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TOPOLOGIES ON $\mathbb{Z}$ DETERMINED BY SEQUENCES

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ABSTRACT. In the first part of this paper, we study nullpotency of linear recurrence sequences of integers and we give a proof of a conjecture of I.Z. Ruzsa.
In the second part, we characterize nullpotent sequences in an abelian ring by an arithmetic property analogous to that known for nullpotent sequences in an abelian group.

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1. Introduction and some basic results

The essential motivation of this work is to determine whether or not there exist group topologies or even ring topologies on $\mathbb{Z}$ satisfying some specified conditions. A typical problem is:
- Given a sequence of integers, when does a norm on the group (resp. ring) of integers $\mathbb{Z}$ exist such that this sequence converges?
Our approach is to use number theory methods, wherever possible, to answer the underlying questions in this context.

In this section, we recall some basic notions and results that will be useful in the sequel.

Let $(G, +)$ be an abelian group written in additive notation. The neutral element is denoted 0.

Definition 1. Let (a_n) be a sequence of elements of G. We say that the sequence (a_n) is nullpotent (or group nullpotent) if there exists a topology T on G such that (G, T) is a Hausdorff topological group and (a_n) converges to 0.

By convention, we say that a sequence of integers is nullpotent if it is nullpotent in the usual group $(\mathbb{Z}, +)$.

Recall that a function $N : G \to [0, +\infty[$ with $N(0) = 0$, is a semi-norm on the group $(G, +)$ if N satisfies the following two properties:
(a) symmetry: for all $u \in G, N(-u) = N(u)$;
(b) triangle inequality (or subadditivity): for all $u, v \in G$, $N(u+v) \leq N(u) + N(v)$.
The semi-norm N is a norm if in addition it satisfies the following property:
(c) separation: for all $u \in G$, $N(u) = 0 \Longrightarrow u = 0$.
Clearly, a norm on the group $(G, +)$ induces a Hausdorff group topology on G.

Simple examples of nullpotent sequences of integers.

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1. Let p be a prime number. The sequence (p^n) is nullpotent since $|p^n|_p \to 0$ where $|.|_p$ is the p-adic norm on $\mathbb{Z}$.
   On the other hand, if $(b_n) = (p^n + 1)$, the relation $b_{n+1} - pb_n = 1 - p$ shows that (b_n) cannot be nullpotent.
2. For any real number x, let $\|x\|$ denote the distance from the real number x to the nearest integer. Let β be a fixed real number in $\mathbb{R} \setminus \mathbb{Q}$. For $u \in \mathbb{Z}$, let $N_\beta(u) := \|\beta u\|$. This defines a norm on $\mathbb{Z}$.
   Let (a_n) be the Fibonacci sequence given by the recurrence relation:
   $$a_0 = a_1 = 1, a_{n+2} = a_{n+1} + a_n \qquad (n \geq 0).$$
   For $\beta = (1 + \sqrt{5})/2$, we have $|a_{n+1} - \beta a_n| \leq \frac{1}{\beta^n} \to 0$ and so $N_\beta(a_n) \to 0$. The Fibonacci sequence is then nullpotent.
3. For any function $\xi(n) \nearrow^{+\infty}$, there exists an increasing nullpotent sequence (a_n) such that $a_n \leq \xi(n)$ for every n. Indeed, we can consider for example: $a_n = 2^{[\frac{\log \xi(n)}{\log 2}]} \to 0$ in the 2-adic topology (where as usual $[.]$ denotes the greatest integer function).
   This gives an answer to problem 3 posed by I. Protasov and E. Zelenyuk in [12].
4. Let P be a polynomial with integer coefficients and with degree $d \geq 1$. Then the sequence $(P(n))$ is not nullpotent (this follows readily by induction on d and by using the fact that $P(n+1) - P(n)$ has degree $< d$).
5. The sequence of primes is not nullpotent. Indeed
   - Method 1. According to Y. Zhang [18], there exists an integer h and infinitely many prime numbers $p < q$ such that $q - p = h$.
   - Method 2. By the classical result of L. Schnirelmann, we know that there exists a positive integer h such that any natural number greater than 1 can be expressed as the sum of no more than h primes. Assume that the sequence (p_n) of primes is nullpotent and consider the compact $K = \{p_n : n \geq 1\} \cup \{0\}$. Let hK denote the compact $K + K + \cdots + K$ (h times). It follows that the space $\mathbb{Z}$ can be written as $\mathbb{Z} = hK \cup (-hK) \cup \{1\}$ which implies that the topological space $\mathbb{Z}$ is compact and this leads to a contradiction with Baire's theorem.

