ such that $i+j \le 4l+3$	$\begin{bmatrix} 2 & 3 & 4 & \dots & l-2 & l-1 & l \\ 1 & 2 & 3 & \dots & l-3 & l-2 & l-1 \end{bmatrix}$
6	$\frac{l^2-l}{2}$	(v_i, v_{i+j}), $l+3 \le i \le 2l+1$, $2l+2 \le j \le \frac{3l}{2}$ such that $i+j \le 4l+3$	$\begin{bmatrix} 2 & 3 & 4 & \dots & l-1 & l \\ l-1 & l-2 & l-3 & \dots & 2 & 1 \end{bmatrix}$
6	l^2+l	(v_i, v_{i+j}), $2 \le i \le l+2$, $l+1 \le j \le 3l+1$	$\begin{bmatrix} 2 & 3 & \dots & l-1 & l & l+1 \\ l+1 & l+1 & \dots & l+1 & l+1 & l+1 \end{bmatrix}$
6	$\frac{l^2+l}{2}$	(v_i, v_{i+j}), $2 \le i \le l+1$, $3l+2 \le j \le 4l+1$ such that $i+j \le 4l+3$	$\begin{bmatrix} 2 & 3 & 4 & \dots & l-1 & l & l+1 \\ 1 & 2 & 3 & \dots & l-2 & l-1 & l \end{bmatrix}$

The degree distance between the vertices corresponding to first row of the table is $10l^2+30l+10$. The degree distance between the vertices of next rows are $9l^3+30l^2+27l+6$, l^3+3l^2+2l, $6l^3+9l^2+3l$,

$2l^3-2l$, l^3+3l^2-4l, $3l^3+12l^2+9l$ and $2l^3+6l^2+4l$ respectively. Hence $DD(C_{4l+3})^P_{(2l+2)(i)}=24l^3+73l^2+69l+16$.

□

Proposition 3.4. *Let $P=\{V_1,V_2,\ldots,V_n\}$ be a partition of the path P_{2n} such that each $\langle V_i\rangle=\langle v_i,v_j\rangle$ in which $d(v_i,v_j)=n$. Then,*

(1) $DD(P_{4l})^P_{(2l+1)(i)}=24l^3+26l^2-20l+4$, *if* $n=2l$

(2) $DD(P_{4l+2})^P_{(2l+2)(i)}=24l^3+62l^2+30l+4$, *if* $n=2l+1$.

Proof. (1) Suppose $n=2l$. Let $P=\{V_1,V_2,\ldots,V_{2l}\}$ be a partition of P_{4l} such that $\langle V_i\rangle=\langle v_i,v_{i+l}\rangle$, $1\le i\le l$. In this partition each partite consists of two vertices which are at distance $2l$.

