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SOME INEQUALITIES ON GENERALIZED DEGREE BASED INDICES: AN (A, B)-KA INDICES AND COINDICES

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Abstract. For any two arbitrary real numbers a and b, the (a, b)-KA indices and coindices lies in the fact that their special cases, for preferential values of the variants a and b, coincide with the vast majority of previously considered vertex degree-based topological indices. The main goal of this paper is to provide some inequalities and their characterizations in terms of the order, size, minimum/maximum degree, Nordhus-Gaddam type and other degree based topological indices of (a, b) - KA indices and coindices.

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1. Introduction

All the graphs considered here are finite and undirected, with no loops and multiple edges. As usual $p = |V|$ and $q = |E|$ denote the number of vertices and edges of a graph $G = (V, E)$, respectively. The edge connecting the vertices u and v will be denoted by $e = uv$. A r-regular graph is a graph where each vertex has the same degree r. i.e., $d_G(u) = d_G(v) = r$. The complement $\overline{G}$ of a simple graph G with the same vertex set $V(G)$ and $uv \in E(G)$ if and only if $uv \notin E(\overline{G})$. Thus $q(G) + q(\overline{G}) = q(K_p) = \frac{p(p-1)}{2}$ and the degree of a vertex u in $\overline{G}$ is given by $d_{\overline{G}}(u) + d_G(u) = p - 1$. For graph theoretic terminology and notation not given here, we refer the reader to Harary [9].

Graph-theoretic indices have been found to be useful in chemical documentation, isomer discrimination, structure-property/ relationships, structure-activity relationships, and pharmaceutical drug design in chemistry. For the historical milestones, some applications and mathematical properties of graph theory, we refer to [2, 3, 5, 7, 8, 10, 22].

The first and second (a, b)-KA indices and coindices are defined as

$$KA^1_{(a,b)}(G) = \sum_{uv \in E(G)} [d_G(u)^a + d_G(v)^a]^b,$$

$$KA^2_{(a,b)}(G) = \sum_{uv \in E(G)} [d_G(u)^a \times d_G(v)^a]^b,$$

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$$\overline{KA^1}_{(a,b)}(G) = \sum_{uv \notin E(G)} [d_G(u)^a + d_G(v)^a]^b, \ and$$

$$\overline{KA^2}_{(a,b)}(G) = \sum_{uv \notin E(G)} [d_G(u)^a \times d_G(v)^a]^b.$$

These indices were initiated by Kulli [11]. In [14], many existing invariants are covered by (a,b)-KA indices and coindices, which are estimated by assigning the specific values of variants a and b. For more details on (a,b)-KA indices and coindices, we refer to [12, 13]. Further, we refer for extra information regarding invariants [1, 4, 6, 15, 16, 18, 20, 21, 23, 24, 25, 26]. Analogously, the third (a,b)-KA indices and coindices are defined as

$$KA^3_{(a,b)}(G) = \sum_{uv \in E(G)} \mid d_G(u)^a - d_G(v)^a \mid^b, and$$

$$\overline{KA^3}_{(a,b)}(G) = \sum_{uv \notin E(G)} \mid d_G(u)^a - d_G(v)^a \mid^b,$$

where a and b are real numbers.

2. Main Results

In this section, we obtain some inequalities of (a,b)-KA indices and coindices in terms of the order, size, minimum/maximum degree and other degree-based topological indices.

2.1. Inequalities in terms of order, size and degrees.

Theorem 2.1. *For any nontrivial graph G with $a > 0$ and $b > 0$,*

(i) $2^b q \leq KA^1_{(a,b)}(G) \leq 2^b q(p-1)^{ab}$

(ii) $q \leq KA^2_{(a,b)}(G) \leq q(p-1)^{2ab}$

(iii) $0 \leq KA^3_{(a,b)}(G) \leq q\left((p-1)^a - 1\right)^b$.