The nullpotency of sequences has been studied by many authors. Nienhuys [9] showed that if (n_k) is a sequence of positive integers such that $n_{k+1}/n_k \to \infty$, then $\mathbb{Z}$ equipped with the finest topology such that (n_k) converges to 0 is a complete topological group. By using a probabilistic method, Ajtai, Havas and Kolmós [1] proved that $\mathbb{Z}$ can be equipped with a Hausdorff (group-)topology such that the principal character is the only continuous character. This result was also proved by I.Z. Ruzsa by showing that the sequence (a_n) defined by $a_{2n} = 2^n, a_{2n+1} = 3^n$ is nullpotent. In the same context, I.Z. Ruzsa and R. Tijdeman [14] showed that there is an integer-valued additive function f, not identically 0, such that the sequence $(f(n+1) - f(n))$ is nullpotent.

An essential tool for the study of the nullpotency of sequences is the following result of I.Z. Ruzsa [13], E. G. Zelenyuk and I. V. Protasov [10, 11] which allows us to insert this study in an arithmetic context.

Theorem 1. *A sequence of integers (a_n) is nullpotent if and only if: for every $b \in \mathbb{Z}^*$, $k \in \mathbb{N}$ and $(\varepsilon_1, \ldots, \varepsilon_k) \in \{-1, 1\}^k$, there is only a finite number of indices*

$j_1 \leq \cdots \leq j_k$ *satisfying*

$$\sum_{i=1}^{k} \varepsilon_i a_{j_i} = b \tag{1}$$

and

$$\sum_{i\in I} \varepsilon_i a_{j_i} \neq 0 \text{ for all non empty subsets } I \text{ of } \{1,\dots,k\}. \tag{2}$$

In fact, under the conditions (1) and (2), I.Z. Ruzsa [13] constructed a norm N on the group $\mathbb{Z}$ such that $N(a_n) \to 0$.

Definition 2. The representation (1) is said to be *primitive* if the condition (2) is satisfied.

The next result provides practical criteria for nullpotency(see [10, 11]).

Theorem 2. *Let* (a_n) *be a sequence of integers such that* $a_n \to +\infty$. *Assume that* $\frac{a_{n+1}}{a_n} \to \beta \in \mathbb{R}^+ \cup \{+\infty\}$. *Then* (a_n) *is nullpotent in each of the following cases:*

(B1) $\beta \in [0,+\infty) \smallsetminus \mathbb{Q}$ *and* $\rho(n) := a_{n+1} - \beta a_n \to 0$;
(B2) $\beta = +\infty$;
(B3) β *is a transcendental number.*

In the first case (B1), we have $N_\beta(a_n) := \|\beta a_n\| \to 0$. In the other cases, the Theorem 2 is obtained by applying Theorem 1.
In [6], N. Hegyvári used Theorem 1 to show that if (a_n) is some linear recurrence sequence, h is a polynomial over the integers of degree ≥ 2 and (b_n) is a sequence with positive upper density, then the sum $x_n = a_n + b_n + h(n)$ is not nullpotent.

Simple applications of the Theorem 2. The following sequences are nullpotent.

- The Fibonacci sequence (mentioned above) is nullpotent since it satisfies (B1).
- Let $p_1 = 2 < p_2 = 3 < p_3 = 5 < \dots$ be the increasing sequence of the prime numbers. Then the sequence $(p_{n!})$ is nullpotent since it satisfies (B2).
- The sequence $(a_n) = ([e^n])$ is nullpotent since it satisfies (B3).

Remarks 1.

1- Let $u_n = 2^n$, $v_n = 2^{n^2} - 2^n + 1$, $w_n = v_n$ if n is a square, $w_n = 0$ otherwise. The two sequences (u_n) and (w_n) are positive nullpotent sequences since $u_n \to 0$ in the 2-adic topology and $\frac{v_{n+1}}{v_n} \to +\infty$, but their sum $(s_n) = (u_n + w_n)$ is not because $s_{m^4+1} - 2s_{m^2} = 2$ for all integers $m \geq 1$.

