Proceedings of the Jangjeon Mathematical Society
26 (2023), No. 1, pp. 23 - 42

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2023.26.1.23

On r-Dynamic Coloring of n-Sunlet Graph Families

G.Nandini[a], M.Venkatachalam[b], Vernold Vivin.J[c,*], Dafik[d,e]

[a] *Department of Mathematics, SNS College of Technology, Coimbatore - 641 035, Tamil Nadu, India*
[b] *Department of Mathematics, Kongunadu Arts and Science College, Coimbatore - 641 029, Tamil Nadu, India*
[c] *University College of Enigneering Nagercoil, (A Constituent College of Anna University, Chennai), Konam, Nagercoil-629 004, Tamil Nadu, India*
[d] *CGANT Research Group, University of Jember, Indonesia*
[e] *Department of Mathematics Education, University of Jember, Indonesia*

Abstract

An r-dynamic coloring of a graph G is a proper coloring c of the vertices such that $|c(N(v))| \geq min\{r, d(v)\}$, for each $v \in V(G)$. The r-dynamic chromatic number of a graph G is the minimum k such that G has an r-dynamic coloring with k colors. In this paper, we determine the r-dynamic chromatic number of n-Sunlet graph and determine sharp bounds of this graph invariant for the four related families of graphs: The middle graph $\chi_r(M(S_n))$, the total graph $\chi_r(T(S_n))$, the central graph $\chi_r(C(S_n))$ and the line graph $\chi_r(L(S_n))$.

Keywords: r-dynamic coloring, n-Sunlet graph, line graph, middle graph, total graph, central graph
2000 MSC: 05C15, 05C75

1. Introduction

In this section, the origin and development of graph theory are succinctly given and a brief literature review on graph coloring have been investigated, by illustrating the application of graph colorings and literature survey related to dynamic coloring.

The concepts of graph theory begins in recreational math problem, then it has grown into considerable area of research with wide applications in chemistry, operations research and computer science. Thereupon, graphs also act as a vital tool describing the molecules structure in biology. Also, theoretical ideas are influenced as a part of operations research.

The Swiss mathematician **Leonhard Euler** is refereed as the father of graph theory. He was the first person to introduce the basic idea of graphs in 18^{th} century. His earnest

*Corresponding author
Email addresses: `nandiniap2006@gmail.com` (G.Nandini), `venkatmaths@gmail.com` (M.Venkatachalam), `vernoldvivin@yahoo.in` (Vernold Vivin.J), `d.dafik@unej.ac.id` (Dafik)

attempts and ultimate solution to the famous Konigsberg bridge problem was the origin of graph theory. The Konigsberg bridge problem was an puzzle which concerned with the possibility of finding a path over every one of seven bridges that span a forked river flowing past an island-but without crossing any bridge twice. Euler proclaimed that no such path prevails.

The study of vertex colorings of graphs in general in which adjacent vertices are colored differently (proper vertex colorings) has become a major area of study in graph theory. Over the years, there have been innumerable changes in properties required for vertex colorings and in ways that certain graph colorings have resulted from other graph colorings.

The r-dynamic chromatic number was first introduced by Montgomery [8]. An r-dynamic coloring of a graph G is a proper coloring and it maps c from $V(G)$ to the set of colors such that

(i) if $uv \in E(G)$, then $c(u) \neq c(v)$, and

(ii) for each vertex $v \in V(G), |c(N(v))| \geq min\{r, d(v)\}$,

where $N(v)$ denotes the set of vertices adjacent to v, $d(v)$ its degree and r is a positive integer. The r-dynamic chromatic number of a graph G, written $\chi_r(G)$, is the minimum k such that G has an r-dynamic proper k-coloring. In this paper we consider only the graphs which are simple, finite, loopless and connected. For all terms and definition which are not specifically defined in this paper, we refer to [2]. The r-dynamic chromatic number has been studied by many authors, for instance in [1, 3, 5, 7, 9, 10, 11, 12].

2. Preliminaries

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$ and $\Delta(G)$ denotes the maximum degree of the vertices of G.

The line graph [4] of G denoted by $L(G)$ is the graph whose vertex set is the edge set of G. Two vertices of $L(G)$ are adjacent whenever the corresponding edges of G are adjacent.

The middle graph [6] of G, denoted by $M(G)$ is defined as follows. The vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices x, y of $M(G)$ are adjacent in $M(G)$ in case one of the following holds: (i) x, y are in $E(G)$ and x, y are adjacent in G. (ii) x is in $V(G)$, y is in $E(G)$, and x, y are incident in G.

The total graph [6] of G, denoted by $T(G)$ is defined in the following way. The vertex set of $T(G)$ is $V(G) \cup E(G)$. Two vertices x, y of $T(G)$ are adjacent in $T(G)$ in case one of the following holds: (i) x, y are in $V(G)$ and x is adjacent to y in G. (ii) x, y are in $E(G)$ and x, y are adjacent in G. (iii) x is in $V(G), y$ is in $E(G)$, and x, y are incident in G.

The central graph [13] $C(G)$ of a graph G is obtained from G by adding an extra vertex on each edge of G, and then joining each pair of vertices of the original graph which were previously non-adjacent.

The n-Sunlet graph [14] on $2n$ vertices is obtained by attaching n pendant edges to the cycle C_n and is denoted by S_n.

Let $V(S_n) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\}$ where v_i are the vertices of cycles taken in cyclic order and u_i are pendant vertices such that $v_i u_i$ is a pendant edge. $E(S_n) = \{e_i : 1 \leq i \leq 2n\}$, where $e_i = v_i v_{i+1}$ $(1 \leq i \leq n-1)$, $e_n = v_n v_1$, $e_{n+i} = v_i u_i$ $(1 \leq i \leq n)$.

3. Main Results

In this paper, we find the r-dynamic chromatic number of n-Sunlet graph, line, middle, total, central graphs of the n-Sunlet graph.

