Advanced Studies in Contemporary Mathematics
33 (2023), No. 1, pp. 95 - 106
www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2023.33.1.95

CERTAIN EXTENDED HYPERGEOMETRIC MATRIX FUNCTIONS OF TWO OR THREE VARIABLES

PRAVEEN AGARWAL[1,2], RAHUL GOYAL[1], TAEKYUN KIM[3] AND SHAHER MOMANI[2,4]

Abstract. Certain generalizations of Appell's hypergeometric matrix functions and Lauricella's hypergeometric matrix function of three variables and their properties are studied in this paper. Also new generalization of RL-fractional derivative operator and their properties are investigated an application point of view.

1. Introduction and Preliminaries

In past year 2014 Atlin et. al [1] have studied the matrix version of the classical Appell's hypergeometric functions as follows:

The first kind of two variables Appell hypergeometric matrix function $F_1(P,Q_1,Q_2,R;u,v)$ is defined as follows:

$$F_1(P,Q_1,Q_2,R;u,v)=\sum_{k,l=0}^{\infty}\frac{B(P+k+l,R-P)(Q_1)_k(Q_2)_l}{B(P,R-P)}\frac{u^k}{k!}\frac{v^l}{l!}, \tag{1.1}$$

where, P, Q_1,Q_2 and R be the matrices in $\mathbb{C}^{r\times r}$ such that $(R+mI)$ is nonsingular for every integer $m\geq 0$ (see[2]) and $\max\{|\ u\ |,|\ v\ |\}<1$.

The second kind of two variables Appell hypergeometric matrix function $F_2(P,Q_1,Q_2,R_1,R_2;u,v)$ is defined as follows:

$$F_2(P,Q_1,Q_2,R_1,R_2;u,v)=\sum_{k,l=0}^{\infty}\frac{(P)_{k+l}B(Q_1+k,R_1-Q_1)B(Q_2+l,R_2-Q_2)}{B(Q_1,R_1-Q_1)B(Q_2,R_2-Q_2)}\frac{u^k}{k!}\frac{v^l}{l!}, \tag{1.2}$$

where, P, Q_1,Q_2, R_1 and R_2 be the matrices in $\mathbb{C}^{r\times r}$ such that (R_1+mI) and (R_2+mI) are nonsingular for every integer $m\geq 0$ (see[2]) and $|\ u\ |+|\ v\ |<1$.

Later in 2021, Dwivedi and Sahai [3] have introduced matrix version of classical Lauricella's hypergeometric function $F_D^{3,}(a,b_1,b_2,b_3,c;u,v,w)$.

2010 *Mathematics Subject Classification.* 26D10, 26D15, 33C05, 33C70, 33C90, 15A16.

Key words and phrases. Beta matrix function, Gauss hypergeometric matrix function, Appell's hypergeometric matrix functions, Lauricella's hypergeometric matrix function, RL-fractional derivative operator.

* corresponding author.

The Lauricella's hypergeometric matrix function $F_D^3(P, Q_1, Q_2, Q_3, R; u, v, w)$

$$F_D^3(P, Q_1, Q_2, Q_3, R; u, v, w) = \sum_{k,l,t=0}^{\infty} \frac{B(P+k+l+t, R-P)(Q_1)_k(Q_2)_l(Q_3)_t}{B(P, R-P)} \frac{u^k}{k!} \frac{v^l}{l!} \frac{w^t}{t!}, \tag{1.3}$$

where, P, Q_1,Q_2, Q_3 and R be the matrices in $\mathbb{C}^{r\times r}$ such that $(R + mI)$ is nonsingular for every integer $k \geq 0$ (see[3]) and $\sqrt{|u|} + \sqrt{|v|} + \sqrt{|w|} < 1$.

In recent past, Goyal et al. [7] have studied new generalization of beta function with matrix argument by using the Wiman matrix function as follows:

Let S, Q_1 and Q_2 are positive stable matrices in $C^{r\times r}$ then extended beta function with matrix argument is represented as [7]:

$$B_{(s_1,s_2)}^{(S)}(Q_1, Q_2) = \int_0^1 z^{Q_1-I}(1-z)^{Q_2-I} E_{(s_1,s_2)}\left(\frac{-S}{z(1-z)}\right) dz. \tag{1.4}$$

where, $\Re(s_1), \Re(s_2) > 0$ and $E_{(s_1,s_2)}(S)$ is Wiman matrix function given in [6].