Proof. Let G be a nontrivial graph G with $a > 0$ and $b > 0$. Then, $1 \leq \{d_G(u), d_G(v)\} \leq p-1$ for each $e = uv \in E(G)$. This implies that $1 \leq d_G(u)^a \leq (p-1)^a$ and $1 \leq d_G(v)^a \leq (p-1)^a$ for $a > 0$. Thus

(i) The sum of $d_G(u)^a$ and $d_G(v)^a$ with $b > 0$ yields

$$2^b \leq (d_G(u)^a + d_G(v)^a)^b \leq (2(p-1)^a)^b.$$

Similarly, we have the q number of inequalities for each edge $e = uv \in E(G)$. Adding all those inequalities, we have

$$2^b q \leq \sum_{uv \in E(G)} (d_G(u)^a + d_G(v)^a)^b \leq q2^b(p-1)^{ab}$$

$$2^b q \leq KA^1_{(a,b)}(G) \leq 2^b q(p-1)^{ab}.$$

(ii) The product of $d_G(u)^a$ and $d_G(v)^a$ yields

$$1 \leq d_G(u)^{ab} d_G(v)^{ab} \leq (p-1)^{2ab}.$$

The above inequality holds good for all $uv \in E(G)$. Therefore

$$q \leq KA^2_{(a,b)}(G) \leq q(p-1)^{2ab}.$$

(iii) The absolute difference of $d_G(u)^a$ and $d_G(v)^a$ with $b > 0$ yields,

$$0 \leq |d_G(u)^a - d_G(v)^a|^b \leq ((p-1)^a - 1)^b .$$

The above inequality holds good for all $uv \in E(G)$. Therefore,

$$0 \leq KA^3_{(a,b)}(G) \leq q\,((p-1)^a - 1)^b .$$

Hence the result follows. □

Similarly, as in the proof techniques of the Theorem 2.1, we have

Theorem 2.2. *Let G be a nontrivial graph. Then*

(i) $2^b\,\overline{q} \leq \overline{KA^1}_{(a,b)}(G) \leq 2^b\,\overline{q}(p-1)^{ab}$

(ii) $\overline{q} \leq \overline{KA^2}_{(a,b)}(G) \leq \overline{q}\,(p-1)^{2ab}$

(iii) $0 \leq \overline{KA^3}_{(a,b)}(G) \leq \overline{q}\,((p-1)^a - 1)^b .$

Theorem 2.3. *Let G be a nontrivial graph with no isolated vertices. If a and b are any two non-zero values, Then the values of*

(i) *$KA^1_{(a,b)}(G)$ lies between $2^b q(\delta(G))^{ab}$ and $2^b q(\Delta(G))^{ab}$*

(ii) *$KA^2_{(a,b)}(G)$ lies $q(\delta(G))^{2ab}$ and $q(\Delta(G))^{2ab}$*

(iii) *$KA^3_{(a,b)}(G)$ lies between 0 and $q|(\Delta(G)^a - \delta(G)^a)|^b$.*

Further, the equality holds if and only if G is regular.

Proof. Let G be a nontrivial graph with no isolated vertices.
If $\delta(G) \leq \{d_G(u), d_G(v)\} \leq \Delta(G)$, then

(i) For any non-zero values of a and b.
Case 1. When $a, b > 0$, or $a, b < 0$, we have
Subcase 1.1. If $a, b > 0$ and $\delta(G)^a \leq \{d_G(u)^a, d_G(v)^a\} \leq \Delta(G)^a$, then the sum of $d_G(u)^a$ and $d_G(v)^a$ with $b > 0$ yields, $2^b\delta(G)^{ab} \leq (d_G(u)^a + d_G(v)^a)^b \leq 2^b\Delta(G)^{ab}$. For each edge $e = uv \in E(G)$ satisfies the above inequality and the sum of all those inequalities, we have $2^b q(\delta(G))^{ab} \leq KA^1_{(a,b)}(G) \leq 2^b q(\Delta(G))^{ab}$. **Subcase 1.2.** If $a, b < 0$, the inequality appears to have flipped, $\delta(G)^a \geq \{d_G(u)^a, d_G(v)\} \geq \Delta(G)^a$ then the sum of $d_G(u)^a$ and $d_G(v)^a$ capitulate $2\delta(G)^a \geq d_G(u)^a + d_G(v)^a \geq 2\Delta(G)^a$. By virtue of the above step the q inequality will change if we choose $b < 0$ for each $e = uv \in E(G)$, then the sum of all those inequalities, we have $2^b q(\delta(G))^{ab} \leq KA^1_{(a,b)}(G) \leq 2^b q(\Delta(G))^{ab}$.
Case 2. When $a > 0$ and $b < 0$ or $a < 0$ and $b > 0$. By the same line of proofs as in Case 1, we have $2^b q(\Delta(G))^{ab} \leq KA^1_{(a,b)}(G) \leq 2^b q(\delta(G))^{ab}$.