Theorem 3.1. *Let $n \geq 6$, the r-dynamic chromatic number of the n-Sunlet graph is*

$$\chi_r(S_n) = \begin{cases} 2, r = 1, n \text{ is even}, \\ 3, r = 1, n \text{ is odd}, \\ 3, r = \Delta - 1, \\ 4, r = \Delta. \end{cases}$$

Proof. **Case 1:** $r = 1$, n is even.

Consider the color function $c : V(S_n) \to \{c_1, c_2\}$ defined as follows.

For $1 \leq i \leq n$,

$$c(v_i) = \begin{cases} c_1, & i \text{ is odd} \\ c_2, & i \text{ is even} \end{cases}$$

$$c(u_i) = \begin{cases} c_2, & i \text{ is odd} \\ c_1, & i \text{ is even} \end{cases}$$

Hence, $\chi_r(S_n) = 2$ for $r = 1$ when n is even.

Case 2: $r = 1$, n is odd.

Consider the color function $c : V(S_n) \to \{c_1, c_2, c_3\}$ defined as follows.

For $1 \leq i \leq n-1$,

$$c(v_i) = \begin{cases} c_1, & i \text{ is odd} \\ c_2, & i \text{ is even} \end{cases}$$

In order to maintain the adjacency condition we need one new color c_3 to color the vertex v_n.

For $1 \leq i \leq n$,

$$c(u_i)= \begin{cases} c_2, & i \text{ is odd} \\ c_1, & i \text{ is even} \end{cases}$$

Hence, $\chi_r(S_n) = 3$ for $r = 1$ when n is odd.

Case 3: $r = \Delta - 1$

Consider the color function $c : V(S_n) \rightarrow \{c_1, c_2, c_3\}$ defined as follows.

When n is even, For $1 \leq i \leq n$,

$$c(v_i)= \begin{cases} c_1, & i \text{ is odd} \\ c_2, & i \text{ is even} \end{cases}$$

$$c(u_i) = c_3$$

When n is odd, For $1 \leq i \leq n-1$,

$$c(v_i)= \begin{cases} c_1, & i \text{ is odd} \\ c_2, & i \text{ is even} \end{cases}$$

Assign $c(v_n) = c_3$, $c(u_i) = c_3$ $(1 \leq i \leq (n-1))$ and $c(u_n) = c_2$.

Hence, $\chi_r(S_n) = 3$ for $r = \Delta - 1$.

Case 4: $r = \Delta$

Consider the color function $c : V(S_n) \rightarrow \{c_1, c_2, c_3, c_4\}$ defined as follows.

Subcase (i): $n \equiv 0 (\text{mod } 3)$

For $1 \leq i \leq n$, color the vertices v_i cyclically with the colors $\{c_1, c_2, c_3\}$ and $c(u_i) = c_4$.

Subcase (ii): $n \equiv 1 (\text{mod } 3)$

For $1 \leq i \leq n-1$, color the vertices v_i cyclically with the colors $\{c_1, c_2, c_3\}$ and assign $c(v_n) = c_4$

For $2 \leq i \leq n-2$, $c(u_i) = c_4$ and color the vertices $\{u_{n-1}, u_n, u_1\}$ cyclically with the colors $\{c_1, c_2, c_3\}$.

Subcase (iii): $n \equiv 2 (\text{mod } 3)$

For $1 \leq i \leq 8$, color the vertices v_i cyclically with the colors $\{c_1, c_2, c_3, c_4\}$. For $9 \leq i \leq n$, color the vertices v_i cyclically with the colors $\{c_1, c_2, c_3\}$.

Color the vertices $\{u_3, u_4, \ldots, u_7, u_9\}$ cyclically with the colors $\{c_1, c_2, c_3, c_4\}$ and color all the remaining u_i with the color c_4.

Hence, $\chi_r(S_n) = 4$ for $r = \Delta$.

□

Theorem 3.2. *Let $n \geq 6$, the r-dynamic chromatic number of the line graph of n-Sunlet graph is*

$$\chi_r(L(S_n)) = \begin{cases} 3, r = 1, 2, \\ 4, r = \Delta - 1, \\ 5, r = \Delta, n \equiv 0 (\bmod\ 3), \quad n \textit{ is even,} \\ 6, r = \Delta, \ \textit{otherwise.} \end{cases}$$

Proof. Let $V(L(S_n)) = \{e_1, e_2, \ldots, e_{2n}\}$. By definition of the line graph, for $1 \leq i \leq n-1$, the vertices $\{e_i, e_{i+1}, e_{n+i+1}\}$ induces a clique of order K_3 in $L(S_n)$. Thus, $\chi_r(L(S_n)) \geq 3$ for any r.

Case 1: $r = 1, 2$

Consider the color function $c : V(L(S_n)) \to \{c_1, c_2, c_3\}$ defined as follows.

When n is even, For $1 \leq i \leq n$,

$$c(e_i) = \begin{cases} c_1, & i \text{ is odd} \\ c_2, & i \text{ is even} \end{cases}$$

For $n + 1 \leq i \leq 2n$, $c(e_i) = c_3$.

When n is odd, For $1 \leq i \leq n - 1$,

$$c(e_i) = \begin{cases} c_1, & i \text{ is odd} \\ c_2, & i \text{ is even} \end{cases}$$

Assign $c(e_n) = c_3$ and $c(e_{n+1}) = c_2$. For $n + 2 \leq i \leq 2n - 1$, assign $c(e_i) = c_3$ and $c(e_{2n}) = c_1$.

Hence, $\chi_r(L((S_n)) \leq 3$. Therefore, $\chi_r(L(S_n)) = 3$ for $r = 1, 2$.

Case 2: $r = \Delta - 1$

Consider the color function $c : V(L(S_n)) \to \{c_1, c_2, c_3, c_4\}$ defined as follows.