Later, inspired and motivated by the above generalization of special matrix functions, Jain et el. [5] have generalised the Gauss hypergeometric and Kummer hypergeometric functions with matrix arguments by using the above extended beta matrix function (1.4) as follows:

$$F_{(s_1,s_2)}^{(S)}(Q_1, Q_2, Q_3; z) = \sum_{l=0}^{\infty} (Q_1)_l B_{(s_1,s_2)}^{(S)}(Q_2 + lI, Q_3 - Q_2)[B(Q_2, Q_3 - Q_2)]^{-1} \frac{z^l}{l!}, \tag{1.5}$$

where, $| z |< 1$.

$$\Phi_{(s_1,s_2)}^{(S)}(Q_2, Q_3; z) = \sum_{l=0}^{\infty} B_{(s_1,s_2)}^{(S)}(Q_2 + lI, Q_3 - Q_2)[B(Q_2, Q_3 - Q_2)]^{-1} \frac{z^l}{l!}, \tag{1.6}$$

where, S, Q_1, Q_2 and Q_3 and $Q_3 - Q_2$ be matrices in $\mathbb{C}^{r\times r}$ such that $(Q_3 + mI)$ is nonsingular for every integer $m \geq 0$, $\Re(s_1), \Re(s_2) > 0$ and $E_{(s_1,s_2)}(S)$ is wiman matrix function given in [6].

Very recently Goyal et.al. [11] have extended classical Appell's hypergeometric functions, $F_{1,(s_1,s_2)}^{(s)}(a, b_1, b_2, c_1; u, v)$ and $F_{2,(s_1,s_2)}^{(s)}(a, b_1, b_2, c_1, c_2; u, v)$ and Lauricella's hypergeometric function, $F_{D,(s_1,s_2)}^{3,(s)}(a, b_1, b_2, b_3, c; u, v, w)$ by the using of generalized Euler's beta function given in [10] and studied important properties of these extended special functions.

2. Main Results

Here, motivated by the above extensions of special functions with matrix arguments, we introduce new extensions of Appell's hypergeometric matrix functions and Lauricella's hypergeometric matrix function of three variable by using the extended beta function with matrix arguments (1.4). These extensions are matrix versions of the classical extensions of Appell's hypergeometric functions and Lauricella's hypergeometric function of three variables studied by Goyal et.al. [11] recently.

Definition 2.1. *Let S be positive stable matrix in $C^{r\times r}$ and $\Re(s_1) > 0$ and $\Re(s_2) > 0$ then new extensions of Appell's hypergeometric matrix functions, $F_1(P, Q_1, Q_2, R; u, v)$ and $F_2(P, Q_1, Q_2, R_1, R_2; u, v)$ are defined as follows:*

$$F_1^{(s_1,s_2,S)}(P, Q_1, Q_2, R; u, v) = \sum_{k,l=0}^{\infty} \frac{B^{(S)}_{(s_1,s_2)}(P + k + l, R - P)(Q_1)_k(Q_2)_l}{B(P, R - P)} \frac{u^k}{k!} \frac{v^l}{l!}, \quad (2.1)$$

where, P, Q_1, Q_2 and R be the matrices in $\mathbb{C}^{r\times r}$ such that $(R + mI)$ is nonsingular for every integer $m \geq 0$ and $\max\{| u |, | v |\} < 1$.

$$F_2^{(s_1,s_2,S)}(P, Q_1, Q_2, R_1, R_2; u, v) = \sum_{k,l=0}^{\infty} \frac{(P)_{k+l} B^{(S)}_{(s_1,s_2)}(Q_1 + k, R_1 - Q_1) B^{(S)}_{(s_1,s_2)}(Q_2 + l, R_2 - Q_2)}{B(Q_1, R_1 - Q_1) B(Q_2, R_2 - Q_2)} \frac{u^k}{k!} \frac{v^l}{l!}, \quad (2.2)$$

where, P, Q_1,Q_2, R_1 and R_2 be the matrices in $\mathbb{C}^{r\times r}$ such that $(R_1 + mI)$ and $(R_2 + mI)$ are nonsingular for every integer $m \geq 0$ and $| u | + | v | < 1$.

Definition 2.2. *Let S be positive stable matrix in $C^{r\times r}$ and $\Re(s_1) > 0$, $\Re(s_2) > 0$ then new extension of Lauricella's hypergeometric matrix function of three variables $F_D^3(P, Q_1, Q_2, Q_3, R; u, v, w)$ is defined as follows:*

$$F^{3,S}_{(D,s_1,s_2),}(P, Q_1, Q_2, Q_3, R; u, v, w) = \sum_{k,l,t=0}^{\infty} \frac{B^{(S)}_{(s_1,s_2)}(P + k + l + t, R - P)(Q_1)_k(Q_2)_l(Q_3)_t}{B(P, R - P)} \frac{u^k}{k!} \frac{v^l}{l!} \frac{w^t}{t!}, \quad (2.3)$$

where, P, Q_1,Q_2, Q_3 and R be the matrices in $\mathbb{C}^{r\times r}$ such that $(R + mI)$ is nonsingular for every integer $m \geq 0$ (see[3]*) and $\sqrt{| u |} + \sqrt{| v |} + \sqrt{| w |} < 1$.*

