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A NOTE ON A SUM OF POWERS OF q-INTEGERS OF SKIP COUNT BY k

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Abstract. In this paper, we introduce a generalized type 2 q-Bernoulli polynomials and a generalized type 2 q-Euler polynomials, and the power sum of q-integers of skip count by k and the alternating sum of powers of q-integers of skip count by k can be represented by the generalized type 2 q-Bernoulli numbers and generalized type 2 q-Euler numbers.

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1. Introduction

Let p be a given prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic rational numbers, and the completions of algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic norm is normalized as $|p|_p = \frac{1}{p}$.

Let $q \in \mathbb{C}_p$ be an indeterminate with $|q-1|_p < p^{-\frac{1}{p-1}}$. Then the *$q$-analogue of number x* are defined as

$$[x]_q = \frac{1-q^x}{1-q}.$$

Note that $\lim_{q\to 1}[x]_q = x$ for each $x \in \mathbb{Z}_p$.

The *Bernoulli polynomials* are defined by the generating function to be

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \text{ (see [?]).}$$

In the special case $x=0$, $B_n = B_n(0)$ are called the *Bernoulli numbers*.

Bernoulli number is one of the very important numbers in combinatorics as well as special function theory (see [6, 7, 8]). In particular, it is well-known that for a give positive integer k,

$$0^k + 1^k + 2^k + \cdots + (n-1)^n = \frac{1}{k+1}\left(B_{k+1}(n) - B_{k+1}\right),\ (n \geq 1).$$

Kim showed the power sums of consecutive nonnegative q-integers can be represented by the q-Bernoulli integers in [9].

Kim and Kim defined *type 2 Bernoulli polynomials* as follows

$$\frac{t}{e^t-e^{-t}}e^{xt} = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}, \tag{1}$$

and, found relations between some special functions or numbers and those polynomials (see [4, 5]). In [4], authors defined the type 2 q-Bernoulli polynomials and type 2 q-Euler polynomials and show that

$$\sum_{l=0}^{n-1} q^{2l+1}[2l+1]_q^m = \frac{1}{2(m+1)}\left(b_{m+1,q}(2n) - b_{m+1,q}\right), \tag{2}$$

and

$$\sum_{l=0}^{n-1}(-1)^l q^l [2l+1]_q^m = \frac{q^n E_{m,q}(2n) + E_{m,q}}{[2]_q}, \tag{3}$$

where $b_{n,q}(x)$ and $E_{n,q}(x)$ are the *type* 2 *q-Bernoulli polynomials* and *type* 2 *q-Euler polynomials*, respectively.

In viewpoint of (2) and (3), we want to know that the sum of powers of q-integers of skip count by k

$$q[1]_q^m + q^{k+1}[k+1]_q^m + q^{2k+1}[2k+1]_q^m + \cdots + q^{kn-1}[kn-1]_q^m = ?,$$

and the alternating sum of powers of q-integers of skip count by k

$$q[1]_q^m - q^{k+1}[k+1]_q^m + q^{2k+1}[2k+1]_q^m + \cdots + (-1)^{n-1} q^{kn-1}[kn-1]_q^m = ?.$$

In this paper, we define a generalized type 2 q-Bernoulli polynomials and a generalized type 2 q-Euler polynomials, and show that the sum of powers of q-integers of skip count by k and the alternating sum of powers of q-integers of skip count by k are related to closely generalized type 2 q-Bernoulli numbers and generalized type 2 q-Euler numbers.

2. A generalized q-Bernoulli polynomials and numbers

Let $UD(\mathbb{Z}_p)$ be the set of all uniformly differentiable on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p$, the *bosonic p-adic q-integral on* $\mathbb{Z}_p$ are defined by the Kim as follows:

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{x=0}^{p^N-1} f(x)q^x, \text{ (see [2, 3])}. \tag{4}$$

By (4), we have

$$q^n I_q(f_n) - I_q(f) = (q-1)\sum_{l=0}^{n-1} q^l f(l) + \frac{q-1}{\log q}\sum_{l=0}^{n-1} q^l f'(l), \tag{5}$$

for each positive integer n where $f'(0) = \left.\frac{df(x)}{dx}\right|_{x=0}$.