When n is even, For $1 \leq i \leq n$,

$$c(e_i) = \begin{cases} c_1, & i \text{ is odd} \\ c_2, & i \text{ is even} \end{cases}$$

For $n+1 \leq i \leq 2n$

$$c(e_i) = \begin{cases} c_3, & i \text{ is odd} \\ c_4, & i \text{ is even} \end{cases}$$

When n is odd, For $1 \leq i \leq n-1$,

$$c(e_i) = \begin{cases} c_1, & i \text{ is odd} \\ c_2, & i \text{ is even} \end{cases}$$

Assign $c(e_n) = c_3$
For $n+1 \leq i \leq 2n$,

$$c(e_i) = \begin{cases} c_3, & i \text{ is odd} \\ c_4, & i \text{ is even} \end{cases}$$

Hence, $\chi_r(L(S_n)) = 4$ for $r = \Delta - 1$.

Case 3: $r = \Delta$, $n \equiv 0(\text{mod } 3)$, n is even.
Consider the color function $c : V\,(L(S_n)) \to \{c_1, c_2, c_3, c_4, c_5\}$ defined as follows.
For $1 \leq i \leq n$, color the vertices e_i cyclically with $\{c_1, c_2, c_3\}$.
For $n+1 \leq i \leq 2n$,

$$c(e_i) = \begin{cases} c_4, & i \text{ is odd} \\ c_5, & i \text{ is even} \end{cases}$$

Hence, $\chi_r(L(S_n)) = 5$ for $r = \Delta$, $n \equiv 0(\text{mod } 3)$ when n is even.

Case 4: $r = \Delta$, otherwise.
Consider the color function $c : V\,(L(S_n)) \to \{c_1, c_2, c_3, c_4, c_5, c_6\}$ defined as follows.

Subcase (i): $n \equiv 0(\text{mod})$, n is odd.
For $1 \leq i \leq n$, color the vertices e_i cyclically with $\{c_1, c_2, c_3\}$.
For $n+1 \leq i \leq 2n-1$,

$$c(e_i) = \begin{cases} c_5, & i \text{ is odd} \\ c_4, & i \text{ is even} \end{cases}$$

In order to maintain the r-adjacency condition we need one new color c_6 to color the vertex e_{2n}.

Subcase (ii): $n \equiv 1(\text{mod } 3)$.

For $1 \leq i \leq 4$, $c(e_i) = c_i$ and for $5 \leq i \leq n$, color the vertices e_i cyclically with $\{c_1, c_2, c_3\}$. Assign $c(e_{n+1}) = c_4$.

For $n + 2 \leq i \leq 2n$,

$$c(e_i) = \begin{cases} c_5, & i \text{ is odd} \\ c_6, & i \text{ is even} \end{cases}$$

Subcase (iii): $n \equiv 2(\text{mod } 3)$.

For $1 \leq i \leq 8$, color the vertices e_i cyclically with $\{c_1, c_2, c_3, c_4\}$ and for $9 \leq i \leq n$, color the vertices e_i cyclically with $\{c_1, c_2, c_3\}$.

Assign $c(e_{n+1}) = c_4$.

For $n + 2 \leq i \leq 2n$,

$$c(e_i) = \begin{cases} c_6, & i \text{ is odd} \\ c_5, & i \text{ is even} \end{cases}$$

Hence, $\chi_r(L(S_n)) = 6$ for $r = \Delta$.

□

Theorem 3.3. *Let $n \geq 9$, the r-dynamic chromatic number of the middle graph of a n-Sunlet graph is*

$$\chi_r(M(S_n)) = \begin{cases} 4, 1 \leq r \leq 3, \\ 5, r = \Delta - 2, \\ 6, r = \Delta - 1, \\ 7, r = \Delta, n \equiv 0(\text{mod } 3),\ n \textit{ is even}, \\ 8, r = \Delta, n \equiv 0(\text{mod } 3),\ n \textit{ is odd or } n \equiv 1, 2(\text{mod } 3). \end{cases}$$

Proof. Let

$$\begin{aligned} V(M(S_n)) &= \{v_1, v_2, \ldots, v_n\} \cup \{u_1, u_2, \ldots, u_n\} \\ &\quad \cup \{e_1, e_2, \ldots, e_n\} \cup \{f_1, f_2, \ldots, f_n\}, \end{aligned}$$

where e_i is the vertex corresponding to the edge $v_i v_{i+1}$, $(1 \leq i \leq (n-1))$, e_n is the vertex corresponding to the edge $v_n v_1$ and f_i is the vertex corresponding to the edge $v_i u_i$, $(1 \leq i \leq n)$.

By definition of the middle graph, for $(1 \leq i \leq (n-1))$, the vertices $\{e_i, e_{i+1}, v_{i+1}, f_{i+1}\}$ and $\{e_n, e_1, v_1, f_1\}$ induce a clique of order K_4 in $M(S_n)$. Thus, $\chi_r(M(S_n)) \geq 4$, for any r.

Case 1: $1 \leq r \leq 3$

Consider the color function $c : V(M(S_n)) \to \{c_1, c_2, c_3, c_4\}$ defined as follows.

If n is odd,

For $1 \leq i \leq n-1$,

$$c(e_i)= \begin{cases} c_1, & i \text{ is odd} \\ c_2, & i \text{ is even} \end{cases}$$

Assign $c(e_n) = c_3$, $c(v_1) = c_2$, $c(v_i) = c_3$ $(2 \leq i \leq n-1)$ and $c(v_n) = c_1$.

For $1 \leq i \leq n$, $c(f_i) = c_4$ and $c(u_i) = c_1$.

If n is even,

For $1 \leq i \leq n$, $c(v_i) = c(u_i) = c_1$,

$$c(e_i)= \begin{cases} c_2, & i \text{ is odd} \\ c_3, & i \text{ is even} \end{cases}$$

Assign $c(f_i) = c_4$.

Hence, $\chi_r(M((S_n)) \leq 4$.

Hence, $\chi_r(M(S_n)) = 4$, for $1 \leq r \leq 3$.

Case 2: $r = \Delta - 2$

Consider the color function $c : V(M(S_n)) \to \{c_1, c_2, c_3, c_4, c_5\}$ defined as follows.