TABLE 4. Distance between pair of vertices of $k-$complement of P_{4l}

deg(v_i) + **deg**(v_j)	**Number of pairs**	**Pair of vertices**	$\begin{bmatrix}\textbf{distance}\\ \textbf{Number of pairs}\end{bmatrix}$
5	$4l$	(v_1,v_i), (v_j,v_{4l}), $2\le i\le 2l+1$, $2l\le j\le 4l-1$	$\begin{bmatrix}1&2&3&\ldots&l-2&l-1&l\\4&4&4&\ldots&4&4&4\end{bmatrix}$
5	$2l$	(v_1,v_i), (v_j,v_{4l}), $2l+2\le i\le 3l+1$, $l\le j\le 2l-1$	$\begin{bmatrix}2&3&4&\ldots&l-1&l&l+1\\2&2&2&\ldots&2&2&2\end{bmatrix}$
5	$2(l-2)$	(v_1,v_i), (v_j,v_{4l}), $3l+2\le i\le 4l-1$, $2\le j\le l-1$	$\begin{bmatrix}4&5&6&\ldots&l-1&l&l+1\\2&2&2&\ldots&2&2&2\end{bmatrix}$
4	1	(v_1,v_{4l})	3
6	$3l^2-4l$	(v_i,v_{i+j}), $2\le i\le 3l-1$, $1\le j\le l$	$\begin{bmatrix}1&2&\ldots&l-1&l\\3l-2&3l-2&\ldots&3l-2&3l-2\end{bmatrix}$
6	$\frac{l^2-l}{2}$	(v_i,v_{i+j}), $3l\le i\le 4l-1$, $1\le j\le l-1$ such that $i+j\le 4l+1$	$\begin{bmatrix}1&2&\ldots&l-2&l-1\\l-1&l-2&\ldots&2&1\end{bmatrix}$
6	$\frac{l^2-l}{2}$	(v_i,v_{i+j}), $2l\le i\le 3l-2$, $l+1\le j\le 2l-1$ such that $i+j\le 4l+1$	$\begin{bmatrix}2&3&4&\ldots&l-2&l-1&l\\1&2&3&\ldots&l-3&l-2&l-1\end{bmatrix}$
6	$2l^2-2l$	(v_i,v_{i+j}), $2\le i\le 2l-1$, $l+1\le j\le 2l$	$\begin{bmatrix}1&2&\ldots&l-1&l\\2l-2&2l-2&\ldots&2l-2&2l-2\end{bmatrix}$

$\mathbf{deg}(v_i)$ + $\mathbf{deg}(v_j)$	**Number of pairs**	**Pair of vertices**	$\begin{bmatrix} \textbf{distance} \\ \textbf{Number of pairs} \end{bmatrix}$
6	$\frac{l^2-l}{2}$	(v_i, v_{i+j}), $l-1 \leq i \leq 2l-2$, $2l+1 \leq j \leq 3l$ such that $i+j \leq 4l+1$	$\begin{bmatrix} 2 & 3 & 4 & \dots & l-1 & l & l+1 \\ l & l-1 & l-2 & \dots & 3 & 2 & 1 \end{bmatrix}$
6	l^2-2l	(v_i, v_{i+j}), $2 \leq i \leq l-2$, $2l+1 \leq j \leq 3l$	$\begin{bmatrix} 2 & 3 & \dots & l-1 & l & l+1 \\ l-3 & l-3 & \dots & l-3 & l-3 & l-3 \end{bmatrix}$
6	$\frac{2l^2-10l+12}{4}$	(v_i, v_{i+j}), $2 \leq i \leq l-2$, $3l+1 \leq j \leq 4l-3$ such that $i+j \leq 4l+1$	$\begin{bmatrix} 5 & 6 & 7 & \dots & l-1 & l & l+1 \\ 1 & 2 & 3 & \dots & l-5 & l-4 & l-3 \end{bmatrix}$

The degree distance between the vertices corresponding to first four rows of the Table 4 are $10l^2+10l$, $5l^2+15l$, $5l^2+15l-50$ and 12 respectively. The degree distance between the vertices of next rows are $9l^3+3l^2-6l$, l^3-l, $2l^3-2l$, $6l^3-6l$, l^3+6l^2+5l, $3l^3-27l$ and $2l^3-3l^2-23l+42$ respectively. Hence $DD(P_{4l})^P_{(2l+1)(i)} = 24l^3+26l^2-20l+4$.

(2) Suppose $n = 2l+1$. Let $P = \{V_1, V_2, \dots, V_{2l+1}\}$ be a partition of P_{4l+2} such that $\langle V_i \rangle = \langle v_i, v_{i+l} \rangle$, $1 \leq i \leq l$. In this partition each partite consists of two vertices which are at distance $2l+1$.