(ii) By the definition of $KA^2_{(a,b)}(G)$ and the same line of proof techniques as in (i). We have
Case 1. $q(\delta(G))^{2ab} \leq KA^2_{(a,b)}(G) \leq q(\Delta(G))^{2ab}$; if $a, b > 0$ or $a, b < 0$.
Case 2. $q(\Delta(G))^{2ab} \leq KA^2_{(a,b)}(G) \leq q(\delta(G))^{2ab}$; if $a > 0$, $b < 0$ or $a < 0$, $b > 0$.

(iii) $0 \leq KA^3_{(a,b)}(G) \leq q|(\Delta(G)^a - \delta(G)^a)|^b$ for all $a \neq 0$ and $b > 0$.

Since G is a regular graph. Hence an, equalities (i), (ii) and (iii) holds if and only if $\delta(G) = \Delta(G)$. Thus the result follows. □

Similarly, as in the proof techniques of the Theorem 2.3, we have

Theorem 2.4. . *Let G be a nontrivial graph with no isolated vertices. If a and b are any two non-zero values, Then the values of*

(i) $\overline{KA^1}_{(a,b)}(G)$ *lies between* $2^b\overline{q}(\delta(G))^{ab}$ *and* $2^b\overline{q}(\Delta(G))^{ab}$

(ii) $\overline{KA^2}_{(a,b)}(G)$ *lies* $\overline{q}(\delta(G))^{2ab}$ *and* $\overline{q}(\Delta(G))^{2ab}$

(iii) $\overline{KA^3}_{(a,b)}(G)$ *lies between* 0 *and* $\overline{q}|(\Delta(G)^a - \delta(G)^a)|^b$.

Further, the equality holds if and only if G is regular.

Theorem 2.5. *Let G be a nontrivial graph. Then the value of*

(i) $KA^1_{(a,b)}(\overline{G})$ *lies between* $\overline{q}\, 2^b(\Delta(\overline{G}))^{ab}$ *and* $\overline{q}\, 2^b(\delta(\overline{G}))^{ab}$

(ii) $KA^2_{(a,b)}(\overline{G})$ *lies between* $\overline{q}\Delta(\overline{G})^{2ab}$ *and* $\overline{q}\delta(\overline{G})^{2ab}$

(iii) $KA^3_{(a,b)}(\overline{G})$ *lies between* $|\ (\Delta(\overline{G})^a - \delta(\overline{G})^a)\ |^b$ *and* 0.

Further, both left and right-hand side equalities holds if and only if G is regular.

Proof. Let G be a nontrivial graph with $(p-1-\delta(G)) = \overline{\delta(G)} = \Delta(\overline{G})$ and $(p-1-\Delta(G)) = \overline{\Delta(G)} = \delta(\overline{G})$. We have $\delta(G) \leq \Delta(G)$ and $\overline{\delta(G)} \geq \overline{\Delta(G)}$.

(i) By Theorem 2.3, and $a, b > 0$ or $a, b < 0$, we have

$$\overline{q}\, 2^b(p-1-\delta(G))^{ab} \leq \overline{KA^1}_{(a,b)}(G) \leq \overline{q}\, 2^b(p-1-\Delta(G))^{ab}.$$

For $a < 0$, $b > 0$ or $a > 0$, $b < 0$, we have

$$\overline{q}\, 2^b(p-1-\Delta(G))^{ab} \leq KA^1_{(a,b)}(\overline{G}) \leq \overline{q}\, 2^b(p-1-\delta(G))^{ab}.$$

(ii) By Theorem 2.3, and $a, b > 0$ or $a, b < 0$, we have

$$\overline{q}\, \delta(\overline{G})^{2ab} \leq \overline{KA^2}_{(a,b)}(G) \leq \overline{q}\, \Delta(\overline{G})^{2ab}.$$

For $a < 0$, $b > 0$ or $a > 0$, $b < 0$, we have

$$\overline{q}\, \Delta(\overline{G})^{2ab} \leq KA^2_{(a,b)}(\overline{G}) \leq \overline{q}\, \delta(\overline{G})^{2ab}.$$

(iii) By Theorem 2.3, we have

$$0 \leq KA^3_{(a,b)}(\overline{G}) \leq|\ (\Delta(\overline{G})^a - \delta(\overline{G})^a)\ |^b\ .$$

Since G is a regular graph. Hence an, equalities (i), (ii) and (iii) holds if and only if $\delta(G) = \Delta(G)$. Thus the result follows. □

2.2. **Inequalities in terms of Nordhaus-Gaddum type.** A Nordhaus-Gaddum-type result is a (tight) left and right hand side inequalities on the sum or product of a degree based topological indices of a graph and its complement. For more details, we refer to [19].