Remark. *(i)If we consider $s_2 = s_1 = 1$ and $S = O$ matrix then new extended Appell's hypergeometric matrix functions defined in* (2.1),(2.2) *and new extended Lauricella's hypergeometric matrix function of three variables defined in* (2.3) *reduces to Appell's hypergeometric matrix functions given by* (1.1),(1.2) *and extended Lauricella's hypergeometric matrix function of three variables given by* (1.3) *respectively.*

Theorem 2.3. *Assume S, P, Q_1, Q_2, R and $R - P$ be the positive stable matrices in $C^{r\times r}$. Then the extended Appell's hypergeometric matrix function, $F_1^{(s_1,s_2,S)}(P, Q_1, Q_2, R; u, v)$*

have following integral representation

$$F_1^{(s_1,s_2,S)}(P,Q_1,Q_2,R;u,v) = \frac{\Gamma(R)}{\Gamma(P)\Gamma(R-P)}\int_0^1 z^{P-I}(1-z)^{R-P-I}(1-uz)^{-Q_1}(1-vz)^{-Q_2}E_{s_1,s_2}\left(\frac{-S}{z(1-z)}\right)dz, \tag{2.4}$$

provided, $\Re(s_1)>0$, $\Re(s_2)>0$ *with* $|\,u\,|<1$ *and* $|\,v\,|<1$.

Proof. From definition of extended Appell's hypergeometric matrix function (2.1), we have:

$$F_1^{(s_1,s_2,S)}(P,Q_1,Q_2,R;u,v) = \sum_{k,l=0}^{\infty}\frac{B_{(s_1,s_2)}^{(S)}(P+k+l,R-P)(Q_1)_k(Q_2)_l}{B(P,R-P)}\frac{u^k}{k!}\frac{v^l}{l!}. \tag{2.5}$$

Then, substituting the value extended beta matrix function (1.4) in the above equation, we get:

$$F_1^{(s_1,s_2,S)}(P,Q_1,Q_2,R;u,v) = \frac{1}{B(P,R-P)}\sum_{k,l=0}^{\infty}\left\{\int_0^1 z^{P+(k+l)I-I}(1-z)^{R-P-I}E_{s_1,s_2}\left(\frac{-S}{z(1-z)}\right)dz\right\}(Q_1)_k(Q_2)_l\frac{u^k}{k!}\frac{v^l}{l!}. \tag{2.6}$$

On interchanging summation and integration in the above equation and after some calculations, we have:

$$F_1^{(s_1,s_2,S)}(P,Q_1,Q_2,R;u,v) = \frac{1}{B(P,R-P)}\int_0^1 z^{P-I}(1-z)^{R-P-I}E_{s_1,s_2}\left(\frac{-S}{z(1-z)}\right)\left\{\sum_{k,l=0}^{\infty}(Q_1)_k(Q_2)_l\frac{(uz)^k}{k!}\frac{(vz)^l}{l!}\right\}dz. \tag{2.7}$$

Then using the matrix identity and relationship between beta and Gamma matrix functions in the above equation, we get our desired result.

$$(1-uz)^{-P} = \sum_{k=0}^{\infty}\frac{(P)_k}{k!}(uz)^k, \tag{2.8}$$

$$B(Q_1,Q_2) = \frac{\Gamma(Q_1)\Gamma(Q_2)}{\Gamma(Q_1+Q_2)} \tag{2.9}$$

$$F_1^{(s_1,s_2,S)}(P,Q_1,Q_2,R;u,v) = \frac{\Gamma(R)}{\Gamma(P)\Gamma(R-P)}\int_0^1 z^{P-I}(1-z)^{R-P-I}(1-uz)^{-Q_1}(1-vz)^{-Q_2}E_{s_1,s_2}\left(\frac{-S}{z(1-z)}\right)dz. \tag{2.10}$$

□

Theorem 2.4. *Consider S, P, Q_1, Q_2, R_1, R_2, $R_1 - Q_1$ and $R_2 - Q_2$ be the positive stable matrices in $C^{r\times r}$. Then the extended Appell's hypergeometric matrix function, $F_2^{(s_1,s_2,S)}(P,Q_1,Q_2,R_1,R_2;u,v)$ have following integral representation*

$$F_2^{(s_1,s_2,S)}(P,Q_1,Q_2,R_1,R_2;u,v) = \frac{\Gamma(R_1)\Gamma(R_2)}{\Gamma(Q_1)\Gamma(Q_2)\Gamma(R_1-Q_1)\Gamma(R_2-Q_2)}\times \times\int_0^1\int_0^1 z^{Q_1-I}(1-z)^{R_1-Q_1-I}w^{Q_2-I}(1-w)^{R_2-Q_2-I}(1-uz-vw)^{-P} \times E_{s_1,s_2}\left(\frac{-S}{z(1-z)}\right)E_{s_1,s_2}\left(\frac{-S}{w(1-w)}\right)dzdw, \tag{2.11}$$

provided, $\Re(s_1) > 0$, $\Re(s_2) > 0$ *with* $|\, u \,| + |\, v \,| < 1$.