Table 5. Distance between pair of vertices of $k-$complement of P_{4l+2}

$\mathbf{deg}(v_i)$ + $\mathbf{deg}(v_j)$	**Number of pairs**	**Pair of vertices**	$\begin{bmatrix} \textbf{distance} \\ \textbf{Number of pairs} \end{bmatrix}$
5	$2(l+1)$	$(v_1, v_i), (v_j, v_{4l+2})$, $2 \leq i \leq l+2$, $3l+1 \leq j \leq 4l+1$	$\begin{bmatrix} 1 & 2 & 3 & \dots & l-1 & l & l+1 \\ 2 & 2 & 2 & \dots & 2 & 2 & 2 \end{bmatrix}$
5	$2l$	(v_1, v_i), (v_j, v_{4l+2}), $l+3 \leq i \leq 2l+2$, $2l+1 \leq j \leq 3l$	$\begin{bmatrix} 1 & 2 & 3 & \dots & l-2 & l-1 & l \\ 2 & 2 & 2 & \dots & 2 & 2 & 2 \end{bmatrix}$
5	$2(l+1)$	(v_1, v_i), (v_j, v_{4l+2}), $2l+3 \leq i \leq 3l+3$, $l \leq j \leq 2l$	$\begin{bmatrix} 2 & 3 & 4 & \dots & l & l+1 & l+2 \\ 2 & 2 & 2 & \dots & 2 & 2 & 2 \end{bmatrix}$

deg(v_i) + **deg**(v_j)	**Number of pairs**	**Pair of vertices**	$\begin{bmatrix} \textbf{distance} \\ \textbf{Number of pairs} \end{bmatrix}$
5	$2(l+1)$	(v_1, v_i), (v_j, v_{4l+2}), $3l+4 \leq i \leq 4l+1$, $2 \leq j \leq l-1$	$\begin{bmatrix} 4 & 5 & 6 & \dots & l-1 & l & l+1 \\ 2 & 2 & 2 & \dots & 2 & 2 & 2 \end{bmatrix}$
4	1	(v_1, v_{4l+2})	3
6	$3l^2+4l$	(v_i, v_{i+j}), $2 \leq i \leq 3l$, $1 \leq j \leq l+1$	$\begin{bmatrix} 1 & 2 & 3 & \dots & l-1 & l & l+1 \\ 3l & 3l & 3l & \dots & 3l & 3l & 3l \end{bmatrix}$
6	$\frac{l^2+l}{2}$	(v_i, v_{i+j}), $3l+1 \leq i \leq 4l$, $1 \leq j \leq l$ such that $i+j \leq 4l+3$	$\begin{bmatrix} 1 & 2 & 3 & \dots & l-2 & l-1 & l \\ l & l-1 & l-2 & \dots & 3 & 2 & 1 \end{bmatrix}$
6	$2l^2$	(v_i, v_{i+j}), $2 \leq i \leq 2l$, $l+2 \leq j \leq 2l+1$	$\begin{bmatrix} 1 & 2 & 3 & \dots & l-2 & l-1 & l \\ 2l & 2l & 2l & \dots & 2l & 2l & 2l \end{bmatrix}$
6	$\frac{l^2-l}{2}$	(v_i, v_{i+j}), $2l+1 \leq i \leq 3l-1$, $l+2 \leq j \leq 2l$ such that $i+j \leq 4l+3$	$\begin{bmatrix} l & l-1 & l-2 & \dots & 4 & 3 & 2 \\ l-1 & l-2 & l-3 & \dots & 3 & 2 & 1 \end{bmatrix}$
6	$\frac{l^2+l}{2}$	(v_i, v_{i+j}), $l-1 \leq i \leq 2l-1$, $2l+2 \leq j \leq 3l+2$ such that $i+j \leq 4l+3$	$\begin{bmatrix} 2 & 3 & 4 & \dots & l & l+1 & l+2 \\ l+1 & l & l-1 & \dots & 3 & 2 & 1 \end{bmatrix}$
6	$\frac{2l^2-2l+1}{2}$	(v_i, v_{i+j}), $2 \leq i \leq l-2$, $2l \leq j \leq 3l+2$	$\begin{bmatrix} 2 & 3 & \dots & l & l+1 & l+2 \\ l-3 & l-3 & \dots & l-3 & l-3 & l-3 \end{bmatrix}$
6	$\frac{l^2-5l+6}{2}$	(v_i, v_{i+j}), $2 \leq i \leq l-2$, $3l+3 \leq j \leq 4l-1$ such that $i+j \leq 4l+3$	$\begin{bmatrix} 5 & 6 & 7 & \dots & l-1 & l & l+1 \\ 1 & 2 & 3 & \dots & l-5 & l-4 & l-3 \end{bmatrix}$