If n is odd,

For $1 \leq i \leq n-1$,

$$c(e_i)= \begin{cases} c_1, & i \text{ is odd} \\ c_2, & i \text{ is even} \end{cases}$$

Assign $c(e_n) = c_3$, $c(v_1) = c_2$, $c(v_i) = c_3$ $(2 \leq i \leq n-1)$ and $c(v_n) = c_4$.

For $1 \leq i \leq n-1$,

$$c(f_i)= \begin{cases} c_4, & i \text{ is odd} \\ c_5, & i \text{ is even} \end{cases}$$

$$c(u_i)= \begin{cases} c_5, & i \text{ is odd} \\ c_4, & i \text{ is even} \end{cases}$$

Assign $c(f_n) = c_5$ and $c(u_n) = c_1$.

If n is even,

For $1 \leq i \leq n$, $c(v_i) = c_1$,

$$c(e_i) = \begin{cases} c_2, & i \text{ is odd} \\ c_3, & i \text{ is even} \end{cases}$$

$$c(f_i) = \begin{cases} c_4, & i \text{ is odd} \\ c_5, & i \text{ is even} \end{cases}$$

$$c(u_i) = \begin{cases} c_5, & i \text{ is odd} \\ c_4, & i \text{ is even} \end{cases}$$

Hence, $\chi_r(M(S_n)) = 5$, for $r = \Delta - 2$.

Case 3: $r = \Delta - 1$

Consider the color function $c : V\left(M\left(S_n\right)\right) \to \{c_1, c_2, c_3, c_4, c_5, c_6\}$ defined as follows.

Subcase (i): $n \equiv 0 (\text{mod } 3)$

For $(1 \leq i \leq n)$, Color the vertices e_i cyclically with $\{c_1, c_2, c_3\}$ and u_i cyclically with $\{c_2, c_3, c_1\}$

If n is odd,

For $1 \leq i \leq n-1$

$$c(v_i) = \begin{cases} c_4, & i \text{ is odd} \\ c_5, & i \text{ is even} \end{cases}$$

$$c(f_i) = \begin{cases} c_5, & i \text{ is odd} \\ c_6, & i \text{ is even} \end{cases}$$

Assign $c(v_n) = c_6$ and $c(f_n) = c_4$.

If n is even,

$$c(v_i) = \begin{cases} c_4, & i \text{ is odd} \\ c_5, & i \text{ is even} \end{cases}$$

$$c(f_i)= \begin{cases} c_5, & i \text{ is odd} \\ c_6, & i \text{ is even} \end{cases}$$

Subcase (ii): $n \equiv 1(\text{mod } 3)$

Color the vertices $\{u_4, u_5, \ldots, u_n, u_1, u_2\}$ cyclically with $\{c_1, c_2, c_3\}$ and $c(u_3) = c_4$.

For $1 \le i \le 4$, $c(e_i) = c_i$ and from e_5 color the remaining vertices e_i cyclically with $\{c_1, c_2, c_3\}$.

When n is odd,

Assign $c(v_1) = c_4$ and for $2 \le i \le n$,

$$c(v_i)= \begin{cases} c_6, & i \text{ is odd} \\ c_5, & i \text{ is even} \end{cases}$$

Color the vertices $\{f_1, f_2, f_3, f_4, f_5\}$ cyclically with the colors $\{c_6, c_4, c_1, c_2, c_3\}$, for $6 \le i \le n-1$, $c(f_i) = c_4$ and $c(f_n) = c_5$.

When n is even, For $1 \le i \le n$,

$$c(v_i)= \begin{cases} c_5, & i \text{ is odd} \\ c_6, & i \text{ is even} \end{cases}$$

Color the vertices $\{f_3, f_4, f_5\}$ cyclically with the colors $\{c_1, c_2, c_3\}$ and color the remaining f_i with the color c_4.

Subcase (iii): $n \equiv 2(\text{mod } 3)$

For $1 \le i \le 8$, color the vertices e_i cyclically with $\{c_1, c_2, c_3, c_4\}$ and for $9 \le i \le n$, color cyclically with $\{c_1, c_2, c_3\}$.

For $1 \le i \le 8$, color the vertices u_i cyclically with $\{c_2, c_3, c_4, c_1\}$ and for $9 \le i \le n$, color cyclically with $\{c_2, c_3, c_1\}$.

When n is odd, Assign $c(v_1) = c_4$ and for $2 \le i \le n$,

$$c(v_i)= \begin{cases} c_6, & i \text{ is odd} \\ c_5, & i \text{ is even} \end{cases}$$

Assign $c(f_1) = c_6$,$c(f_2) = c_4$,$c(f_n) = c_5$.

For $3 \le i \le 10$, color the vertices f_i cyclically with $\{c_1, c_2, c_3, c_4\}$ and color the remaining f_i with c_4.

When n is even, For $1 \le i \le n$,

$$c(v_i)= \begin{cases} c_5, & i \text{ is odd} \\ c_6, & i \text{ is even} \end{cases}$$

Assign $c(f_1) = c_6$, $c(f_2) = c_4$.

For $3 \leq i \leq 10$, color the vertices f_i cyclically with $\{c_1, c_2, c_3, c_4\}$ and color the remaining f_i with c_4.

Hence, $\chi_r(M(S_n)) = 6$, for $r = \Delta - 1$

Case 4: $r = \Delta$, $n \equiv 0(\text{mod } 3)$, n is even.

Consider the color function $c : V(M(S_n)) \rightarrow \{c_1, c_2, c_3, c_4, c_5, c_6, c_7\}$ defined as follows.

For $1 \leq i \leq n$, color the vertices e_i cyclically with $\{c_1, c_2, c_3\}$ and u_i cyclically with $\{c_2, c_3, c_1\}$.