Theorem 2.6. *For any nontrivial graph G with $a > 0$ and $b > 0$,*

(i) $2^{b-1}p(p-1) \leq KA^1_{(a,b)}(G) + \overline{KA^1}_{(a,b)}(G) \leq 2^{b-1}p(p-1)^{ab+1}$

(ii) $\frac{p(p-1)}{2} \leq KA^2_{(a,b)}(G) + \overline{KA^2}_{(a,b)}(G) \leq \frac{p(p-1)^{2ab+1}}{2}$

(iii) $0 \leq KA^3_{(a,b)}(G)\ +\ \overline{KA^3}_{(a,b)}(G) \leq \frac{p(p-1)}{2}\ ((p-1)^a - 1)^b$.

Proof. By Theorems 2.1 and 2.2, we have

(i) $2^bq + 2^b\overline{q} \leq KA^1_{(a,b)}(G) + \overline{KA^1}_{(a,b)}(G) \leq 2^bq(p-1)^{ab} + 2^b\overline{q}(p-1)^{ab}$

$$2^b(q+\overline{q}) \leq KA^1_{(a,b)}(G) + \overline{KA^1}_{(a,b)}(G) \leq 2^b(p-1)^{ab}(q+\overline{q})$$

$$2^{b-1}p(p-1) \leq KA^1_{(a,b)}(G) + \overline{KA^1}_{(a,b)}(G) \leq 2^{b-1}p(p-1)^{ab+1}.$$

(ii) $q+\overline{q} \leq KA^2_{(a,b)}(G)+\overline{KA^2}_{(a,b)}(G) \leq q(p-1)^{2ab}+\overline{q}(p-1)^{2ab}$. Similarly, as in the proof technique of (i), we have

$$\frac{p(p-1)}{2} \leq KA^2_{(a,b)}(G) + \overline{KA^2}_{(a,b)}(G) \leq \frac{p(p-1)^{2ab+1}}{2}.$$

(iii) $0 \leq KA^3_{(a,b)}(G) + \overline{KA^3}_{(a,b)}(G) \leq q((p-1)^a-1)^b + \overline{q}((p-1)^a-1)^b$. Similarly, as in the proof technique of (i), we have

$$0 \leq KA^3_{(a,b)}(G) + \overline{KA^3}_{(a,b)}(G) \leq \frac{p(p-1)}{2}((p-1)^a-1)^b.$$

Thus the result follows. □

Theorem 2.7. *For any nontrivial graph G with $a > 0$ and $b > 0$,*

(i) $2^{2(b-1)}\,p(p-1) \leq KA^1_{(a,b)}(G)\,\overline{KA^1}_{(a,b)}(G) \leq 2^{2b-3}\,p^2(p-1)^{2(ab+1)}$

(ii) $\frac{p(p-1)}{4} \leq KA^2_{(a,b)}(G)\,\overline{KA^2}_{(a,b)}(G) \leq \frac{p^2(p-1)^{4ab+2}}{8}$

(iii) $0 \leq KA^3_{(a,b)}(G)\,\overline{KA^3}_{(a,b)}(G) \leq \frac{p^2(p-1)^2}{8}\,((p-1)^a-1)^{2b}$.

Proof. By Theorems 2.1 and 2.2, we have

(i) $2^b\,q\,2^b\,\overline{q} \leq KA^1_{(a,b)}(G)\,\overline{KA^1}_{(a,b)}(G) \leq 2^b\,q(p-1)^{ab}\,2^b\,\overline{q}(p-1)^{ab}$
$2^{2b}(q\overline{q}) \leq KA^1_{(a,b)}(G)\overline{KA^1}_{(a,b)}(G) \leq 2^{2b}(p-1)^{2ab}(q\overline{q})$.
Here, the size q and $\overline{q}$ are positive integers, which implies

$$\frac{q+\overline{q}}{2} \leq q\overline{q} \leq \left(\frac{q+\overline{q}}{2}\right)^2.$$

$$2^{2b}\frac{\frac{p(p-1)}{2}}{2} \leq KA^1_{(a,b)}(G)\overline{KA^1}_{(a,b)}(G) \leq 2^{2b}(p-1)^{2ab}\frac{\left(\frac{p(p-1)}{2}\right)^2}{2}$$

$$2^{2(b-1)}p(p-1) \leq KA^1_{(a,b)}(G)\overline{KA^1}_{(a,b)}(G) \leq 2^{2b-3}p^2(p-1)^{2(ab+1)}.$$