The degree distance between the vertices corresponding to first five rows of the Table 5 are $5l^2+15l+10$, $5l^2+5l$, $5l^2+25l+20$, $5l^2+15l-50$ and 12 respectively. The degree distance between the vertices of next rows are $9l^3+24l^2+9l-6$, l^3+3l^2+2l, $6l^3+3l^2-3l$,

$2l^3-2l$, $l^3+9l^2+20l+12$, $3l^3+6l^2-33l-36$ and $2l^3-3l^2-23l+42$ respectively. Hence $DD(P_{4l+2})^P_{(2l+2)(i)} = 24l^3+62l^2+30l+4$.

□

Proposition 3.5. *Let $P = \{V_1, V_2, \ldots, V_{n+1}\}$ be a partition of path $v_1, v_2, \ldots, v_{2n+1}$ such that $\langle V_1\rangle = \{v_1\}$ and $\langle V_i\rangle = \langle v_i, v_j\rangle$, such that $d(v_i, v_j) = n$. Then,*

(1) $DD(P_{4l+1})^P_{(2l+1)(i)} = 24l^3+36l^2+16l-10$, *if* $n = 2l$

(2) $DD(P_{4l+3})^P_{(2l+2)(i)} = 24l^3+72l^2+76l+13$, *if* $n = 2l+1$.

Proof. (1) Suppose $n = 2l$. Let $v_1, v_2, \ldots, v_{4l+1}$ be the vertex set of a path P_{4l+1}. Let $P = \{V_1, V_2, \ldots, V_{2l+1}\}$ be a partition of P_{4l+1} graph so that $\langle V_1\rangle = \{v_{4l+1}\}$ and $\langle V_i\rangle = \langle v_i, v_{i+l}\rangle$, $1 \le i \le l$. The $d(v_i, v_j)$, $1 \le i < j \le 4l$ remains same as in Table 4. $d(v_i, v_{4l+1})$, $1 \le i \le 4l$ are $4, l+2, l+1, \ldots, 7, 6, 5, l+2, l+1, \ldots, 5, 4, 3, 2, 3, \ldots, l-1, l, l+1, l+1, l, \ldots, 3, 2, 1$. $deg(v_1)+deg(v_j) = 5$ for $2 \le j \le 4l$, $deg(v_i)+deg(v_j) = 6$ for $2 \le i < j \le 4l$, $deg(v_i)+deg(v_{4l+1}) = 4$ for $2 \le i \le 4l$ and $deg(v_1)+deg(v_{4l+1}) = 3$.

The degree distance between the vertices in the first four rows are $5l^2+5l+6l^2+6l$, $\frac{5}{2}(l^2+3l)+3l^2+9l$, $\frac{5}{2}(l^2+3l-10)+3l^2+9l-30$ and 15 respectively. The degree distance between the vertices of next rows are $9l^3+3l^2-6l$, l^3-l, $2l^3-2l$, $6l^3-6l$, l^3+6l^2+5l, $3l^3-27l$ and $2l^3-3l^2-23l+42$ respectively. The degree distance between (v_i, v_{4l+1}), $1 \le i \le 4l$ is $8l^2+32l-12$. Hence $DD(P_{4l+1})^P_{(2l+1)(i)} = 24l^3+36l^2+16l-10$.