For $1 \leq i \leq n$,

$$c(v_i)= \begin{cases} c_4, & i \text{ is odd} \\ c_5, & i \text{ is even} \end{cases}$$

$$c(f_i)= \begin{cases} c_6, & i \text{ is odd} \\ c_7, & i \text{ is even} \end{cases}$$

Hence, $\chi_r(M(S_n)) = 7$, for $r = \Delta$, $n \equiv 0(\text{mod } 3)$ when n is even.

Case 5: $r = \Delta$, $n \equiv 0(\text{mod } 3)$, n is odd or $n \equiv 1$ (mod 3), or n $\equiv 2(\text{mod } 3)$

Consider the color function $c : V(M(S_n)) \rightarrow \{c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8\}$ defined as follows.

Subcase (i): $n \equiv 0(\text{mod } 3)$, n is odd.

For $1 \leq i \leq n$, color the vertices e_i cyclically with $\{c_1, c_2, c_3\}$ and u_i cyclically with $\{c_2, c_3, c_1\}$.

For $1 \leq i \leq n-1$,

$$c(v_i)= \begin{cases} c_4, & i \text{ is odd} \\ c_5, & i \text{ is even} \end{cases}$$

Assign $c(v_n) = c_6$.

Assign $c(f_1) = c(f_{n-1}) = c_8$, $c(f_n) = c_7$ and for $2 \leq i \leq n-2$,

$$c(f_i)=\begin{cases} c_7, & i \text{ is odd} \\ c_6, & i \text{ is even} \end{cases}$$

Subcase (ii): $n \equiv 1(\text{mod } 3)$

When n is even, For $1 \leq i \leq 4$, color the vertices e_i with c_i and from e_5 color the remaining e_i cyclically with $\{c_1, c_2, c_3\}$.

Assign $c(u_3) = c_4$ and color the remaining vertices $\{u_4, u_5, \ldots, u_n, u_1, u_2\}$ cyclically with $\{c_1, c_2, c_3\}$.

Assign $c(v_1) = c_4, c(f_1) = c_6$ and for $(2 \leq i \leq n)$,

$$c(v_i)=\begin{cases} c_6, & i \text{ is odd} \\ c_5, & i \text{ is even} \end{cases}$$

$$c(v_i)=\begin{cases} c_8, & i \text{ is odd} \\ c_7, & i \text{ is even} \end{cases}$$

When n is odd, Color the vertices $\{v_1, e_1, v_2, e_2, \ldots, v_{n-1}, e_{n-1}\}$ cyclically with $\{c_1, c_2, c_3, c_4, c_5, c_6\}$, $c(v_n) = c_7$ and $c(e_n) = c_8$.

Assign $c(f_1) = c_5$ and for $2 \leq i \leq n-2$,

$$c(f_i)=\begin{cases} c_8, & i \text{ is odd} \\ c_7, & i \text{ is even} \end{cases}$$

Assign $c(f_{n-1}) = c_1$ and $c(f_n) = c_3$.

Assign $c(u_1) = c_3$, color the vertices $\{u_2, u_3, \ldots, u_{n-1}\}$ cyclically with $\{c_1, c_2, c_3\}$ and $c(u_n) = c_4$.

Subcase (iii): $n \equiv 2(\text{mod } 3)$

Color the vertices $\{v_1, e_1, v_2, e_2, \ldots, v_{n-2}, e_{n-2}\}$ cyclically with $\{c_1, c_2, c_3, c_4, c_5, c_6\}$ and $\{v_{n-1}, e_{n-1}, v_n, e_n\}$ with $\{c_2, c_3, c_4, c_5\}$.

Assign $c(u_1) = c_3$, color the vertices $\{u_2, u_3, \ldots, u_n\}$ cyclically with $\{c_1, c_2, c_3\}$

When n is odd, Assign $c(f_1) = c_6$ and for $(2 \leq i \leq n)$,

$$c(f_i)=\begin{cases} c_8, & i \text{ is odd} \\ c_7, & i \text{ is even} \end{cases}$$

When n is even, For $(1 \leq i \leq n)$,

$$c(f_i)=\begin{cases} c_7, & i \text{ is odd} \\ c_8, & i \text{ is even} \end{cases}$$

Hence, $\chi_r(M(S_n)) = 8$ for $r = \Delta$, $n \equiv 0 (\text{mod } 3)$, n is odd or $n \equiv 1, 2 (\text{mod } 3)$.

□

Theorem 3.4. *Let $n \geq 9$, the r-dynamic chromatic number of the total graph of a n-Sunlet graph is*

$$\chi_r(T(S_n)) = \begin{cases} 4, 1 \leq r \leq 3, \\ 5, r = \Delta - 2, \\ 6, r = \Delta - 1, \\ 7, r = \Delta,\ n \equiv 0(\text{mod } 5)\ ,\ n \textit{ is even,} \\ 8, r = \Delta,\ n \equiv 0(\text{mod } 5),\ n \textit{ is odd or } n \equiv 0(\text{mod } 3), \\ \qquad n \textit{ is even or } n \equiv 1,2(\text{mod } 3), \\ 9, r = \Delta,\ n \equiv 0(\text{mod } 3),\ n \textit{ is odd.} \end{cases}$$

Proof. Let

$$\begin{aligned} V(T(S_n)) &= \{v_1, v_2, \ldots, v_n\} \cup \{u_1, u_2, \ldots, u_n\} \\ &\cup \{e_1, e_2, \ldots, e_n\} \cup \{f_1, f_2, \ldots, f_n\}, \end{aligned}$$

where e_i is the vertex corresponding to the edge $v_i v_{i+1}$, $(1 \leq i \leq (n-1))$, e_n is the vertex corresponding to the edge $v_n v_1$ and f_i is the vertex corresponding to the edge $v_i u_i$, $(1 \leq i \leq n)$. By definition of the total graph, for $(1 \leq i \leq (n-1))$, the vertices $\{e_i, e_{i+1}, v_{i+1}, f_{i+1}\}$ and $\{e_n, e_1, v_1, f_1\}$ induces a clique of order K_4 in $M(S_n)$.Thus, $\chi_r(T(S_n)) \geq 4$, for any r.