Let k be a given nonnegative integer. As a generalization of the type 2 q-Bernoulli polynomials, we define the *type k q-Bernoulli polynomials* to be

$$\mathcal{B}_{n,k,q}(x) = \int_{\mathbb{Z}_p} q^{-y}[ky+x+1]_q^n d\mu_q(y) \ (n \geq 0). \tag{6}$$

In the special case $= 0$, $b_{n,k,q} = b_{n,k,q}(0)$ are called the *type k q-Bernoulli numbers.*

By (6), we see that the generating function of generalized type 2 q-Bernoulli polynomials are

$$\sum_{n=0}^{\infty} B_{n,k,q}(x)\frac{t^n}{n!} = \int_{\mathbb{Z}_p} q^{-y} e^{[ky+x+1]_q t} d\mu_q(y). \tag{7}$$

From (4), we have

$$\begin{aligned}
\mathcal{B}_{n,k,q} =& \frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-q)^l \frac{kl}{[kl]_q} \\
=& \frac{kn}{(1-q)^{n-1}}\sum_{l=1}^{n}\binom{n-1}{l-1}(-q)^l \sum_{m=0}^{\infty} q^{klm} \\
=& \frac{kn}{(1-q)^{n-1}}\sum_{l=0}^{n-1}\binom{n-1}{l}(-q)^{l+1} \sum_{m=0}^{\infty} q^{km(l+1)} \\
=& \frac{-kn}{(1-q)^{n-1}}\sum_{m=0}^{\infty} q^{km+1}\sum_{l=0}^{n-1}\binom{n-1}{l}\left(-q^{km+1}\right)^l \\
=& -kn\sum_{m=0}^{\infty} q^{km+1}\left(\frac{1-q^{km+1}}{1-q}\right)^{n-1} \\
=& -kn\sum_{m=0}^{\infty} q^{km+1}\left[km+1\right]_q^{n-1}.
\end{aligned} \tag{8}$$

Hence, by (8), we obtain the following theorem.

Theorem 2.1. *For each nonnegative integer n, we have*

$$\begin{aligned}
\mathcal{B}_{n,k,q} =& \frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-q)^l \frac{kl}{[kl]_q} \\
=& -kn\sum_{m=0}^{\infty} q^{km+1}\left[km+1\right]_q^{n-1}.
\end{aligned}$$

By the Theorem 2.1, the generating function of the type k q-Bernoulli numbers can be derived as follows:

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{B}_{n,k,q}\frac{t^n}{n!} =& -kt\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} q^{km+1}[km+1]_q^{n-1}\frac{t^{n-1}}{(n-1)!} \\
=& -kt\sum_{m=0}^{\infty} q^{km+1}e^{[km+1]_q t}.
\end{aligned} \tag{9}$$

Since

$$[ky+x+1]_q = q^x[ky+1]_q[x]_q, \tag{10}$$

by (6) and (10), we see that

$$
\begin{aligned}
\mathcal{B}_{n,k,q}(x) =& \int_{\mathbb{Z}_p} q^{-y}[ky+x+1]_q^n d\mu(y) \\
=& \int_{\mathbb{Z}_p} q^{-y} \sum_{l=0}^{n} \binom{n}{l} q^{lx}[ky+1]_q^l [x]_q^{n-l} d\mu_q(y) \\
=& \sum_{l=0}^{n} \binom{n}{l} q^{lx} \mathcal{B}_{l,k,q}[x]_q^{n-l}.
\end{aligned} \tag{11}
$$

By the similar calculations of (8), we note that

$$
\begin{aligned}
\mathcal{B}_{n,k,q}(x) =& \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{(x+1)l} \frac{kl}{[kl]_q} \\
=& -kn \sum_{m=0}^{\infty} q^{km+x+1}[km+x+1]_q^{n-1},
\end{aligned} \tag{12}
$$

for each nonnegative integer n.