Proof. From the definition of extended Appell's hypergeometric matrix function (2.2), we have:

$$F_2^{(s_1,s_2,S)}(P,Q_1,Q_2,R_1,R_2;u,v) = \sum_{k,l=0}^{\infty}\frac{(P)_{k+l}B^{(S)}_{(s_1,s_2)}(Q_1+k,R_1-Q_1)B^{(S)}_{(s_1,s_2)}(Q_2+l,R_2-Q_2)}{B(Q_1,R_1-Q_1)B(Q_2,R_2-Q_2)}\frac{u^k}{k!}\frac{v^l}{l!}. \tag{2.12}$$

Then on following the similar steps as the proof of Theorem (2.3), using the definition extended beta matrix function given in (1.4) and interchanging summation and integration sign in above equation, we get:

$$F_2^{(s_1,s_2,S)}(P,Q_1,Q_2,R_1,R_2;u,v) = \frac{1}{B(Q_1,R_1-Q_1)B(Q_2,R_2-Q_2)}\int_0^1\int_0^1 z^{Q_1-I}(1-z)^{R_1-Q_1-I}w^{Q_2-I}(1-w)^{R_2-Q_2-I} \times E_{s_1,s_2}\left(\frac{-S}{z(1-z)}\right)E_{s_1,s_2}\left(\frac{-S}{w(1-w)}\right)\left(\sum_{k,l=0}^{\infty}(P)_{k+l}\frac{u^k}{k!}\frac{v^l}{l!}\right)dzdw. \tag{2.13}$$

Then using the matrix identity and result given in (2.9) in the above equation, we get our desired result.

$$(1-uz-vw)^{-P} = \sum_{n=0}^{\infty}\frac{(P)_n}{n!}(uz+vw)^n = \sum_{k,l=0}^{\infty}(P)_{k+l}\frac{u^k}{k!}\frac{v^l}{l!}, |\, u+v \,|< 1, \tag{2.14}$$

$$F_2^{(s_1,s_2,S)}(P,Q_1,Q_2,R_1,R_2;u,v) = \frac{\Gamma(R_1)\Gamma(R_2)}{\Gamma(Q_1)\Gamma(Q_2)\Gamma(R_1-Q_1)\Gamma(R_2-Q_2)}\times \times\int_0^1\int_0^1 z^{Q_1-I}(1-z)^{R_1-Q_1-I}w^{Q_2-I}(1-w)^{R_2-Q_2-I}(1-uz-vw)^{-P} \times E_{s_1,s_2}\left(\frac{-S}{z(1-z)}\right)E_{s_1,s_2}\left(\frac{-S}{w(1-w)}\right)dzdw. \tag{2.15}$$

□

Theorem 2.5. *Let S, P, Q_1, Q_2, Q_3, R and $R-P$ be the positive stable matrices in $\mathbb{C}^{r\times r}$. Then the extended Lauricella's hypergeometric matrix function of three variables $F^{3,S}_{(D,s_1,s_2),}(P,Q_1,Q_2,Q_3,R;u,v,w)$ have following integral representation:*

$$F^{3,S}_{(D,s_1,s_2),}(P,Q_1,Q_2,Q_3,R;u,v,w) = \\ = \frac{\Gamma(R)}{\Gamma(P)\Gamma(R-P)}\int_0^1 z^{P-I}(1-z)^{R-P-I}(1-uz)^{-Q_1}(1-vz)^{-Q_2}(1-wz)^{-Q_3}E_{s_1,s_2}\left(\frac{-S}{z(1-z)}\right)dz, \tag{2.16}$$

provided, $\Re(s_1)>0$, $\Re(s_2)>0$ with $|u|<1$, $|v|<1$ and $|w|<1$.

Proof. By the following same parallel line of proof as Theorem (2.3), we get our desired result. □

Theorem 2.6. *Assume S, P, Q_1, Q_2, R and $R-P$ be the positive stable matrices in $C^{r\times r}$. Then the extended Appell's hypergeometric matrix function, $F_1^{(s_1,s_2,S)}(P,Q_1,Q_2,R;u,v)$ have following differential formula:*

$$\frac{d^{k+l}}{du^k dv^l}\left[F_1^{(s_1,s_2,S)}(P,Q_1,Q_2,R;u,v)\right] \\ = \frac{(Q_1)_k(Q_2)_l(P)_{k+l}}{(R)_{k+l}}F_1^{(s_1,s_2,S)}\Big(P+(k+l)I,Q_1+kI,Q_2+lI,R+(k+l)I;u,v\Big), \tag{2.17}$$

provided, $\Re(s_1)>0$, $\Re(s_2)>0$ with $|u|<1$ and $|v|<1$.