(2) Suppose $n = 2l+1$. Let $v_1, v_2, \ldots, v_{4l+3}$ be the vertex set of a path P_{4l+3}. Let $P = \{V_1, V_2, \ldots, V_{2l+2}\}$ be a partition of P_{4l+3} in such that $\langle V_1\rangle = \{v_{4l+3}\}$ and $\langle V_i\rangle = \langle v_i, v_{i+l}\rangle$, $1 \le i \le l+1$. In such a partition one partite consists of one vertex and remaining partites consist of two vertices which are at distance $2l+1$. The $d(v_i, v_j)$, $1 \le i < j \le 4l+2$ remains same as in Table 5. It is noted that $d(v_i, v_{4l+3})$, $1 \le i \le 4l+2$ is 4, $l+2, l+1, \ldots, 7, 6, 5$, $l+3, l+2, \ldots, 5, 4, 3$, $2, 3, \ldots, l-1, l, l+1$, $l+2, l+1, \ldots, 3, 2, 1$. Also $deg(v_1)+deg(v_j) = 5$ for $2 \le j \le 4l+1$, $deg(v_i)+deg(v_j) = 6$ for $2 \le i < j \le 4l+1$, $deg(v_i)+deg(v_{4l+3}) = 4$ for $2 \le i \le 4l+1$ and $deg(v_1)+deg(v_{4l+3}) = 3$.

The degree distance between the vertices of first five rows are $\frac{5}{2}(l^2+3l+2)+3l^2+9l+6$, $\frac{5}{2}(l^2+l)+3l^2+3l$, $\frac{5}{2}(l^2+5l+4)+3l^2+15l+12$, $\frac{5}{2}(l^2+3l-10)+3l^2+9l-30$ and 15 respectively. The degree distance between the vertices of next rows are $9l^3+24l^2+9l-6$, l^3+3l^2+2l, $6l^3+3l^2-3l$, $2l^3-2l$, $l^3+9l^2+20l+12$, $3l^3+6l^2-33l-36$ and $2l^3-3l^2-23l+42$ respectively. The degree distance between (v_i, v_{4l+3}), $1 \le i \le 4l+1$ is $8l^2+40l+8$. Hence $DD(P_{4l+3})^P_{(2l+2)(i)} = 24l^3+72l^2+76l+13$.

□

Proposition 3.6. *Let G be a complete bipartite graph $K_{m,n}$ with vertex set $V(G) = \{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n\}$.*

(1) *If m and n both are even and $K_{m,n}$ is partitioned into $\frac{m+n}{2}$ partites such that each partite has exactly two non-adjacent vertices. Then,* $DD\left(G^P_{(\frac{m+n}{2})(i)}\right) = 3mn(m+n) + 2(m^2+n^2) - 4mn - 3(m+n)$.

(2) *If m is even and n is odd and $K_{m,n}$ is partitioned into $\frac{m+n-1}{2}$ partites such that $\langle V_1\rangle = \{v_n\}$ and remaining partites have exactly two non-adjacent vertices. Then,* $DD\left(G^P_{(\frac{m+n-1}{2})(i)}\right) = 3mn(m+n)+2(m^2+n^2) - 4mn - 2m - 5n + 4$.

(3) *If m and n both are odd and $K_{m,n}$ is partitioned into $\frac{m+n-2}{2}$ partites such that $\langle V_1\rangle = \{v_1\}$, $\langle V_2\rangle = \{v_2\}$ and remaining partites have exactly two non-adjacent vertices. Then,* $DD\left(G^P_{(\frac{m+n-2}{2})(i)}\right) = \frac{1}{2}(6mn(m+n) + 4(m^2+n^2) - 8mn - 11(m+n) + 14)$.

Proof. Let G be a complete bipartite graph $K_{m,n}$.

(1) Suppose m and n both are even and $K_{m,n}$ is partitioned into $\frac{m+n}{2}$ partites such that each partite has exactly two non-adjacent vertices. Then,

TABLE 6. Degree distance of $k(i)-$complement of complete bipartite graph of even order

Number of Pairs of vertices	**deg(u)+ deg(v)**	**d(u,v)**	**DD(u,v)**
mn	$m+n+2$	1	$m^2n + mn^2 + 2mn$
$\frac{m}{2}$	$2n+2$	1	$mn+m$
$\frac{n}{2}$	$2m+2$	1	$mn+n$
$\binom{m}{2} - \frac{m}{2}$	$2n+2$	2	$2m^2n + 2m^2 - 4mn - 4m$
$\binom{n}{2} - \frac{n}{2}$	$2m+2$	2	$2mn^2 + 2n^2 - 4mn - 4n$

Thus, $DD\left(G^P_{(\frac{m+n}{2})(i)}\right) = 3mn(m+n)+2(m^2+n^2)-4mn-3(m+n)$.

