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Forgotten Index of Duplication and Double Duplication of Graphs by Means of Degree Sequences

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Abstract

The concept of forgotten index of a graph G was introduced by Furtula and Gutman in 2015 as $F(G) = \sum_{u\in V(G)} d_u^3$ where d_u is the degree of a vertex u in G. It is a degree based topological index that correlates well with many physio-chemical properties of organic compounds. Duplication of a vertex by an edge and duplication of an edge by a vertex of graphs were introduced by Vaidya et al. in 2009. Double duplication of graphs were introduced by Shobana and Roopa in 2017. In this paper, some general results and inequalities on forgotten index of duplication and double duplication of graphs using degree sequence and edge and vertex partitions of these duplication graphs are obtained.

Keywords: forgotten index, degree sequence, duplication, double duplication

1 Introduction

Graph theory plays an important role in chemistry with the help of mathematical models. The mathematical chemistry is used to guess the chemical and physical properties of molecules such as the boiling and melting points which are related to the geometric structure of the organic compound called alkanes composed of carbon and hydrogen atoms. Molecular structures are represented graphically, atoms named as vertices and bonds named as edges. Then deleting all the pendant vertices corresponding to hydrogen atoms results in a carbon graph without losing any information on the molecular structure.

In this paper, we consider only the simple connected graphs. The degree of a vertex d_u is the number of edges incident with it. The degree sequence of a graph is the set of all vertex degrees. A list of vertices and their vertex degrees together with multiplicities will be called the vertex partition of the graph. SİmIlarly, a list of the pairs of the neighbor vertices together with the pairs of vertex degrees and the number of such edges will be called as the edge partition.

The first Zagreb index $M_1(G)$ of a graph G was first introduced by Gutman and Trinajstic in 1972, [13], defined by

$$M_1(G) = \sum_{u \in V(G)} d_u^2. \tag{1}$$

Together with the second Zagreb index, it is probably the most intensively studied topological graph index, see e.g. [2, 4, 5, 6, 15, 16, 18].

The forgotten index $F(G)$ of a graph G was introduced by Furtula and Gutman in 2015, [11], which is defined by

$$F(G) = \sum_{u \in V(G)} d_u^3. \tag{2}$$

Forgotten index was defined to analyse the drug molecule structure which is helpful for medical scientists to get the chemical and biological characteristics of drugs.

Recently, a new and useful graph invariant is defined in [7] and intensively studied in many papers to determine the graph theoretical, combinatorial and topological properties of the graphs and given degree sequences, see [8, 9, 10], can be applied to study duplication and double duplication of graphs.

Definition 1.1. [19] Duplication of a vertex by an edge was introduced by Vaidya et al. in 2009 as duplication of a vertex v_k by a new edge $e = v'v''$

in graph G which produces a new graph G' such that $N(v') = \{v_k, v''\}$ and $N(v'') = \{v_k, v'\}$. We denote the duplication of all vertices by edges of G by $D_V(G)$.

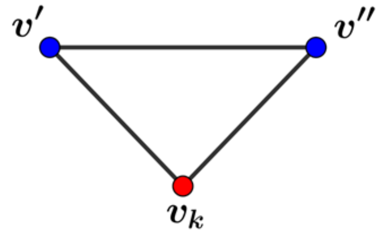

Figure 1: $D_V(v_k)$

Definition 1.2. [19] Duplication of edge by a vertex is defined as duplication of an edge $e = v_i v_{i+1}$ by a vertex v' in a graph G which produces a new graph G' such that $N(v') = \{v_i, v_{i+1}\}$. We denote the duplication of all edges by vertices of G by $D_E(G)$.