Note that the generating function of the type k q-Bernoulli polynomials are

$$
\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{B}_{n,k,q}(x)\frac{t^n}{n!} =& -kt \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} q^{km+x+1}[km+x+1]_q^{n-1} \frac{t^{n-1}}{(n-1)!} \\
=& -kt \sum_{m=0}^{\infty} q^{km+x+1} e^{[km+x+1]_q t}.
\end{aligned} \tag{13}
$$

By (5), we have

$$
\begin{aligned}
&\mathcal{B}_{m,k,q}(kn) - \mathcal{B}_{m,k,q} \\
=& \int_{\mathbb{Z}_p} q^{-y}[ky+kn+1]_q^m d\mu_q(y) - \int_{\mathbb{Z}_p} q^{-y}[ky+1]_q^m d\mu_q(y) \\
=& (q-1)\sum_{l=0}^{n-1} q^l \left(q^{-l}[kl+1]_q^m\right) + \frac{q-1}{\log q}\sum_{l=0}^{n-1} q^l \left(-q^{-l}\log q \frac{\left(1-q^{kl+1}\right)^{m-1}}{(1-q)^m}\left(1-q^{kl+1}+kmq^{kl+1}\right)\right) \\
=& km \sum_{l=0}^{n-1} q^{kl+1}[kl+1]_q^{m-1}.
\end{aligned} \tag{14}
$$

Hence, by (14), we obtain the following theorem.

Theorem 2.2. *For each nonnegative integer m and each positive integer n, we have*

$$
\sum_{l=0}^{n-1} q^{kl+1}[kl+1]_q^m = \frac{1}{k(m+1)}\left(\mathcal{B}_{m+1,k,q}(kn) - \mathcal{B}_{m+1,k,q}\right).
$$

In the special case of the Theorem 2.2, if we put $k=2$, then we obtain the following corollary which is the Theorem 2.2 in [8].

Corollary 2.3. *For $m \geq 0$ and $k \in \mathbb{N}$, we have*

$$\frac{1}{m+1}\left(\mathcal{B}_{m+1,q}(kn)-\mathcal{B}_{m+1,q}\right)=\sum_{l=0}^{n-1} q^{kl+1}[kl+1]_q^m.$$

By (11) and the Theorem 2.2, we get

(15)

$$\begin{aligned}&\sum_{l=0}^{n-1} q^{kl+1}[kl+1]_q^m\\ =&\frac{1}{k(m+1)}\left(\sum_{l=0}^{m+1}\binom{m+1}{l} q^{knl}\mathcal{B}_{l,k,q}[kn]_q^{m+1-l}-\mathcal{B}_{m+1,k,q}\right)\\ =&\frac{1}{k(m+1)}\left(\sum_{l=0}^{m}\binom{m+1}{l} q^{knl}\mathcal{B}_{l,k,q}[kn]_q^{m+1-l}+(q-1)[kn(m+1)]_q\mathcal{B}_{m+1,k,q}\right),\end{aligned}$$

and thus, we obtain the following corollary.

Corollary 2.4. *For each nonnegative integer m and each positive integer n, we have*

$$\begin{aligned}&\sum_{l=0}^{n-1} q^{kl+1}[kl+1]_q^m\\ =&\frac{1}{k(m+1)}\sum_{l=0}^{m}\binom{m+1}{l} q^{knl}\mathcal{B}_{l,k,q}[kn]_q^{m+1-l}+\frac{(q-1)}{k(m+1)}[kn(m+1)]_q\mathcal{B}_{m+1,k,q}.\end{aligned}$$

3. A generalized q-Euler polynomials and numbers

In this section, we consider the alternating sum of powers of q-integers of skip count by k :

$$\sum_{l=0}^{n-1}(-q)^l[kl+1]_q^k=[1]_q^k-q[k+1]_q^k+q^2[2k+1]_q^k-\cdots+(-1)^{n-1}q^{n-1}[2n-1]_q^k.$$

Let p be a given odd prime number, and let $C(\mathbb{Z}_p)$ be the set of all continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, The *fermionic p-adic q-integral* are defined as follows:

$$I_{-q}(f)=\int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x)=\lim_{N\to\infty}\frac{1}{[p^N]_{-q}}\sum_{x=0}^{p^N-1} f(x)(-q)^x,\ \text{(see [?])}. \tag{16}$$

By (16), we see that

$$q^n I_{-q}(f_n)+(-1)^{n+1}I_{-q}(f)=[2]_q\sum_{l=0}^{n-1}(-1)^{n-l-1}q^l f(l),\ \text{(see [?])}. \tag{17}$$

In viewpoint of (7), we introduce a *generalized type* 2 *q-Euler polynomials* which are defined by the generating function to be

$$\sum_{n=0}^{\infty}\mathcal{E}_{n,k,q}(x)\frac{t^n}{n!}=\int_{\mathbb{Z}_p} e^{[ky+x+1]_q t}d\mu_{-q}(y). \tag{18}$$

In the special case $x = 0$, $\mathcal{E}_{n,k,q} = \mathcal{E}_{n,k,q}(0)$ are called a *generalized type* 2 *q-Euler numbers*.