Proof. On taking derivative term by term of extended Appell's hypergeometric matrix function (2.1) with respect to u and v, we get:

$$\frac{d^2}{dudv}\left[F_1^{(s_1,s_2,S)}(P,Q_1,Q_2,R;u,v)\right] = \frac{d^2}{dudv}\left[\sum_{k,l=0}^{\infty}\frac{B^{(S)}_{(s_1,s_2)}(P+k+l,R-P)(Q_1)_k(Q_2)_l}{B(P,R-P)}\frac{u^k}{k!}\frac{v^l}{l!}\right] \\ = \sum_{k,l=1}^{\infty}\frac{B^{(S)}_{(s_1,s_2)}(P+k+l,R-P)(Q_1)_k(Q_2)_l}{B(P,R-P)}\frac{u^{k-1}}{(k-1)!}\frac{v^{l-1}}{(l-1)!}, \tag{2.18}$$

Then replace $k\longrightarrow k+1$ and $l\longrightarrow l+1$, and after some calculations, we have:

$$\frac{d^2}{dudv}\left[F_1^{(s_1,s_2,S)}(P,Q_1,Q_2,R;u,v))\right] = \frac{Q_1,Q_2(P)_2}{(R)_2}F_1^{(s_1,s_2,S)}\Big(P+2I,Q_1+I,Q_2+I,R+2I;u,v\Big). \tag{2.19}$$

Then follow the same procedure, derivative term by term of extended Appell's hypergeometric matrix function (2.1) with respect to u and v with k and l times respectively, we get our desired result.

$$\frac{d^{k+l}}{du^k dv^l}\left[F_1^{(s_1,s_2,S)}(P,Q_1,Q_2,R;u,v)\right]$$
$$= \frac{(Q_1)_k(Q_2)_l(P)_{k+l}}{(R)_{k+l}} F_1^{(s_1,s_2,S)}\bigg(P+(k+l)I, Q_1+kI, Q_2+lI, R+(k+l)I; u, v\bigg). \tag{2.20}$$

□

Here, we omit the proof of the given below two Theorems due to proofs are same as Theorem (2.6).

Theorem 2.7. *Consider S, P, Q_1, Q_2, R_1, R_2, $R_1 - Q_1$ and $R_2 - Q_2$ be the positive stable matrices in $C^{r\times r}$. Then the extended Appell's hypergeometric matrix function, $F_2^{(s_1,s_2,S)}(P,Q_1,Q_2,R_1,R_2;u,v)$ have following derivative formula*

$$\frac{d^{k+l}}{du^k dv^l}\left[F_2^{(s_1,s_2,S)}(P,Q_1,Q_2,R_1,R_2;u,v)\right]$$
$$= \frac{(P)_{k+l}(Q_1)_k(Q_2)_l}{(R_1)_k(R_2)_l} F_2^{(s_1,s_2,S)}\bigg(P+(k+l)I, Q_1+kI, Q_2+lI, R_1+kI, R_2+lI; u, v\bigg), \tag{2.21}$$

provided, $\Re(s_1) > 0$, $\Re(s_2) > 0$ *with* $|u| + |v| < 1$.

Theorem 2.8. *Let S, P, Q_1, Q_2, Q_3, R and $R - P$ be the positive stable matrices in $\mathbb{C}^{r\times r}$. Then the extended Lauricella's hypergeometric matrix function of three variables $F^{3,S}_{(D,s_1,s_2),}(P,Q_1,Q_2,Q_3,R;u,v,w)$ have following derivative formula:*

$$\frac{d^{k+l+t}}{du^k dv^l dw^t}\left[F^{3,S}_{(D,s_1,s_2),}(P,Q_1,Q_2,Q_3,R;u,v,w)\right] =$$
$$= \frac{(P)_{k+l+t}(Q_1)_k(Q_2)_l(Q_3)_t}{(R)_{k+l+t}}$$
$$\times F^{3,S}_{(D,s_1,s_2),}\bigg(P+(k+l+t)I, Q_1+kI, Q_1+lI, Q_3+tI, R+(k+l+t)I; u, v, w\bigg), \tag{2.22}$$

provided, $\Re(s_1) > 0$, $\Re(s_2) > 0$ *with* $|u| < 1$, $|v| < 1$ *and* $|w| < 1$.

3. Applications in fractional calculus

In this section, here we study new extension of Riemann-Liouville(RL) fractional derivatives with matrix arguments and image formulas for extended gauss hypergeometric matrix function in terms of extended Appell's hypergeometric matrix functions and Lauricella's hypergeometric matrix function of three variables.

