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ON THE PARTITION FUNCTION

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ABSTRACT. In this paper, we exhibit short deductions of the Jha and Malenfant expressions for the partition function, and we show several connections between the Euler, divisor and partition functions via the partial Bell polynomials and Hessenberg determinants. Besides, we show that the Gandhi's recurrence relation for colour partitions implies the Osler-Hassen-Chandrupatla's expression for the divisor function.

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1. INTRODUCTION

We know the following result [11, 7, 19, 5, 12] for the partition function $p(n)$:

$$\sum_{n=0}^{\infty} p(n)t^n = \frac{1}{\sum_{r=0}^{\infty} a_r t^r} = \frac{1}{E(q)} \equiv \frac{1}{(q;q)_\infty}, \qquad p(0) = a_0 = 1, \tag{1}$$

where:

$$a_j = \begin{cases} 0, & if \;\; j \neq \frac{N}{2}(3N+1), \\ (-1)^N, & if \;\; j = \frac{N}{2}(3N+1), \end{cases} \qquad N = 0, \pm 1, \pm 2, ..., \tag{2}$$

and $E(q)$ is the Euler function; therefore [4, 2, 21, 15]:

$$p(n) = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \, k! \, B_{n,k}(1! \, a_1, \, 2! \, a_s, ..., (n-k+1)! \, a_{n-k+1}!), \tag{3}$$

expression recently deduced by Jha [15], involving the partial Bell polynomials [4, 15, 6, 16], with the recurrence relation [19, 2, 18]:

$$\sum_{k=0}^{n} a_k \, p(n-k) = 0, \tag{4}$$

discovered by MacMahon [7, 19, 18].

On the other hand, from [23, 10] we have that relations type (1) are equivalent to the following Hessenberg determinant:

$$p(n)=(-1)^n \begin{vmatrix} a_1 & a_0 & 0 & 0 & \dots & \dots & 0 \\ a_2 & a_1 & a_0 & 0 & \dots & \dots & 0 \\ a_3 & a_2 & a_1 & a_0 & \dots & \dots & 0 \\ . & . & . & . & \dots & \dots & \dots \\ . & . & . & . & \dots & \dots & \dots \\ . & . & . & . & \dots & \dots & a_0 \\ a_n & a_{n-1} & a_{n-2} & \dots & \dots & \dots & a_1 \end{vmatrix}, \tag{5}$$

obtained by Malenfant [18].

Remark.- The definition (2) gives the values:

$$a_j = \begin{cases} 1, \ j=0,5,7,22,26,51,57,92,100,145,155,\dots \\ & 0 \quad otherwise. \\ -1, \ j=1,2,12,15,35,40,70,77,117,126,176,\dots \end{cases} \tag{6}$$

and from (1):

$$E(q)=\sum_{n=0}^{\infty}\frac{E^{(n)}(0)}{n!}\,q^n, \qquad E^{(n)}(0)=n!\,a_n. \tag{7}$$

2. Applications of (1), (3), (4) and (5)

As examples of similar expressions to (1), (3), (4) and (5), we know the results [12]:

$$(x;q)_n=\sum_{k=0}^{n}(-1)^k\binom{n}{k}_q q^{\binom{k}{2}}\,x^k, \qquad \frac{1}{(x;q)_{n+1}}=\sum_{k=0}^{\infty}\binom{n+k}{k}_q x^k, \tag{8}$$

then it is immediate the relation:

$$\sum_{k=0}^{\infty}\frac{1}{(q;q)_k}x^k=\frac{1}{\sum_{k=0}^{\infty}\frac{(-1)^k\,q^{\binom{k}{2}}}{(q;q)_k}x^k}, \qquad (q;q)_0=1, \tag{9}$$

therefore [4, 2, 21]:

$$\frac{1}{(q;q)_n}=\frac{1}{n!}\sum_{k=1}^{n}(-1)^k\,k!\,B_{n,k}\times$$

$$\times\left(-\frac{1!\,q^{\binom{1}{2}}}{(q;q)_1},\frac{2!\,q^{\binom{2}{2}}}{(q;q)_2},-\frac{3!\,q^{\binom{3}{2}}}{(q;q)_3},\frac{4!\,q^{\binom{4}{2}}}{(q;q)_4},\dots,\frac{(-1)^{n-k+1}(n-k+1)!\,q^{\binom{n-k+1}{2}}}{(q;q)_{n-k+1}}\right), \tag{10}$$

involving the partial Bell polynomials, with the recurrence property:

$$\sum_{k=0}^{n}\frac{(-1)^k\,q^{\binom{k}{2}}}{(q;q)_k\,(q;q)_{n-k}}=0, \qquad n\geq 1. \tag{11}$$

We can apply to (10) the Birmajer-Gil-Weiner's inversion process [3] to obtain:

(12)
$$\frac{(-1)^k\, q^{\binom{k}{2}}}{(q;q)_n} = \frac{1}{n!}\sum_{k=1}^{n}(-1)^k\, k!\, B_{n,k}\left(\frac{1!}{(q;q)_1},\frac{2!}{(q;q)_2},\frac{3!}{(q;q)_3},\frac{4!}{(q;q)_4},\ldots,\frac{(n-k+1)!}{(q;q)_{n-k+1}}\right).$$

Besides, from [23, 10] we have that relations type (9) are equivalent to the following Hessenberg determinant:

(13)
$$\frac{q^{\binom{n}{2}}}{(q;q)_n} = \begin{vmatrix} \frac{1}{(q;q)_1} & 1 & 0 & 0 & \ldots & \ldots & 0 \\ \frac{1}{(q;q)_2} & \frac{1}{(q;q)_1} & 1 & 0 & \ldots & \ldots & 0 \\ \frac{1}{(q;q)_3} & \frac{1}{(q;q)_2} & \frac{1}{(q;q)_1} & 1 & \ldots & \ldots & 0 \\ . & . & . & . & \ldots & \ldots & \ldots \\ . & . & . & . & \ldots & \ldots & \ldots \\ . & . & . & . & \ldots & \ldots & 1 \\ \frac{1}{(q;q)_n} & \frac{1}{(q;q)_{n-1}} & \frac{1}{(q;q)_{n-2}} & \ldots & \ldots & \ldots & \frac{1}{(q;q)_1} \end{vmatrix},$$

Similarly, Jha [13] deduced the following connection between the sum of divisors function and the numbers (6):

(14)
$$(n-1)!\,\sigma(n) = \sum_{k=1}^{n}(-1)^k\,(k-1)!\,B_{n,k}\left(1!a_1,\ 2!a_2,\ldots,(n-k+1)!\,a_{n-k+1}\right)$$

therefore, its inverse is given by:

(15)
$$n!\,a_n = \sum_{k=1}^{n}(-1)^k\,B_{n,k}\left(0!\sigma(1),\ 1!\sigma(2),\ 2!\sigma(3),\ldots,(n-k)!\,0!\sigma(n-k+1)\right).$$

The r-coloured partition function $p_r(n)$ is defined in terms of the Euler function [8, 14]:

(16)
$$\left[E(q)\right]^r = \sum_{n=0}^{\infty} p_r(n) q^n,$$

for example, if $r = 0, 1$ then (7) and (16) imply the values:

(17)
$$p_0(n) = \begin{cases} 1, & n = 0 \\ & , \\ 0, & n \geq 1 \end{cases} \qquad p_1(n) = a_n.$$

From (1) and (16):