(ii) $q\,\overline{q} \leq KA^2_{(a,b)}(G)\,\overline{KA^2}_{(a,b)}(G) \leq q(p-1)^{2ab}\,\overline{q}(p-1)^{2ab}$. Similarly, as in the proof technique of (i), we have

$$\frac{p(p-1)}{4} \leq KA^2_{(a,b)}(G)\,\overline{KA^2}_{(a,b)}(G) \leq \frac{p^2(p-1)^{4ab+2}}{8}.$$

(iii) $0 \leq KA^3_{(a,b)}(G)\overline{KA^3}_{(a,b)}(G) \leq q((p-1)^a-1)^b\overline{q}((p-1)^a-1)^b$. Similarly, as in the proof technique of (i), we have

$$0 \leq KA^3_{(a,b)}(G)\ \overline{KA^3}_{(a,b)}(G) \leq \frac{p^2(p-1)^2}{8}\,((p-1)^a-1)^{2b}.$$

Hence the desired results. □

2.3. **Inequalities among the (a,b)-KA indices and coindices.**

Theorem 2.8. *Let G be a nontrivial graph with $a>0$ and $b>0$. Then*

$$\frac{2^b}{\Delta(G)^{ab}}KA^2_{(a,b)}(G) \le KA^1_{(a,b)}(G) \le \frac{2^b}{\delta(G)^{ab}}KA^2_{(a,b)}(G).$$

Further, the equality holds if and only if G is regular.

Proof. Let G be a nontrivial graph with $a>0$ and $b>0$. Then

$$[d_G(u)^a + d_G(v)^a]^b = d_G(u)^{ab}d_G(v)^{ab}\left[\frac{1}{d_G(v)^a}+\frac{1}{d_G(u)^a}\right]^b$$

$$d_G(u)^{ab}d_G(v)^{ab}\left[\frac{2}{\Delta(G)^a}\right]^b \le [d_G(u)^a+d_G(v)^a]^b \le d_G(u)^{ab}d_G(v)^{ab}\left[\frac{2}{\delta(G)^a}\right]^b.$$

The above inequality satisfies for each edge $e=uv\in E(G)$ and the sum of all those inequalities capitulate

$$\Rightarrow \sum_{uv\in E(G)} d_G(u)^{ab}d_G(v)^{ab}\left[\frac{2}{\Delta(G)^a}\right]^b \le \sum_{uv\in E(G)}[d_G(u)^a+d_G(v)^a]^b$$

$$\le \left[\frac{2}{\delta(G)^a}\right]^b \sum_{uv\in E(G)} d_G(u)^{ab}d_G(v)^{ab}$$

$$\frac{2^b}{\Delta(G)^{ab}}KA^2_{(a,b)}(G) \le KA^1_{(a,b)}(G) \le \frac{2^b}{\delta(G)^{ab}}KA^2_{(a,b)}(G).$$

Since $\delta(G)=\Delta(G)$. Hence an, equality attains on both sides if and only if G is regular.
Thus the result follows. □

Theorem 2.9. *Let G be a nontrivial graph with $a>0$ and $b>0$,*

$$0 \le KA^3_{(a,b)}(G) \le \left[\frac{1}{\delta(G)^a}-\frac{1}{\Delta(G)^a}\right]^b KA^2_{(a,b)}(G).$$

Proof. Let G be a nontrivial graph with $a>0$ and $b>0$. Then

$$|d_G(u)^a - d_G(v)^a|^b = d_G(u)^{ab}d_G(v)^{ab}\left|\frac{1}{d_G(v)^a}-\frac{1}{d_G(u)^a}\right|^b$$

$$0 \le |d_G(u)^a - d_G(v)^a|^b \le d_G(u)^{ab}d_G(v)^{ab}\left[\frac{1}{\delta(G)^a}-\frac{1}{\Delta(G)^a}\right]^b.$$