Riemann-Liouville(RL) fractional derivative of order u is given as [8, 9]:

$$\mathbf{D}^u[x^m] = \frac{1}{\Gamma(-u)}\int_0^x (x-t)^{-(u+1)} t^m dt. \tag{3.1}$$

where, $u \in \mathbb{C}$ such that $\Re(u) < 0$.

Recently Jain et.al [4] have introduced Riemann-Liouville(RL) fractional derivative with matrix arguments as follows:

$$\mathbf{D}^{u}[x^{A}] = \frac{1}{\Gamma(-u)} \int_{0}^{x} (x-t)^{-(u+1)} t^{A} dt. \tag{3.2}$$

where, A is positive stable matrix in $\mathbb{C}^{r\times r}$ and $u \in \mathbb{C}$ such that $\Re(u) < 0$.

Definition 3.1. *Assume S be positive stable matrix in $C^{r\times r}$ then new extension of Riemann-Liouville(RL) fractional derivative with matrix arguments is defined as follows:*

$$\mathbf{D}^{u,S}_{(s_1,s_2)}[x^{A}] = \frac{1}{\Gamma(-u)} \int_{0}^{x} (x-t)^{-(u+1)} E_{s_1,s_2}\left(\frac{-Sx^2}{t(x-t)}\right) t^{A} dt. \tag{3.3}$$

where, $\Re(s_1) > 0$ and $\Re(s_2) > 0$ and $\Re(u) < 0$.

Remark. *If we consider $s_2 = s_1 = 1$ and $S = O$ matrix then new extended Riemann-Liouville(RL) fractional derivative with matrix arguments defined in* (3.3) *reduces to Riemann-Liouville(RL) fractional derivative with matrix argument given by* (3.2).

Theorem 3.2. *The following result hold true:*

$$\mathbf{D}^{u,S}_{(s_1,s_2)}[x^{A}] = \frac{x^{A-uI}}{\Gamma(-u)} B^{(S)}_{(s_1,s_2)}(A+I,-uI) \tag{3.4}$$

where, $\Re(s_1) > 0$ and $\Re(s_2) > 0$ and $\Re(u) < 0$.

Proof. From the definition of Riemann-Liouville(RL) fractional derivative with matrix, we have

$$\mathbf{D}^{u,S}_{(s_1,s_2)}[x^{A}] = \frac{1}{\Gamma(-u)} \int_{0}^{x} (x-t)^{-(u+1)} E_{s_1,s_2}\left(\frac{-Sx^2}{t(x-t)}\right) t^{A} dt. \tag{3.5}$$

Then substitute $t = xz$ in the above equation and after some calculation, we get

$$\mathbf{D}^{u,S}_{(s_1,s_2)}[x^{A}] = \frac{x^{A-uI}}{\Gamma(-u)} \int_{0}^{1} z^{A}(1-z)^{-uI-I} E_{(s_1,s_2)}\left(\frac{-S}{z(1-z)}\right) dz. \tag{3.6}$$

By the use of extended beta matrix function (1.4), we get our desired result.

$$\mathbf{D}^{u,S}_{(s_1,s_2)}[x^{A}] = \frac{x^{A-uI}}{\Gamma(-u)} B^{(S)}_{(s_1,s_2)}(A+I,-uI) \tag{3.7}$$

□

Theorem 3.3. *Consider S, A, B, C be the positive stable matrices in $C^{r\times r}$, then following result hold true:*

$$\mathbf{D}^{A-uI,S}_{(s_1,s_2)}\left[z^{A}(1-az)^{-B}(1-bz)^{-C}\right] = \frac{\Gamma(A)}{\Gamma(u)} z^{uI-I} F_1^{(s_1,s_2,S)}(A,B,C,uI;az,bz). \tag{3.8}$$

provided, $\mid az \mid < 1$ and $\mid bz \mid < 1$.

Proof. From the definition of Riemann-Liouville(RL) fractional derivative with matrix, we have

$$\begin{aligned}&\mathbf{D}^{A-uI,S}_{(s_1,s_2)}\left[z^A(1-az)^{-BI}(1-bz)^{-CI}\right]\\&=\frac{1}{\Gamma(uI-A)}\int_0^z(z-t)^{uI-A-I}E_{s_1,s_2}\left(\frac{-Sz^2}{t(z-t)}\right)t^A(1-at)^{-B}(1-bt)^{-C}dt.\end{aligned}\tag{3.9}$$

On taking out z from integral we have:

$$\begin{aligned}&\mathbf{D}^{A-uI,S}_{(s_1,s_2)}\left[z^A(1-az)^{-B}(1-bz)^{-C}\right]\\&=\frac{z^{uI-AI-I}}{\Gamma(uI-A)}\int_0^z(1-\frac{t}{z})^{uI-AI-I}E_{s_1,s_2}\left(\frac{-Sz^2}{t(z-t)}\right)t^A(1-at)^{-B}(1-bt)^{-C}dt.\end{aligned}\tag{3.10}$$