(18)
$$\sum_{n=0}^{\infty} p_r(n) q^n = \frac{1}{[\sum_{k=0}^{\infty} p(k)\, q^k]^r},$$

thus [10]:

$$\begin{vmatrix} rp(1) & 1 & 0 & . \; . \; . & 0 \\ 2rp(2) & (r+1)p(1) & 2 & . \; . \; . & 0 \\ 3rp(3) & (2r+1)p(2) & (r+2)p(1) & . \; . \; . & 0 \\ . & (3r+1)p(3) & (2r+2)p(2) & . \; . \; . & . \\ . & . & (3r+2)p(3) & . \; . \; . & . \\ . & . & . & . \; . \; . & . \\ nrp(n) & ((n-1)r+1)p(n-1) & ((n-2)r+2)p(n-2) & . \; . \; . & (r+n-1)p(1) \end{vmatrix},$$

$$= (-1)^n \, n! \, p_r(n), \quad n \geq 1, \tag{19}$$

with its inverse [15, 13]:

$$p(n) = \sum_{r=0}^{n} (-1)^r \binom{n+1}{r+1} p_r(n), \quad n \geq 0. \tag{20}$$

For example, if $n = 1, 2, 3, 4$ then (19) implies the relations:

$$p_r(1) = -r, \;\; p_r(2) = \frac{r}{2!}(r-3), \;\; p_r(3) = -\frac{r}{3!}(r-1)(r-8),$$
$$p_r(4) = \frac{r}{4!}(r-1)(r-3)(r-14). \tag{21}$$

Johnson [12] exhibits the following connection between the Euler and divisors functions:

$$\sum_{n=1}^{\infty} \sigma(n) q^n = -q\frac{d}{dq}\log\,(q;q)_\infty = -\frac{q}{E(q)}\frac{d}{dq}\;E(q) \overset{(7)}{=} -\frac{1}{E(q)}\Big(a_1 q + 2a_2 q^2 + 3a_a q^3 + ...\Big)$$

$$\overset{(1)}{=} -\Big[\sum_{k=0}^{\infty} p(k)\;q^k\Big]\Big[\sum_{m=0}^{\infty} m\;a_m\;q^m\Big] \overset{[19]}{=} -\sum_{n=1}^{\infty} q^n \Big[\sum_{j=0}^{n} p(j)\;(n-j)\;a_{n-j}\Big],$$

where we can apply (4) to obtain the formula:

$$\sigma(n) = \sum_{k=1}^{n} k\;a_{n-k}\;p(k) = -\sum_{k=1}^{n} k\;a_k\;p(n-k), \tag{22}$$

deduced by Osler-Hassen-Chandrupatla [19]; with (4) and (22) it is immediate the known expression:

$$p(n) = \frac{1}{n}\sum_{j=0}^{n-1} p(j)\sigma(n-j), \quad n \geq 1, \tag{23}$$

finally, we indicate the interesting recurrence relation [19]:

$$\sigma(n) = -n\;a_n - \sum_{k=1}^{n-1} a_{n-k}\;\sigma(k), \quad n \geq 2, \tag{24}$$

and the identity [13]:

$$B_{n,k}\;\Big(1!\;a_1,\;2!\;a_2, ..., (n-k+1)!\;a_{n-k+1}\Big) = \frac{n!}{k!}\sum_{r=1}^{k} (-1)^{k-r}\binom{k}{r} p_r(n). \tag{25}$$

It is simple to show that (14) and (25) imply the relationship [13]:

$$\sigma(n) = n \sum_{k=1}^{n} \frac{(-1)^r}{r} \binom{n}{r} p_r(n), \quad n \geq 1, \tag{26}$$

Remark.- From (19) for $r = 9, n = 4$:

$$p_9(4) = \frac{1}{4!} \begin{vmatrix} 9 & 1 & 0 & 0 \\ 36 & 10 & 2 & 0 \\ 81 & 38 & 11 & 3 \\ 180 & 84 & 40 & 12 \end{vmatrix} = -90 \equiv 0 \; mod \; 5,$$

in agreement with the congruence $p_9(5m + 4) \equiv 0 \; mod \; 5$ indicated by Forbes [14]; besides, from (19) with $r = -1, n = 5$:

$$p_{-1}(5) = \frac{1}{5!} \begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ 4 & 0 & 2 & 0 & 0 \\ 9 & -2 & 1 & 3 & 0 \\ 20 & -6 & 0 & 2 & 4 \\ 35 & -15 & -3 & 2 & 3 \end{vmatrix} = 7 = p(5),$$

which it is correct because $p_{-1}(n) = p(n)$. We note that $p_{24}(n-1)$ is the Ramanujan's $\tau-$function.