(2) Suppose m is even and n is odd and n is odd and $K_{m,n}$ is partitioned into $\frac{m+n-1}{2}$ partites such that $\langle V_1\rangle = \{v_n\}$ and remaining partites have exactly two non-adjacent vertices. Then,

TABLE 7. Degree distance of $k-$complement of complete bipartite graph of odd order

Number of Pairs of vertices	**deg(u)+ deg(v)**	**d(u,v)**	**DD(u,v)**
$m(n-1)$	$m+n+2$	1	$m^2n + mn^2 + mn - m^2 - 2m$
m	$m+n+1$	1	$mn + m^2 + m$
$\frac{m}{2}$	$2n+2$	1	$mn+m$
$\frac{n-1}{2}$	$2m+2$	1	$mn+n$

Number of Pairs of vertices	**deg(u)+ deg(v)**	**d(u,v)**	**DD(u,v)**
$\binom{m}{2} - \frac{m}{2}$	$2n+2$	2	$2m^2n + 2m^2 - 4mn - 4m$
$n-1$	$2m+1$	2	$4mn + 2n - 4m - 2$
$\binom{n}{2} - \frac{n-1}{2} - (n-1)$	$2m+2$	2	$2mn^2 + 2n^2 - 8mn - 8n + 6m + 6$

Thus, $DD\left(G^P_{(\frac{m+n-1}{2})(i)}\right) = 3mn(m+n) + 2(m^2+n^2) - 4mn - 2m - 5n + 4.$

(3) Suppose m and n both are odd and $K_{m,n}$ is partitioned into $\frac{m+n-2}{2}$ partites such that $\langle V_1 \rangle = \{v_1\}$, $\langle V_2 \rangle = \{v_2\}$ and remaining partites have exactly two non-adjacent vertices. Then,

Table 8. Degree distance of $k-$complement of complete bipartite graph of even order

Number of Pairs of vertices	**deg(u)+ deg(v)**	**d(u,v)**	**DD(u,v)**
$(m-1)(n-1)$	$m+n+2$	1	$m^2n + mn^2 - m^2 - n^2 - m - n + 2$
$n-1$	$m+n+1$	1	$n^2 + mn - m - 1$
$m-1$	$m+n+1$	1	$m^2 + mn - n - 1$
1	$m+n$	1	$m+n$
$\frac{m-1}{2}$	$2n+1$	1	$\frac{1}{2}(2mn + m - 2n - 1)$
$\frac{n-1}{2}$	$2m+1$	1	$\frac{1}{2}(2mn + n - 2m - 1)$
$m-1$	$2n+1$	2	$4mn + 2m - 4n - 2$
$n-1$	$2m+1$	2	$4mn + 2n - 4m - 2$
$\binom{m}{2} - \frac{m-1}{2} - (m-1)$	$2n+2$	2	$2m^2n + 2m^2 - 8mn + 6n - 8m + 6$
$\binom{n}{2} - \frac{n-1}{2} - (n-1)$	$2m+2$	2	$2mn^2 + 2n^2 - 8mn + 6m - 8n + 6$

Thus, $DD\left(G^P_{(\frac{m+n-2}{2})(i)}\right) = \frac{1}{2}(6mn(m+n) + 4(m^2+n^2) - 8mn - 11(m+n) + 14).$

□

4. Conclusion

In section 2 the authors obtained the characterization of G^P_k with respect to connectedness and calculated the degree distance index of $k-$complement of path graph, cycle and complete bipartite graph. The characterization of $k(i)-$complement and degree distance index is obtained in section 3.

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