Then set value of $t=xz$ in above equation and changing the limit from $t=0, t=z$ to $x=0, x=1$, $dt=zdx$ with some re-arranging the terms, we get:

$$\begin{aligned}&\mathbf{D}^{A-uI,S}_{(s_1,s_2)}\left[z^A(1-az)^{-BI}(1-bz)^{-CI}\right]\\&=\frac{z^{uI-I}}{\Gamma(uI-A)}\int_0^1x^{A-I}(1-x)^{uI-A-I}E_{s_1,s_2}\left(\frac{-S}{x(1-x)}\right)(1-axz)^{-B}(1-bxz)^{-C}dx.\end{aligned}\tag{3.11}$$

By the integral representation of extended Appell's hypergeometric matrix function $F_1^{(s_1,s_2,S)}(A,B,B',C;x,y)$ (2.4), we get our desired result.

$$\mathbf{D}^{A-uI,S}_{(s_1,s_2)}\left[z^A(1-az)^{-B}(1-bz)^{-C}\right]=\frac{\Gamma(A)}{\Gamma(u)}z^{uI-I}F_1^{(s_1,s_2,S)}(A,B,C,uI;az,bz).\tag{3.12}$$

□

Theorem 3.4. *Assume S, A, B, C and D be the positive stable matrices in* $C^{r\times r}$*, then following result hold true:*

$$\begin{aligned}\mathbf{D}^{A-uI,S}_{(s_1,s_2)}&\left[z^{A-I}(1-z)^{-B}F^{(S)}_{(s_1,s_2)}\left(B,C,D;\frac{x}{1-z}\right)\right]=\\&=\frac{\Gamma(A)}{\Gamma(u)}z^{uI-I}F_2^{(s_1,s_2,S)}(B,C,A,D,uI;x,z).\end{aligned}\tag{3.13}$$

where, $\Re(u)<0$*, and* $\mid\frac{x}{1-z}\mid<1$.

Proof. From the definition of extended hypergeometric function (1.5), we get:

$$\begin{aligned}&\mathbf{D}^{A-uI,S}_{(s_1,s_2)}\left[z^{A-I}(1-z)^{-B}F^{(S)}_{(s_1,s_2)}\left(B,C,D;\frac{x}{1-z}\right)\right]=\\&=\mathbf{D}^{A-uI,S}_{(s_1,s_2)}\left[z^{A-I}(1-z)^{-B}\left\{\sum_{m=0}^{\infty}(B)_mB^{(S)}_{(s_1,s_2)}(C+mI,D-C)[B(C,D-C)]^{-1}\frac{x^m(1-z)^{-m}}{m!}\right\}\right]\end{aligned}\tag{3.14}$$

After some calculation, we have:

$$\mathbf{D}_{(s_1,s_2)}^{A-uI,S}\left[z^{A-I}(1-z)^{-B}F_{(s_1,s_2)}^{(S)}\left(B,C,D;\frac{x}{1-z}\right)\right]=$$
$$=\frac{1}{B(C,D-C)}\mathbf{D}_{(s_1,s_2)}^{A-uI,S}\left[z^{A-I}\sum_{m=0}^{\infty}(B)_mB_{(s_1,s_2)}^{(S)}(C+mI,D-C)\frac{x^m(1-z)^{-B+mI}}{m!}\right]$$
$$=\frac{1}{B(C,D-C)}\mathbf{D}_{(s_1,s_2)}^{A-uI,S}\left[z^{A+nI-I}\sum_{m,n=0}^{\infty}(B)_m(B+mI)_nB_{(s_1,s_2)}^{(S)}(C+mI,D-C)\frac{x^m}{m!n!}\right]$$
$$\frac{1}{B(C,D-C)}\sum_{m,n=0}^{\infty}(B)_m(B+mI)_nB_{(s_1,s_2)}^{(S)}(C+mI,D-C)\frac{x^m}{m!n!}\mathbf{D}_{(s_1,s_2)}^{A-uI,S}\left[z^{A+nI-I}\right] \tag{3.15}$$

Then using Theorem (3.2),we get

$$\mathbf{D}_{(s_1,s_2)}^{A-uI,S}\left[z^{A-I}(1-z)^{-B}F_{(s_1,s_2)}^{(S)}\left(B,C,D;\frac{x}{1-z}\right)\right]=$$
$$=\frac{1}{B(C,D-C)}\sum_{m,n=0}^{\infty}(B)_{n+m}B_{(s_1,s_2)}^{(S)}(C+mI,D-C)\frac{x^m}{m!n!}\frac{z^{nI+uI-I}}{\Gamma(uI-A)}B_{(s_1,s_2)}^{(S)}(A+nI,uI-A)$$
$$\frac{z^{uI-I}\Gamma(u)}{\Gamma(A)}\sum_{m,n=0}^{\infty}(B)_{n+m}\frac{B_{(s_1,s_2)}^{(S)}(A+nI,uI-A)B_{(s_1,s_2)}^{(S)}(C+mI,D-C)}{B(A,uI-A)B(C,D-C)}\frac{x^m}{m!}\frac{z^n}{n!} \tag{3.16}$$