3. Relationships between the sum of divisors and partition functions via determinants

Gould [10] exhibits the following result:

$$If \qquad e_n = \sum_{j=0}^{n} b_j^{(n)} c_{n-j}, \qquad n \geq 0, \quad b_0^{(k)} \neq 0, \tag{27}$$

then:

$$c_n = \frac{1}{\Pi_{r=0}^{n} b_0^{(r)}} \begin{vmatrix} e_n & b_1^{(n)} & b_2^{(n)} & b_3^{(3)} & \dots & b_n^{(n)} \\ e_{n-1} & b_0^{(n-1)} & b_1^{(n-1)} & b_2^{(n-1)} & \dots & b_{n-1}^{(n-1)} \\ e_{n-2} & 0 & b_0^{(n-2)} & b_1^{(n-2)} & \dots & b_{n-2}^{(n-2)} \\ e_{n-3} & 0 & 0 & b_0^{(n-3)} & \dots & b_{n-3}^{(n-3)} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ e_0 & 0 & 0 & \dots & \dots & b_0^{(0)} \end{vmatrix}, \tag{28}$$

with interesting applications.

a). Jha [15, 13] obtained the relation (20) with the structure (27):

$$\frac{p(n)}{(n+1)!} = \sum_{j=0}^{n} \frac{p_{n-j}(n)}{j!} \frac{(-1)^{n-j}}{(n-j+1)!}, \quad e_n = \frac{p(n)}{(n+1)!},$$

$$b_j^{(n)} = \frac{p_{n-j}(n)}{j!}, \quad c_{n-j} = \frac{(-1)^{n-j}}{(n-j+1)!}, \tag{29}$$

then (28) and (29) imply the connection:

$$\frac{(-1)^n}{(n+1)!} = \frac{1}{\Pi_{r=0}^{n} \, p_r(r)} \times$$

(30)
$$\begin{vmatrix} \frac{p(n)}{(n+1)!} & \frac{p_{n-1}(n)}{1!} & \frac{p_{n-2}(n)}{2!} & \frac{p_{n-3}(n)}{3!} & . & . & \cdots & \cdots & \frac{p_0(n)}{n!} \\ \frac{p(n-1)}{n!} & \frac{p_{n-1}(n-1)}{0!} & \frac{p_{n-2}(n-1)}{1!} & \frac{p_{n-3}(n-1)}{2!} & . & . & \cdots & \cdots & \frac{p_0(n-1)}{(n-1)!} \\ \frac{p(n-2)}{(n-1)!} & 0 & \frac{p_{n-2}(n-2)}{0!} & \frac{p_{n-3}(n-2)}{1!} & . & . & \cdots & \cdots & \frac{p_0(n-2)}{(n-2)!} \\ \frac{p(n-3)}{(n-2)!} & 0 & 0 & \frac{p_{n-3}(n-3)}{0!} & . & . & \cdots & \cdots & \frac{p_0(n-3)}{(n-3)!} \\ \cdots & \cdots & \cdots & \cdots & . & . & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & . & . & \cdots & \cdots & \cdots \\ \frac{p(2)}{3!} & 0 & 0 & 0 & . & . & \frac{p_2(2)}{0!} & \frac{p_1(2)}{1!} & \frac{p_0(2)}{2!} \\ \frac{p(1)}{2!} & 0 & 0 & 0 & . & . & \cdots & \frac{p_1(1)}{0!} & \frac{p_0(1)}{1!} \\ \frac{p(0)}{1!} & 0 & 0 & 0 & . & . & \cdots & \cdots & \frac{p_0(0)}{0!} \end{vmatrix}.$$