The above inequality satisfies each edge $e=uv\in E(G)$ and the sum of all those inequalities is as follows.

$$0 \le \sum_{uv\in E(G)}|d_G(u)^a-d_G(v)^a|^b \le \sum_{uv\in E(G)} d_G(u)^{ab}d_G(v)^{ab}\left[\frac{1}{\delta(G)^a}-\frac{1}{\Delta(G)^a}\right]^b$$

$$0 \le KA^3_{(a,b)}(G) \le \left[\frac{1}{\delta(G)^a}-\frac{1}{\Delta(G)^a}\right]^b KA^2_{(a,b)}(G).$$

Thus the result follows. □

Theorem 2.10. *For any nontrivial graph G with $a > 0$ and $b > 0$,*

$$2^b KA^2_{\left(a,\frac{b}{2}\right)}(G) \le 2^b KA^1_{(a,b)}(G) \le 2^b KA^2_{(a,b)}(G).$$

Proof. Let G be a nontrivial graph with a non-zero real number a. Since $d_G(u), d_G(v) \ge 0$ for all $e = uv \in E(G)$. By the relationship between Arithmetic and Geometric mean inequality (i.e., $AM \ge GM$) in terms of $d_G(u)^a$ and $d_G(v)^a$, we have

$$\sqrt{d_G(u)^a d_G(v)^a} \le \frac{d_G(u)^a + d_G(v)^a}{2}.$$

For any $b > 0$, we have $2^b \left(d_G(u)d_G(v)\right)^{\frac{ab}{2}} \le \left(d_G(u)^a + d_G(v)^a\right)^b$. The above inequality holds good for each edge $e = uv \in E(G)$. Adding all those inequalities, we have

$$\begin{aligned} 2^b \sum_{uv\in E(G)} \left(d_G(u)d_G(v)\right)^{\frac{ab}{2}} &\le \sum_{uv\in E(G)} \left(d_G(u)^a + d_G(v)^a\right)^b \\ 2^b KA^2_{\left(a,\frac{b}{2}\right)}(G) &\le KA^1_{(a,b)}(G). \end{aligned}$$

Now we prove the second part.

The graph G with $d_G(u), d_G(v) \ge 0$ and $a > 0$, which implies $d_G(u)^a \ge 0$ and $d_G(v)^a \ge 0$. Clearly, $d_G(u)^a \le d_G(u)^a d_G(v)^a$ and $d_G(v)^a \le d_G(u)^a d_G(v)^a$ for all $e = uv \in E(G)$. Adding above inequalities, we have $d_G(u)^a + d_G(v)^a \le 2d_G(u)^a d_G(v)^a$. For any $b > 0$, $\left(d_G(u)^a + d_G(v)^a\right)^b \le \left(2d_G(u)^a d_G(v)^a\right)^b$.

The above inequalities holds good for all $e = uv \in E(G)$. Adding all of these inequalities together, we get

$$\begin{aligned} \sum_{uv\in E(G)} \left(d_G(u)^a + d_G(v)^a\right)^b &\le 2^b \sum_{uv\in E(G)} \left(d_G(u)d_G(v)\right)^{ab} \\ KA^1_{(a,b)}(G) &\le 2^b KA^2_{(a,b)}(G). \end{aligned}$$

Thus the result follows. □

Theorem 2.11. *For any connected graph G with $a > 0$ and $b > 0$,*

$$2^b(11q - 12p + KA^2_{\left(a,\frac{b}{2}\right)}(G)) \le KA^1_{(a,b)}(G) \le 2^b(2q^2 - \frac{1}{2}KA^2_{(a,b)}(G)).$$

Proof. By Theorems 2.1, we have desired bounds of $KA^1_{(a,b)}(G)$ in terms of order, size and $KA^2_{(a,b)}(G)$. □

Theorem 2.12. *Let G be a connected graph with a real number a. Then*

(i) $KA^1_{(a,b)}(G) > KA^3_{(a,b)}(G)$; *if* $b > 0$

(ii) $KA^1_{(a,b)}(G) < KA^3_{(a,b)}(G)$; *if* $b < 0$.

Proof. Let G be a connected graph with a real number a with $d_G(u)^a, d_G(v)^a \ge 1$. Then $\mid d_G(u)^a - d_G(v)^a \mid < d_G(u)^a + d_G(v)^a$.