Then from definition of extended Appell's hypergeometric matrix function (2.2), we get our desired result.

$$\mathbf{D}_{(s_1,s_2)}^{A-uI,S}\left[z^{A-I}(1-z)^{-B}F_{(s_1,s_2)}^{(S)}\left(B,C,D;\frac{x}{1-z}\right)\right]=$$
$$=\frac{\Gamma(A)}{\Gamma(u)}z^{uI-I}F_2^{(s_1,s_2,S)}(B,C,A,D,uI;x,z). \tag{3.17}$$

□

Theorem 3.5. *Let S, A, B, C and D be the positive stable matrices in $C^{r\times r}$, then following result hold true:*

$$\mathbf{D}_{(s_1,s_2)}^{A-uI,S}\left[z^{A}(1-az)^{-B}(1-bz)^{-C}(1-cz)^{-D}\right]=\frac{\Gamma(A)}{\Gamma(u)}z^{uI-I}F_{(D,s_1,s_2),}^{3,S}(A,B,C,D,uI;az,bz,cz). \tag{3.18}$$

provided, $|\,az\,|<1$, $|\,bz\,|<1$ *and* $|\,cz\,|<1$.

Proof. By the following similar parallel line of proof as the Theorem (3.3), we get our desired result. □

Concluding Remark:

In this research, firstly we presented new extensions of Appell's hypergeometric matrix functions and Lauricella's hypergeometric matrix function of three variables by using the extended beta matrix function. Then we have studied some important properties of these extended special matrix functions. Furthermore, we have established new generalization of RL-fractional derivative operator and derived some results containing extended Appell's hypergeometric matrix functions and Lauricella's hypergeometric matrix function of three variables.

Acknowledgement

The authors are thankful to NBHM (DAE), Grant/Award Number: 02011/12/2020 NBHM (R.P)/RD II/7867 for their necessary support.

References

[1] Altin, A., Çekim, B., and Şahin, R., On the matrix versions of Appell hypergeometric functions, Quaest. Math., 37(1), 31-38, 2014.
[2] Batahan, R.S., and Metwally, M.S., Differential and integral operators on Appell's matrix functions, 7-25, 2009.
[3] Dwivedi, R., and Sahai, V., A note on the Lauricella matrix functions. arXiv preprint arXiv:2105.00438, 2021.
[4] Jain, S., Goyal, R., Agarwal, P. and Momami, S., An extended Struve matrix functions and their applications, under review process, 2022.
[5] Jain, S., Goyal, R., Oros, G. I., Agarwal, P., and Momani, S., A study of generalized hypergeometric Matrix functions via two-parameter Mittag–Leffler matrix function, Open Physics, 20, 1-10, 2022.
[6] Garrappa, R. and Popolizio, M., Computing the matrix Mittag-Leffler function with applications to fractional calculus., J. Sci. Comput., 2018.
[7] Goyal, R., Agarwal, P., Oros, G.I. and Jain, S., Extended Beta and Gamma Matrix Functions via 2-Parameter Mittag-Leffler Matrix Function. Mathematics, 10(6), 892, 2022.
[8] Samko, S.G., Kilbas, A.A. and Marichev, O.I., Fractional integrals and derivatives (Vol. 1). Yverdon-les-Bains, Switzerland: Gordon and breach science publishers, Yverdon, 1993.
[9] Kilbas, A.A., Srivastava, H.M. and Trujillo, J.J., Theory and applications of fractional differential equations (Vol. 204). elsevier, 2006.
[10] Goyal, R., Momani, S., Agarwal, P.and Rassias, M. T., An Extension of Beta Function by Using wiman's function. Axioms, 10(3), 187, 2021.
[11] Goyal, R. and Agarwal, P., Certain extended type hypergeometric functions of two and three variables, under review , 2022.

P.Agarwal: [1]Department of Mathematics, Anand International College of Engineering, Jaipur 303012,India.

[2]Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman, United Arab Emirates
Email address: goyal.praveen2011@gmail.com

R. Goyal: [1]Department of Mathematics, Anand International College of Engineering, Jaipur 303012, India.
Email address: rahul.goyal01@anandice.ac.in, rahulgoyal4692@gmail.com

T. Kim: [3]Department of Mathematics, College of Natural Science, Kwangwoon University, Seoul, Republic of Korea
Email address: tkkim@kw.ac.kr

S. Momani: [4]Department of Mathematics, The University of Jordan, Jordan
Email address: shahermm@yahoo.com