Case 1: $1 \leq r \leq 3$

Consider the color function $c : V(T(S_n)) \to \{c_1, c_2, c_3, c_4\}$ defined as follows.

If n is odd, For $1 \leq i \leq n-1$,

$$c(e_i)= c(u_i)=\begin{cases} c_1, & i \text{ is odd} \\ c_2, & i \text{ is even} \end{cases}$$

$$c(v_i)=\begin{cases} c_4, & i \text{ is odd} \\ c_3, & i \text{ is even} \end{cases}$$

Assign $c(e_n) = c_3$, $c(v_n) = c_1$ and $c(u_n) = c_2$.

Assign $c(f_1) = c_2$ and $c(f_n) = c_4$.

For $2 \leq i \leq n-1$,

$$c(f_i)= \begin{cases} c_3, & i \text{ is odd} \\ c_4, & i \text{ is even} \end{cases}$$

If n is even, For $1 \leq i \leq n$, $c(u_i) = c_1$,

$$c(e_i)= \begin{cases} c_1, & i \text{ is odd} \\ c_2, & i \text{ is even} \end{cases}$$

$$c(v_i)= \begin{cases} c_3, & i \text{ is odd} \\ c_4, & i \text{ is even} \end{cases}$$

$$c(f_i)= \begin{cases} c_3, & i \text{ is odd} \\ c_4, & i \text{ is even} \end{cases}$$

Hence, $\chi_r(T((S_n)) \leq 4$.Thus, $\chi_r(T(S_n)) = 4$, for $1 \leq r \leq 3$.

Case 2: $r = \Delta - 2$

Consider the color function $c : V(T(S_n)) \to \{c_1, c_2, c_3, c_4, c_5\}$ defined as follows.

If n is odd, For $1 \leq i \leq n-1$,

$$c(e_i)= \begin{cases} c_1, & i \text{ is odd} \\ c_2, & i \text{ is even} \end{cases}$$

$$c(v_i)= \begin{cases} c_4, & i \text{ is odd} \\ c_5, & i \text{ is even} \end{cases}$$

Assign $c(e_n) = c_3$ and $c(v_n) = c_1$.

Assign $c(f_1) = c_2$, $c(f_i) = c_3$ $(2 \leq i \leq n-2)$, $c(f_{n-1}) = c_4$ and $c(f_n) = c_5$.

For $1 \leq i \leq n-2$,

$$c(u_i)= \begin{cases} c_5, & i \text{ is odd} \\ c_4, & i \text{ is even} \end{cases}$$

Assign $c(u_{n-1}) = c_3$ and $c(u_n) = c_4$.

If n is even, For $1 \leq i \leq n$, $c(v_i) = c_1$,

$$c(e_i) = \begin{cases} c_2, & i \text{ is odd} \\ c_3, & i \text{ is even} \end{cases}$$

$$c(f_i) = \begin{cases} c_4, & i \text{ is odd} \\ c_5, & i \text{ is even} \end{cases}$$

$$c(u_i) = \begin{cases} c_5, & i \text{ is odd} \\ c_4, & i \text{ is even} \end{cases}$$

Hence, $\chi_r(T(S_n)) = 5$, for $r = \Delta - 2$.

Case 3: $r = \Delta - 1$

Consider the color function $c : V(M(S_n)) \to \{c_1, c_2, c_3, c_4, c_5, c_6\}$ defined as follows.

Subcase (i): $n \equiv 0(\mod 3)$

For $1 \leq i \leq n$, color the vertices e_i cyclically with $\{c_1, c_2, c_3\}$, u_i cyclically with $\{c_2, c_3, c_1\}$, v_i cyclically with $\{c_4, c_5, c_6\}$ and f_i cyclically with $\{c_6, c_4, c_5\}$.

Subcase (ii): $n \equiv 1(\text{mod})$

For $1 \leq i \leq 4$, color the vertices e_i with c_i and from e_5 color the remaining e_i cyclically with $\{c_1, c_2, c_3\}$. Color the vertices $\{u_n, u_1, u_2, u_3\}$ with $\{c_1, c_2, c_3, c_4\}$ and from u_4 color the remaining u_i cyclically with $\{c_1, c_2, c_3\}$.

Assign $c(v_1) = c_4$ and for $(2 \leq i \leq n)$,

$$c(v_i) = \begin{cases} c_6, & i \text{ is odd} \\ c_5, & i \text{ is even} \end{cases}$$

When n is odd, Color the vertices $\{f_n, f_1, f_2, f_2, f_4, f_5\}$ with $\{c_5, c_6, c_4, c_1, c_2, c_3\}$ and remaining f_i with the color c_4.

When n is even, Color the vertices $\{f_1, f_2, f_2, f_4, f_5\}$ with $\{c_6, c_4, c_1, c_2, c_3\}$ and remaining f_i with the color c_4.

Subcase (iii): $n \equiv 2(\text{mod } 3)$

For $1 \leq i \leq 8$, color the vertices e_i cyclically with $\{c_1, c_2, c_3, c_4\}$ and from e_9 color the remaining e_i cyclically with $\{c_1, c_2, c_3\}$.

For $3 \leq i \leq 10$, color the vertices u_i cyclically with $\{c_1, c_2, c_3, c_4\}$ and from u_{11} color the remaining u_i cyclically with $\{c_1, c_2, c_3\}$.

Assign $c(v_1) = c_4$ and for $(2 \leq i \leq n)$

$$c(v_i)= \begin{cases} c_6, & i \text{ is odd} \\ c_5, & i \text{ is even} \end{cases}$$

When n is odd, Assign $c(f_1) = c_5, c(f_2) = c_6, c(f_3) = c_4$, for $4 \leq i \leq 11$ color the vertices f_i cyclically with $\{c_1, c_2, c_3, c_4\}$ and color the remaining f_i with c_4.