(i) If $b > 0$, then $\left(\mid d_G(u)^a - d_G(v)^a \mid\right)^b < \left(d_G(u)^a + d_G(v)^a\right)^b$. The above inequalities holds good for each $uv \in E(G)$. Adding all those inequalities, we have

$$\sum_{uv\in E(G)} \left(\mid d_G(u)^a - d_G(v)^a \mid\right)^b < \sum_{uv\in E(G)} \left(d_G(u)^a + d_G(v)^a\right)^b$$

$$KA^1_{a,b}(G) > KA^3_{a,b}(G).$$

(ii) If $b < 0$, then by result (i), we have $KA^1_{a,b}(G) < KA^3_{a,b}(G)$. □

Theorem 2.13. *Let G be a simple graph with $p \geq 2$ vertices. Then $KA^3_{(a,b)}(G) = KA^1_{(a,b)}(G)$ if and only if the graph G is isomorphic to a totally disconnected graph.*

Proof. By the definitions of $KA^1_{(a,b)}(G)$ and $KA^3_{(a,b)}(G)$ with due to the fact of $d_G(u) = 0$ for all $u \in V(G)$, the desired result follows. □

Theorem 2.14. *Let G be a nontrivial graph with a real number a. Then*

(i) $\overline{KA^1}_{(a,b)}(G) \geq \overline{KA^3}_{(a,b)}(G)$; *if* $b > 0$

(ii) $\overline{KA^1}_{(a,b)}(G) \leq \overline{KA^3}_{(a,b)}(G)$; *if* $b < 0$.

Further, the equalities of (i) and (ii) holds if and only if the graph G is isomorphic to a totally disconnected graph.

Proof. Let G be a nontrivial graph with a real number a. In the Theorem 2.12, replace $d_G(u)$ with 0. Thus the results follow. □

By Theorem 2.10, we have the following result.

Theorem 2.15. *For any nontrivial graph G with $a > 0$ and $b > 0$,*

$$2^b\overline{KA^2}_{(a,\frac{b}{2})}(G) \leq 2^b\overline{KA^1}_{(a,b)}(G) \leq 2^b\overline{KA^2}_{(a,b)}(G).$$

2.4. **Inequalities between (a,b)-KA indices and coindices.**

Theorem 2.16. *Let G be a nontrivial graph with $d_G(u) \leq \left\lfloor \frac{p-1}{2} \right\rfloor$ for all $u \in V(G)$. Then*

(i) $KA^1_{(a,b)}(G) \leq \overline{KA^1}_{(a,b)}(G)$ *if* $a, b > 0$ *or* $a, b < 0$

(ii) $KA^2_{(a,b)}(G) \leq \overline{KA^2}_{(a,b)}(G)$ *if* $a, b > 0$ *or* $a, b < 0$.

Further, the equalities of (i) and (ii) holds if and only if G is regular with vertex degree $\frac{p-1}{2}$, where $\lfloor x \rfloor$ denote the greatest integer less than or equal to x.

Proof. Let G be a nontrivial graph with $d_G(u) \leq \left\lfloor \frac{p-1}{2} \right\rfloor$ for all $u \in V(G)$. Then $q(G) + q(\overline{G}) = \frac{p(p-1)}{2}$ and $d_G(u) \leq p - 1 - d_{\overline{G}}(u)$ for all $u \in V(G)$. We have

Case 1. When $a, b > 0$, we know that $d_G(u)^a \leq (p - 1 - d_{\overline{G}}(u))^a$ and $d_G(v)^a \leq (p - 1 - d_{\overline{G}}(v))^a$ for all $u, v \in V(G)$ with $a > 0$. Adding these inequalities, we have

$$d_G(u)^a + d_G(v)^a \leq (p - 1 - d_{\overline{G}}(u))^a + (p - 1 - d_{\overline{G}}(v))^a.$$

For any $b > 0$, the above inequality becomes

$$(d_G(u)^a + d_G(v)^a)^b \leq \left((p - 1 - d_{\overline{G}}(u))^a + (p - 1 - d_{\overline{G}}(v))^a\right)^b.$$