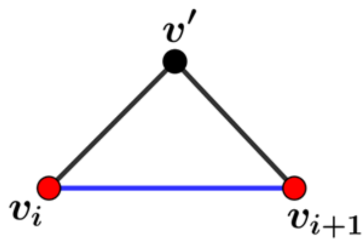

Figure 2: $D_E(v_i v_{i+1})$

Definition 1.3. [17] The double duplication of a vertex by an edge of a graph was introduced by Roopa and Shobana in 2017, [17], as duplication of a vertex v_k by an edge $e = v'_k v''_k$ in a graph G produces a graph G', in which $N(v'_k) = \{v_k, v''_k\}$ and $N(v''_k) = \{v_k, v'_k\}$. Again duplication of vertices v_k, v'_k and v''_k by edges $e' = u_k w_k$, $e'' = u'_k w'_k$ and $e''' = u''_k w''_k$ respectively in G' produces a new graph G'' such that, $N(u_k) = \{w_k, v_k\}$, $N(w_k) = \{u_k, v_k\}$, $N(u'_k) = \{w'_k, v'_k\}$, $N(w'_k) = \{u'_k, v'_k\}$, $N(u''_k) = \{w''_k, v''_k\}$ and $N(w''_k) = \{u''_k, v''_k\}$. Double duplication of vertices by edges respectively of a graph G is denoted by $DD_{VV}(G)$.

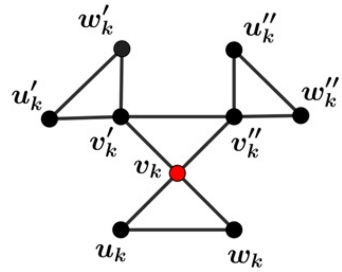

Figure 3: $DD_{VV}(v_k)$

Definition 1.4. [17] The double duplication of a vertex by an edge followed by edge by vertex of a graph is defined as the duplication of a vertex v_k by an edge $e = v'_k v''_k$ in a graph G which produces a graph G' in which $N(v'_k) = \{v_k, v''_k\}$ and $N(v''_k) = \{v_k, v'_k\}$. Similarly, duplication of edges $v_k v'_k, v'_k v''_k$ and $v_k v''_k$ by vertices u_k, u'_k and u''_k respectively in G' produces a new graph G'' such that $N(u_k) = \{v_k, v'_k\}$, $N(u'_k) = \{v'_k, v''_k\}$ and $N(u''_k) = \{v_k, v''_k\}$. Double duplication of a graph G is vertices by edges followed with edges by vertices, respectively, and is denoted by $DD_{VE}(G)$.

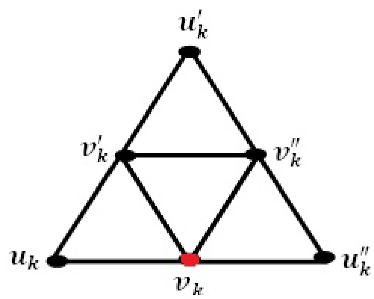

Figure 4: $DD_{VE}(v_k)$

Definition 1.5. [17] The double duplication of an edge by a vertex followed with a vertex by an edge of a graph is defined as duplication of an edge $e_k = v_k v_{k+1}$ by a vetex v'_k in a graph G which produces a graph G' in which $N(v'_k) = \{v_k, v_{k+1}\}$. Again the duplication of v_k, v_{k+1} and v'_k by the edges $e' = u_k w_k$, $e'' = u_{k+1} w_{k+1}$ and $e''' = u'_k w'_k$ in G' produces a new graph G'' such that $N(u_k) = \{v_k, w_k\}$, $N(w_k) = \{u_k, v_k\}$, $N(u_{k+1}) = \{v_{k+1}, w_{k+1}\}$, $N(w_{k+1}) = \{u_{k+1}, v_{k+1}\}$, $N(u'_k) = \{v'_k, w'_k\}$ and $N(w'_k) = \{u'_k, v'_k\}$. Double duplication of a graph G is edges by vertices followed by the vertices by edges, respectively and is denoted by $DD_{EV}(G)$.

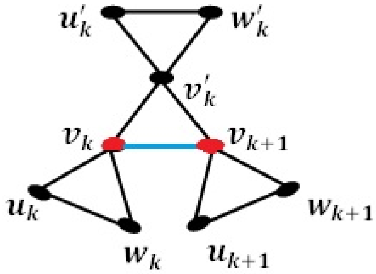

Figure 5: $DD_{EV}(v_k v_{k+1})$

Definition 1.6. [17] The double duplication of an edge by a vertex followed with an edge by a vertex of a graph G is defined as the duplication of an edge $e = v_k v'_k$ by a vertex v''_k produces a graph G' in which $N(v''_k) = \{v_k, v'_k\}$. Again duplication of the edges $v_k v'_k$, $v'_k v''_k$ and $v_k v''_k$ by the vertices u''_k, u'_k and u_k respectively in G' produces a new graph G'' such that $N(u_k) = \{v_k, v''_k\}$, $N(u'_k) = \{v'_k, v''_k\}$ and $N(u''_k) = \{v_k, v'_k\}$. Double duplication of edges by vertices, respectively, of a graph G is denoted by $DD_{EE}(G)$.