When n is even, Assign $c(f_n) = c(f_1) = c(f_2) = c_6, c(f_3) = c_4$, for $4 \leq i \leq 11$ color the vertices f_i cyclically with $\{c_1, c_2, c_3, c_4\}$ and from u_{12} color the remaining f_i with the color c_4.

Hence, $\chi_r(T(S_n)) = 6$, for $r = \Delta - 1$.

Case 4: $r = \Delta$, $n \equiv 0 (\text{mod } 5)$, n is even.

Consider the color function $c : V(T(S_n)) \rightarrow \{c_1, c_2, c_3, c_4, c_5, c_6, c_7\}$ defined as follows.

Color the vertices $\{v_1, e_1, v_2, e_2, v_3, \ldots, v_n, e_n\}$ cyclically with the colors $\{c_1, c_2, c_3, c_4, c_5\}$.

For $(1 \leq i \leq n)$

$$c(f_i)= \begin{cases} c_6, & i \text{ is odd} \\ c_7, & i \text{ is even} \end{cases}$$

$$c(u_i)= \begin{cases} c_7, & i \text{ is odd} \\ c_6, & i \text{ is even} \end{cases}$$

Hence, $\chi_r(T(S_n)) = 7$, for $r = \Delta$,$n \equiv 0 (\text{mod } 5)$, n is even.

Case 5: $r = \Delta$, $n \equiv 0 (\text{mod } 5)$, n is odd or $n \equiv 0 (\text{mod } 3)$, n is even or $n \equiv 1, 2 (\text{mod } 3)$

Consider the color function $c : V(T(S_n)) \rightarrow \{c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8\}$ defined as follows.

Subcase (i): $n \equiv 0 (\text{mod } 5)$, n is odd.

Color the vertices $\{v_1, e_1, v_2, e_2, v_3, \ldots, v_n, e_n\}$ cyclically with the colors $\{c_1, c_2, c_3, c_4, c_5\}$.

For $1 \leq i \leq n - 1$,

$$c(f_i)= \begin{cases} c_6, & i \text{ is odd} \\ c_7, & i \text{ is even} \end{cases}$$

$$c(u_i)=\begin{cases} c_7, & i \text{ is odd} \\ c_6, & i \text{ is even} \end{cases}$$

In order to maintain the adjacency condition we need one new color c_8 to color the vertex f_n and assign $c(u_n) = c_6$.

Subcase (ii): $n \equiv 0(\text{mod } 3)$, n is even.

For $1 \le i \le n$, color the vertices e_i cyclically with $\{c_1, c_2, c_3\}$, v_i cyclically with $\{c_4, c_5, c_6\}$.

$$c(f_i)=\begin{cases} c_7, & i \text{ is odd} \\ c_8, & i \text{ is even} \end{cases}$$

$$c(u_i)=\begin{cases} c_8, & i \text{ is odd} \\ c_7, & i \text{ is even} \end{cases}$$

Subcase (iii): $n \equiv 1(\text{mod } 3)$.

Color the vertices $\{v_1, e_1, v_2, e_2, v_3, \ldots, v_{n-1}, e_{n-1}\}$ cyclically with the colors $\{c_1, c_2, c_3, c_4, c_5, c_6\}$, $c(v_n) = c_7$ and $c(e_n) = c_8$. Assign $c(f_1) = c_5$, for $2 \le i \le n-2$ color the vertices f_i alternatively with c_7 and c_8 , $c(f_{n-1}) = c_1$ and $c(f_n) = c_3$. For $(1 \le i \le n)$ color the vertices u_i cyclically with $\{c_4, c_6, c_2\}$.

Subcase (iv): $n \equiv 2(\text{mod } 3)$.

Color the vertices $\{v_1, e_1, v_2, e_2, v_3, \ldots, v_{n-2}, e_{n-2}\}$ cyclically with the colors $\{c_1, c_2, c_3, c_4, c_5, c_6\}$ and $\{v_{n-1}, e_{n-1}, v_n, e_n\}$ cyclically with the colors $\{c_2, c_3, c_4, c_5\}$. Assign $c(f_1) = c_6$ and color the remaining f_i alternatively with c_7 and c_8. For $(1 \le i \le n)$ color the vertices u_i alternatively with c_7 and c_8. Now, an easy check shows that r- adjacency condition is fulfilled. Hence, $\chi_r(T(S_n)) = 8$, for $r = \Delta$, $n \equiv 5(\text{mod } 3)$, n is odd or $n \equiv 0(\text{mod } 3)$, n is even or $n \equiv 1, 2(\text{mod } 3)$

Case 6: $r = \Delta$, $n \equiv 0(\text{mod } 3)$, n is odd. Consider the color function $c : V(T(S_n)) \to \{c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9\}$ defined as follows. For $1 \le i \le n$, color the vertices e_i cyclically with $\{c_1, c_2, c_3\}$, v_i cyclically with $\{c_4, c_5, c_6\}$ and u_i cyclically with $\{c_2, c_3, c_1\}$.

For $1 \le i \le n-1$,

$$c(f_i)=\begin{cases} c_7, & i \text{ is odd} \\ c_8, & i \text{ is even} \end{cases}$$

In order to maintain the adjacency condition we need one new color c_9 to color the vertex f_n. Hence, $\chi_r(T(S_n)) = 9$, $r = \Delta$, $n \equiv 0(\text{mod } 3)$ when n is odd.