By (18) and (16), we see that

$$\begin{aligned}\mathcal{E}_{n,k,q}(x) =&\int_{\mathbb{Z}_p}[ky+x+1]_q^n d\mu_{-q}(y)\\ =&\frac{q+1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}\frac{(-1)^l q^{(x+1)l}}{1+q^{kl+1}}\\ =&(1+q)\sum_{l=0}^{\infty}(-q)^l[kl+x+1]_q^n.\end{aligned} \tag{19}$$

In addition, by (18), we get

$$\sum_{n=0}^{\infty}E_{n,kq}(x)\frac{t^n}{n!} = [2]_q\sum_{l=0}^{\infty}(-q)^l e^{[kl+x+1]_q t}. \tag{20}$$

Furthermore, by (10), we get

$$\begin{aligned}\mathcal{E}_{n,k,q}(x) =&\int_{\mathbb{Z}_p}[ky+x+1]_q^n d\mu_{-q}(y)\\ =&\sum_{l=0}^{n}\binom{n}{l}q^{xl}[x]_q^{n-l}\int_{\mathbb{Z}_p}[ky+1]_q^l d\mu_{-q}(y)\\ =&\sum_{l=0}^{n}\binom{n}{l}q^{xl}[x]_q^{n-l}\mathcal{E}_{n,k,q}.\end{aligned} \tag{21}$$

By (19), (20) and (21), we obtain the following theorem.

Theorem 3.1. *For each nonnegative integer n, we have*

$$\begin{aligned}\mathcal{E}_{n,k,q}(x) =&\frac{[2]_q}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}\frac{(-1)^l q^{(x+1)l}}{1+q^{kl+1}}\\ =&[2]_q\sum_{l=0}^{\infty}(-q)^l[kl+x+1]_q^n\\ =&\sum_{l=0}^{n}\binom{n}{l}q^{xl}[x]_q^{n-l}\mathcal{E}_{l,k,q},\end{aligned}$$

and

$$\sum_{n=0}^{\infty}E_{n,k,q}(x)\frac{t^n}{n!} = [2]_q\sum_{l=0}^{\infty}(-q)^l e^{[kl+x+1]_q t}.$$

By (17), we get

$$\begin{aligned}&q^n\int_{\mathbb{Z}_p}[ky+kn+1]_q^n d\mu_{-q}(y)+(-1)^{n+1}\int_{\mathbb{Z}_p}[ky+1]_q^n d\mu_{-q}(y)\\ =&[2]_q\sum_{l=0}^{n-1}(-1)^{n-1-l}q^l[kl+1]_q^n,\ (n\geq 1).\end{aligned} \tag{22}$$

In particular, if we put $n = 1$, we see that

$$q\mathcal{E}_{n,k,q}(k) - \mathcal{E}_{n,k,q} = (1+q), \ (n \geq 0). \tag{23}$$

Let n be a positive odd integer. Then by (22), we see that

$$\begin{aligned} &q^n \int_{\mathbb{Z}_p} [ky + kn + 1]_q^l d\mu_{-q}(y) + \int_{\mathbb{Z}_p} [ky + 1]_q^l d\mu_{-1}(y) \\ =&(1+q) \sum_{m=0}^{n-1} (-q)^l [km+1]_q^l. \end{aligned} \tag{24}$$

By the definition of generalized type 2 q-Euler polynomials and (24), we obtain the following theorem.

Theorem 3.2. *For each odd positive integer n and each nonnegative integer l, we have*

$$\sum_{m=0}^{n-1} (-1)^m q^m [km+1]_q^l = \frac{q^n \mathcal{E}_{l,k,q}(kn) + \mathcal{E}_{l,k,q}}{1+q}.$$

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