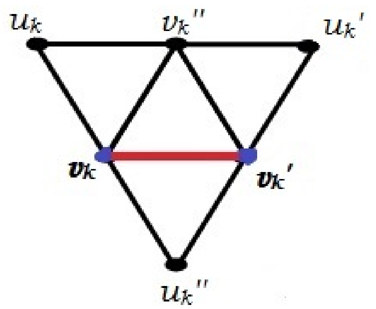

Figure 6: $DD_{EE}(v_k v'_k)$

In this paper, we proved some general results and inequalities on the forgotten index of duplication and double duplication of graphs.

2 Main Results

In this section, by G, we mean a graph having n vertices and m edges.

Theorem 2.1. For a graph G, we have

$$F(D_V(G)) = F(G) + 6M_1(G) + 24(n + m).$$

Proof. Let the graph $D_V(G)$ be obtained by duplication of all vertices by edges of G. It contains $3n$ vertices and $m+3n$ edges. By the definition of forgotten index, we have

$$\begin{aligned} F(D_V(G)) &= \sum_{i=1}^{n}(d_i+2)^3 + 16n \\ &= \sum_{i=1}^{n} d_i^3 + 8n + 6\sum_{i=1}^{n} d_i^2 + 12\sum_{i=1}^{n} d_i + 16n \\ &= F(G) + 6M_1(G) + 24(n+m) \end{aligned}$$

by the handshaking lemma. □

Theorem 2.2. For a graph G, we have

$$F(D_E(G)) = 8F(G) + 8m.$$

Proof. Let the graph $D_E(G)$ be obtained by duplication of all edges by vertices of G which contains $n+m$ vertices and $3m$ edges. By the definition of forgotten index, it follows that

$$\begin{aligned} F(D_E(G)) &= \sum_{i=1}^{n}(d_i+d_i)^3 + 8m \\ &= 8\sum_{i=1}^{n} d_i^3 + 8m \\ &= 8F(G) + 8m. \end{aligned}$$

□

Theorem 2.3. For a graph G, we have

$$F(DD_{VV}(G)) = F(G) + 12M_1(G) + 240n + 96m.$$

Proof. Let $DD_{VV}(G)$ be obtained by double duplication of all vertices by edges of G which contains $9n$ vertices and $12n+m$ edges. Then

$$\begin{aligned} F(DD_{VV}(G)) &= \sum_{i=1}^{n}(d_i+4)^3 + 176n \\ &= \sum_{i=1}^{n} d_i^3 + \sum_{i=1}^{n} 64 + 12\sum_{i=1}^{n} d_i^2 + 48\sum_{i=1}^{n} d_i + 176n \\ &= F(G) + 12M_1(G) + 240n + 96m. \end{aligned}$$

□

Theorem 2.4. For a graph G, we have

$$F(DD_{EE}(G)) = 64F(G) + 88m.$$

Proof. Let $DD_{EE}(G)$ is obtained by double duplication of all edges by vertices of G which contains $n + 4m$ vertices and $9m$ edges. Then we have

$$\begin{aligned} F(DD_{EE}(G)) &= \sum_{i=1}^{n}(d_i + d_i + 2d_i)^3 + 4^3m + 24m \\ &= \sum_{i=1}^{n}(4d_i)^3 + 88m \\ &= 64F(G) + 88m \end{aligned}$$

□

Theorem 2.5. For a graph G, we have

$$F(DD_{VE}(G)) = 8(F(G) + 6M_1(G) + 27n + 25m).$$

Proof. Let G' be a graph obtained by duplication of all vertices by edges of G and let $G'' = DD_{VE}(G)$ be a graph which is obtained by duplication of all edges by vertices of G' which contains $6n + m$ vertices and $9n + 3m$ edges. Then

$$\begin{aligned} F(G'') &= \sum_{i=1}^{n}(2d_i + 4)^3 + 128n + 8(3n + m) \\ &= 8\sum_{i=1}^{n}({d_i}^3 + 8 + 6d_i(d_i + 2)) + 152n + 8m \\ &= 8F(G) + 64n + 192m + 48M_1(G) + 152n + 8m \\ &= 8(F(G) + 6M_1(G) + 27n + 25m). \end{aligned}$$