For example, from (30) for $n = 4$:

$$\frac{1}{5!} = -\frac{1}{25} \begin{vmatrix} \frac{p(4)}{5!} & \frac{p_3(4)}{1!} & \frac{p_2(4)}{2!} & \frac{p_1(4)}{3!} & \frac{p_0(4)}{4!} \\ \frac{p(3)}{4!} & \frac{p_3(3)}{0!} & \frac{p_2(3)}{1!} & \frac{p_1(3)}{2!} & \frac{p_0(3)}{3!} \\ \frac{p(2)}{3!} & 0 & \frac{p_2(2)}{0!} & \frac{p_1(2)}{1!} & \frac{p_0(2)}{2!} \\ \frac{p(1)}{2!} & 0 & 0 & \frac{p_1(1)}{0!} & \frac{p_0(1)}{1!} \\ p(0) & 0 & 0 & 0 & p_0(0) \end{vmatrix}$$

$$= -\frac{1}{25} \begin{vmatrix} \frac{5}{5!} & 0 & \frac{1}{2!} & 0 & 0 \\ \frac{3}{4!} & 5 & \frac{2}{1!} & 0 & 0 \\ \frac{2}{3!} & 0 & \frac{-1}{0!} & \frac{-1}{1!} & 0 \\ \frac{1}{2!} & 0 & 0 & \frac{-1}{0!} & 0 \\ 1 & 0 & 0 & 0 & 1 \end{vmatrix} = \frac{1}{120}.$$

b). Jha [13] deduced the expression (26), which can be written in the form (27):

$$\frac{\sigma(n)}{n! \, n} = \sum_{j=0}^{n} \frac{p_{n-j}(n)}{j!} \, \frac{(-1)^{n-j}}{(n-j)! \, (n-j)}, \quad e_n = \frac{\sigma(n)}{n! \, n},$$

$$b_j^{(n)} = \frac{p_{n-j}(n)}{j!}, \quad c_{n-j} = \frac{(-1)^{n-j}}{(n-j)! \, (n-j)}, \tag{31}$$

therefore:

$$\frac{(-1)^n}{n! \, n} = \frac{1}{\Pi_{r=0}^{n} \, p_r(r)} \times$$

(32)

$$\begin{vmatrix}
\frac{\sigma(n)}{(n)!n} & \frac{p_{n-1}(n)}{1!} & \frac{p_{n-2}(n)}{2!} & \frac{p_{n-3}(n)}{3!} & . & . & \dots & \dots & \frac{p_0(n)}{(n)!} \\
\frac{\sigma(n-1)}{(n-1)!(n-1)} & \frac{p_{n-1}(n-1)}{0!} & \frac{p_{n-2}(n-1)}{1!} & \frac{p_{n-3}(n-1)}{2!} & . & . & \dots & \dots & \frac{p_0(n-1)}{(n-1)!} \\
\frac{\sigma(n-2)}{(n-2)!(n-2)} & 0 & \frac{p_{n-2}(n-2)}{0!} & \frac{p_{n-3}(n-2)}{1!} & . & . & \dots & \dots & \frac{p_0(n-2)}{(n-2)!} \\
\frac{\sigma(n-3)}{(n-3)!(n-3)} & 0 & 0 & \frac{p_{n-3}(n-3)}{0!} & . & . & \dots & \dots & \frac{p_0(n-3)}{(n-3)!} \\
\dots & \dots & \dots & \dots & . & . & \dots & \dots & \frac{p_0(n-4)}{(n-4)!} \\
\dots & \dots & \dots & \dots & . & . & \dots & \dots & \dots \\
\frac{\sigma(2)}{2!2} & 0 & 0 & 0 & . & . & \frac{p_2(2)}{0!} & \frac{p_1(2)}{1!} & \frac{p_0(2)}{2!} \\
\frac{\sigma(1)}{1!} & 0 & 0 & 0 & . & . & \dots & \frac{p_1(1)}{0!} & \frac{p_0(1)}{1!} \\
0 & 0 & 0 & 0 & . & . & \dots & \dots & \frac{p_0(0)}{0!}
\end{vmatrix}$$