The above inequality holds good for each pair of adjacent and non-adjacent vertices. Taking the sum of all those inequalities, we have

$$\sum_{uv\in E(G)} (d_G(u)^a + d_G(v)^a)^b \leq \sum_{un\notin E(G)} (d_G(u)^a + d_G(v)^a)^b$$

$$KA^1_{(a,b)}(G) \leq \overline{KA^1}_{(a,b)}(G).$$

Case 2. When $a, b < 0$. By above case, we have $KA^1_{(a,b)}(G) \leq \overline{KA^1}_{(a,b)}(G)$. Thus (i) follows.
Similarly, when $a, b > 0$ or $a, b < 0$, the result (ii) follows. □

Further, the equalities of (i) and (ii) holds if and only if G is regular with vertex degree $\frac{p-1}{2}$.

Remark 2.4.1. *The converse of Theorem2.16 is need not to be true. Because, if there exist $d_G(u) \geq p-1-d_{\overline{G}}(u)$ or $d_G(u) \geq \left\lfloor \frac{p-1}{2} \right\rfloor$ then $KA^1_{(a,b)}(G)$ need not be greater than $\overline{KA^1}_{(a,b)}(G)$.*

Theorem 2.17. *Let G be a nontrivial graph with $d_G(u) \geq p-1-d_{\overline{G}}(u)$ for all $u \in V(G)$. Then*

(i) $KA^2_{(a,b)}(G) \geq \overline{KA^2}_{a,b}(G)$; *if $a, b > 0$ or $a, b < 0$.*
(ii) $KA^2_{(a,b)}(G) \leq \overline{KA^2}_{a,b}(G)$; *if $a > 0$ and $b < 0$ or $a < 0$ and $b > 0$.*

2.5. **Inequalities in terms of first general Zagreb index.** In 2005, Xueliang Li and Jie Zheng [17] defined the first general Zagreb index for $a \in R$ and $a \neq 0$ as

$$M_1^{a+1}(G) = \sum_{uv\in E(G)} [d_G(u)^a + d_G(v)^a].$$

Theorem 2.18. *Let G be a nontrivial graph. Then*

(i) $M_1^{a+1}(G) \leq KA^1_{(a,b)}(G)$; *if $a > 0$, $b \geq 1$.*
(ii) $M_1^{a+1}(G) \geq KA^1_{(a,b)}(G)$; *if $a > 0$, $b \leq 1$ and $b \neq 0$.*

Farther, equality holds if $b = 1$.

Proof. Let G be a graph with $p \geq 2$ vertices. If $d_G(u), d_G(v) \geq 1$ and $d_G(u)^a d_G(v)^a \geq 0$ (tends to 0 as a tends to $-\infty$).
Case 1. When $a > 0$, $b \geq 1$, we have

$$d_G(u)^a + d_G(v)^a \leq [d_G(u)^a + d_G(v)^a]^b.$$

The above inequality holds good for each edge $e = uv \in E(G)$. Therefore the sum of all those inequalities, we have

$$\sum_{uv\in E(G)} [d_G(u)^a + d_G(v)^a] \leq \sum_{uv\in E(G)} [d_G(u)^a + d_G(v)^a]^b.$$

$$M_1^{a+1}(G) \leq KA^1_{(a,b)}(G).$$

Case 2. When $a > 0$, $b \leq 1$ and $b \neq 0$, we have

$$d_G(u)^a + d_G(v)^a \geq [d_G(u)^a + d_G(v)^a]^b.$$

The above inequality holds good for each edge $e = uv \in E(G)$. Therefore the sum of all those inequalities, we have

$$M_1^{a+1}(G) \geq KA^1_{(a,b)}(G).$$

□

3. Conclusions

Being a new generalized version of vertex degree-based topological indices of a graph G, the (a, b)-KA indices and coindices lies in the fact that their special cases, for pertinently chosen values of the parameters a and b, coincide with the vast majority of previously considered vertex degree-based topological indices. For the comparative advantages, applications, and mathematical point of view, many questions are suggested by this research, among them the following.

1. Find the extremal values and extremal graphs of the (a, b)-KA indices and coindices.
2. Find the values of (a, b)-KA indices and coindices of certain classes of chemical graphs and explore some results towards QSPR / QSAR / QSTR Model.
3. Obtain the relationship between (a, b)-KA indices /coindices in terms of other degree/distance/spectral based topological indices.

4. **Conflicts of Interest**

The authors declare no conflict of interest.

5. **Acknowledgements**

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