□

Theorem 2.6. For a graph G, we have

$$F(DD_{EV}(G)) = 8(F(G) + 3M_1(G) + 3n + 16m).$$

Proof. Let G^* be a graph obtained by duplication of all edges by vertices of G and let $G^{**} = DD_{EV}(G)$ be a graph which is obtained by duplication of all vertices by edges of G' which contains $3n+3m$ vertices and $3n+6m$ edges. Then we have

$$\begin{aligned}F(G^{**}) &= \sum_{i=1}^{n}(2d_i+2)^3 + 64m + 16(n+m)\\ &= 8\sum_{i=1}^{n}(d_i{}^3 + 1 + 3d_i(d_i+1)) + 16n + 80m\\ &= 8F(G) + 8n + 24M_1(G) + 48m + 16n + 80m\\ &= 8(F(G) + 3M_1(G) + 3n + 16m).\end{aligned}$$

□

As an application we have

Theorem 2.7. Let $K_n, (n \geq 6)$ be a complete graph with n vertices. Then

$$\begin{aligned}F(D_V(K_n)) < F(DD_{VV}(K_n)) < F(D_E(K_n)) < F(DD_{EV}(K_n))\\ < F(DD_{VE}(K_n)) < F(DD_{EE}(K_n)).\end{aligned}$$

Proof. We act as in the following cases:

To obtain these inequalities, we compare the values of all the indices $F(DD_{VV}(K_n))$, $F(D_E(K_n))$, $F(DD_{EV}(K_n))$, $F(DD_{VE}(K_n))$ and $F(DD_{EE}(K_n))$ by considering their differences:

- $F(D_V(K_n)) - F(DD_{VV}(K_n)) = -6n^3 - 24n^2 - 186n < 0$
  $\Longrightarrow F(D_V(K_n)) < F(DD_{VV}(K_n))$;
- $F(DD_{VV}(K_n)) - F(D_E(K_n)) = -7n^4 + 33n^3 - n^2 + 215n < 0$
  $\Longrightarrow F(DD_{VV}(K_n)) < F(D_E(K_n))$;
- $F(D_E(K_n)) - F(DD_{EV}(K_n)) = -24n^3 - 12n^2 + 12n < 0$
  $\Longrightarrow F(D_E(K_n)) < F(DD_{EV}(K_n))$;
- $F(D_{EV}(K_n)) - F(DD_{VE}(K_n)) = -24n^3 + 12n^2 - 180n < 0$
  $\Longrightarrow F(D_{EV}(K_n)) < F(DD_{VE}(K_n))$;
- $F(DD_{VE}(K_n)) - F(DD_{EE}(K_n)) = -56n^4 + 192n^3 - 196n^2 + 84n < 0$
  $\Longrightarrow F(DD_{VE}(K_n)) < F(DD_{EE}(K_n))$.

From all above cases, it follows that $F(D_V(K_n)) < F(DD_{VV}(K_n)) < F(D_E(K_n)) < F(DD_{EV}(K_n)) < F(DD_{VE}(K_n)) < F(DD_{EE}(K_n))$. □

Observation. Theorem 2.7 is applicable for all values of $n \geq 6$. The following inequalitites are deduced for $n = 2, 3, 4$ and 5:
(i) For $n = 2$, $F(D_E(K_n)) < F(D_V(K_n)) < F(DD_{EE}(K_n)) < F(DD_{EV}(K_n)) < F(DD_{VV}(K_n)) < F(DD_{VE}(K_n))$.
(ii) For $n = 3$, $F(D_E(K_n)) < F(D_V(K_n)) < F(DD_{EV}(K_n)) < F(DD_{VV}(K_n)) < F(DD_{EE}(K_n)) < F(DD_{VE}(K_n))$.
(iii) For $n = 4$ and $n = 5$, $F(D_V(K_n)) < F(D_E(K_n)) < F(DD_{VV}(K_n)) < F(DD_{EV}(K_n)) < F(DD_{VE}(K_n)) < F(DD_{EE}(K_n))$.

3 Conclusion

In this paper, we obtained some general results and inequalities on the forgotten index of the duplication and double duplication graphs by means of degree sequence, edge and vertex partitions. Also finding some more inequalities in our future work which can be implemented as a real time application in encrypt and decrypt the messages.

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