From (32) for $n = 4$:

$$\frac{1}{4!\,4} = -\frac{1}{25}\begin{vmatrix}
\frac{7}{96} & 0 & \frac{1}{2} & 0 & 0 \\
\frac{4}{18} & 5 & 2 & 0 & 0 \\
\frac{3}{4} & 0 & -1 & -1 & 0 \\
1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{vmatrix} = \frac{1}{96}.$$

c). Osler-Hassen-Chandrupatla [19] showed the property (22):

$$\sigma(n) = \sum_{k=0}^{n} k\; a_{n-k}\; p(k) = \sum_{j=0}^{n} a_j\; (n-j)\; p(n-j), \tag{33}$$

then (27), (28) and (33) imply the determinant:

$$n\; p(n) = \begin{vmatrix}
\sigma(n) & a_1 & a_2 & a_3 & \dots & \dots & \dots & a_n \\
\sigma(n-1) & 1 & a_1 & a_2 & \dots & \dots & \dots & a_{n-1} \\
\sigma(n-2) & 0 & 1 & a_1 & a_2 & \dots & \dots & a_{n-2} \\
\dots & \dots & 0 & 1 & a_1 & \dots & \dots & a_{n-3} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\sigma(1) & 0 & 0 & 0 & \dots & \dots & \dots & a_1 \\
\sigma(0) & 0 & 0 & 0 & 0 & \dots & \dots & 1
\end{vmatrix}. \tag{34}$$

From (34) for $n = 4$:

$$4\,p(4) = \begin{vmatrix}
7 & -1 & -1 & 0 & 0 \\
4 & 1 & -1 & -1 & 0 \\
3 & 0 & 1 & -1 & -1 \\
1 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 1
\end{vmatrix} = 20 \;\rightarrow\; p(4) = 5,$$

and for $n = 5$:

$$5\,p(5) = \begin{vmatrix}
6 & -1 & -1 & 0 & 0 \\
7 & 1 & -1 & -1 & 0 \\
4 & 0 & 1 & -1 & -1 \\
3 & 0 & 0 & 1 & -1 \\
1 & 0 & 0 & 0 & 1
\end{vmatrix} = 35 \;\rightarrow\; p(5) = 7.$$

4. On the Gandhi's recurrence relation for colour partitions

Gandhi [9, 17] showed the following recurrence relation for the colour partitions:

$$p_r(n) = -\frac{r}{n}\sum_{k=0}^{n} p_r(n-k)\,\sigma(k), \quad n \geq 1. \tag{35}$$

For $r = -1$ the relation (35) implies the known property (23) [19, 22, 1]:

$$np(n) = \sum_{k=0}^{n} p(n-k)\,\sigma(k) = \sum_{j=0}^{n} p(j)\,\sigma(n-j); \tag{36}$$

if $r = 1$, the Gandhi's expression (35) gives the Osler-Hassen-Chandrupatla's recurrence (24) [19], where it was applied the result (17). Jha [13] obtained the interesting identity (26) which can be considered as the inversion of (35).

Finally, with (24) and (35) it is possible to deduce the following property:

$$\frac{r}{r+1}p_{r+1}(n) = \frac{1}{n}\sum_{k=0}^{n} k\,a_{n-k}\,p_r(k), \quad r \geq 0,\ n \geq 1, \tag{37}$$

similar the relation:

$$p_{r+1}(n) = \sum_{k=0}^{n} a_{n-k}\,p_r(k). \tag{38}$$

Thus, we see that the Gandhi's formula involving the colour partitions and the divisor function implies interesting identities related with the partition function.

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