□

Theorem 3.5. *Let $n \geq 6$, the r-dynamic chromatic number of the central graph of a n-Sunlet graph is*

$$\chi_r(C(S_n)) = \begin{cases} n, \ r = 1, \\ 2n, \ 2 \leq r \leq \Delta - 1, \\ 2n+3, \ r = \Delta, n \text{ is even}, \\ 2n+4, \ r = \Delta, n \text{ is odd}. \end{cases}$$

Proof. Let

$$\begin{aligned} V\left(C\left(S_n\right)\right) &= \left\{v_1, v_2, \ldots, v_n\right\} \cup \left\{u_1, u_2, \ldots, u_n\right\} \\ &\cup \left\{e_1, e_2, \ldots, e_n\right\} \cup \left\{f_1, f_2, \ldots, f_n\right\}, \end{aligned}$$

where e_i is the vertex corresponding to the edge v_iv_{i+1} , $(1 \leq i \leq (n-1))$, e_n is the vertex corresponding to the edge v_nv_1 and f_i is the vertex corresponding to the edge v_iu_i , $(1 \leq i \leq n)$. By definition of the central graph, the vertices $\{u_1, u_2, \ldots, u_n\}$ induce a clique of order K_n in $C(S_n)$. Thus, $\chi_r(C(S_n)) \geq n$, for any r.

Case 1: $r = 1$

Consider the color function $c : V\left(C\left(S_n\right)\right) \rightarrow \{c_1, c_2, \ldots, c_n\}$ defined as follows. For $1 \leq i \leq n$, $c(v_i) = c(u_i) = c_i$ and color the vertices e_i cyclically with $\{c_3, c_4, \ldots, c_n, c_1, c_2\}$. For $(1 \leq i \leq n-1)$ $c(f_i) = c_{i+1}$ and $c(f_n) = c_1$. Hence, $\chi_r(C((S_n)) \leq n$. Therefore, $\chi_r(C(S_n)) = n$, for $r = 1$.

Case 2: $2 \leq r \leq \Delta - 1$

Consider the color function $c : V\left(C\left(S_n\right)\right) \rightarrow \{c_1, c_2, \ldots, c_{2n}\}$ defined as follows. For $1 \leq i \leq n$, $c(u_i) = c(e_i) = c_i$, $c(v_i) = c_{n+i}$. For $1 \leq i \leq n-1$, $c(f_i) = c_{n+i+1}$ and $c(f_n) = c_{n+1}$. Hence, $\chi_r(C(S_n)) = 2n$, $2 \leq r \leq \Delta - 1$.

Case 3: $r = \Delta$, n is even.

Consider the color function $c : V\left(C\left(S_n\right)\right) \rightarrow \{c_1, c_2, \ldots, c_{2n+3}\}$ defined as follows. For $1 \leq i \leq n$, $c(u_i) = c_i$, $c(v_i) = c_{n+i}$ and $c(f_i) = c_{2n+1}$.

For $1 \leq i \leq n$,

$$c(e_i) = \begin{cases} c_{2n+2}, \ i \text{ is odd} \\ c_{2n+3}, \ i \text{ is even} \end{cases}$$

Hence, $\chi_r(C(S_n)) = 2n+3$, $r = \Delta$ for n is even.

Case 4: $r = \Delta$, n is odd.

Consider the color function $c : V(C(S_n)) \to \{c_1, c_2, \ldots, c_{2n+4}\}$ defined as follows. For $1 \le i \le n$, $c(u_i) = c_i$, $c(v_i) = c_{n+i}$ and $c(f_i) = c_{2n+1}$.

For $1 \le i \le n-1$,

$$c(e_i)= \begin{cases} c_{2n+2}, & i \text{ is odd} \\ c_{2n+3}, & i \text{ is even} \end{cases}$$

In order to maintain the r-adjacency condition we need one new color c_{2n+4} to color the vertex e_n. Hence, $\chi_r(C(S_n)) = 2n+4$, $r = \Delta$ for n is odd.

□

REFERENCES

[1] I. H. Agustin, Dafik, A.Y.Harsya, On r-dynamic coloring of some graph operations, Indonesian Journal of Combinatorics, 1(1), (2016), 22–30.

[2] J.A. Bondy, U.S.R. Murty, Graph theory with applications, New York: Macmillan Ltd. Press, 1976.

[3] H. Furmańczyk, J. Vernold Vivin, N. Mohanapriya, r-dynamic chromatic number of some line graphs, Indian Journal of Pure and Applied Mathematics 49(4), (2018), 591–600.

[4] F. Harary, Graph Theory, Narosa Publishing home, New Delhi 1969.

[5] K. Kaliraj, H. Naresh Kumar, J. Vernold Vivin, On dynamic colouring of cartesian product of complete graph with some graphs, Journal of Taibah University for Science, 14(1), (2020), 168–171.

[6] D. Michalak, On middle and total graphs with coarseness number equal 1, Springer Verlag Graph Theory, Lagow proceedings, Berlin Heidelberg, New York, Tokyo, (1981), 139–150.

[7] N.Mohanapriya, J.Vernold Vivin, M.Venkatachalam, δ-dynamic chromatic number of Helm graph families, Cogent Mathematics, (2016), 3: 1178411.

[8] B. Montgomery, Dynamic coloring of graphs, ProQuest LLC, Ann Arbor, MI, (2001), Ph.D Thesis, West Virginia University.

[9] G.Nandini, M.Venkatachalam, S. Gowri, On r-dynamic coloring of the family of bistar graphs, Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 68(1), (2019), 923–928.

[10] G.Nandini, M.Venkatachalam, Raul M. Falcon, On r-dynamic coloring of Subdivision-Edge Corronation by Path, AIMS Mathematics, 5(5), (2020), 4546-4562.

[11] G.Nandini and M.Venkatachalam, On r-dynamic coloring of of Para-Line Graph of Some Standard Graphs, Palestine Journal of Mathematics, 10, (2021), 12-22.

[12] J. Vernold Vivin, N. Mohanapriya, J. Kok, M. Venkatachalam, On dynamic coloring of certain cycle-related graphs, Arabian Journal of Mathematics, 9, (2020), 213–221,

[13] J. Vernold Vivin , Ph.D Thesis, Harmonious coloring of total graphs, $n-$leaf, central graphs and circumdetic graphs, Bharathiar University, (2007), Coimbatore, India.

[14] J. Vernold Vivin, M.Venkatachalam, On b-chromatic number of sunlet graph and wheel graph families, Journal of the Egyptian Mathematical Society, 23, (2015), 215–218.