2- The square of a nullpotent sequence cannot, in general, be nullpotent. Let us consider the sequence $(\frac{h_n}{k_n})$ of convergents of the continued fraction expansion of $\sqrt{2}$. We know that

$$h_{-1} = h_0 = 1, \qquad k_{-1} = 0, k_0 = 1,$$

$$h_n = 2h_{n-1} + h_{n-2}, \qquad 2k_n = h_n + h_{n-1} \tag{3}$$

and

$$|k_n\sqrt{2} - h_n| \leq \frac{1}{2k_n} \qquad (n \geq 0).$$

So for $\beta = 1/\sqrt{2}$, $N_\beta(h_n) := \|\beta h_n\| \to 0$.
On the other hand, we know that the pair of positive integers $(x, y) = (h_{2m+1}, k_{2m+1})$ is a solution of the Pell-Fermat.

$$x^2 - 2y^2 = 1. \tag{4}$$

It follows that

$$6h_{2n+1}^2 - h_{2n+2}^2 - h_{2n}^2 = 4 \qquad (n \geq 0)$$

which shows that the sequence (h_n^2) is not nullpotent.

Several non-trivial nullpotent sequences are provided by applying the following theorem of J-H. Evertse [4] on S-unit equations.

Theorem 3. *Let c, d be constants with $c > 0, 0 \leq d < 1$, let S_0 be a finite set of prime numbers and let k be a positive integer. Then there are only finitely many $(k+1)$-tuples $x = (x_0, x_1, \ldots, x_k)$ of rational integers such that*

(a) $\displaystyle\sum_{i=0}^{k} x_i = 0$;

(b) $\displaystyle\sum_{j\in J} x_j \neq 0 \qquad$ *for all* $\qquad \emptyset \subsetneq J \subsetneq \{0, 1, \ldots, k\}$;

(c) $\gcd(x_0, x_1, \ldots, x_k) = 1$;

(d) $\displaystyle\prod_{i=0}^{k} |x_i| \prod_{p\in S_0} |x_i|_p \leq c\big(\max_{1\leq i\leq k} |x_i|\big)^d$.

An important application of this theorem is the following result of I.Z. Ruzsa [13].

Theorem 4. *Let $\mathcal{P}$ be a finite set of primes numbers. Let (a_n) be a sequence of distinct integers > 1 such that $a_n \to +\infty$ and for all n,*

$$(p \mid a_n \text{ and } p \text{ prime }) \Longrightarrow p \in \mathcal{P}.$$

Then (a_n) is nullpotent.

Example. Let $u_1, \ldots, u_k$ be integers greater than 1 and $c_1, \ldots, c_k$ be nonzero integers. Let $(\varphi_1(n)), \ldots, (\varphi_k(n))$ be strictly increasing sequences of positive integers. Let (a_n) the sequence defined, for all $n \geq 1$, by: $a_{kn+j-1} = u_j^{\varphi_j(n)}$, where $j = 1, \ldots, k$. Then (a_n) is nullpotent. This implies in particular that the sequence $(\sum_{j=1}^{k} c_j u_j^{\varphi_j(n)})$ is also nullpotent.

In what follows we will recall the well known result of Baker [16] (see [15], p. 30) concerning estimates of linear forms in logarithms. In our context, we need it to show, in appropriate situation, that the number of solutions of (1) is at most finite.

Let δ be a non-zero algebraic number and let $P(x) = a_0x^d + a_1x^{d-1} + \cdots + a_d$ be its minimal polynomial over the integers. Recall the definitions:
the degree of δ : $\deg(\delta) := \mathrm{d}^\circ P = d$,
the height of δ : $H(\delta) := \max(|a_0|, |a_1|, \ldots, |a_d|)$.
Furthermore, if $\delta_1 = \delta, \delta_2, \ldots, \delta_d$ are all conjugates of δ, we set the notation $|\overline{\delta}| = \max_{1\leq i\leq d} |\delta_i|$.
Let us recall also the following well known properties of $|\overline{\delta}|$ and the height function

H ([15], pp 10-11): if β and δ are non-zero algebraic numbers of degrees at most d then

$$\begin{aligned} |\overline{\beta+\delta}| &\leq |\overline{\beta}| + |\overline{\delta}| && (5)\\ |\overline{\beta\delta}| &\leq |\overline{\beta}||\overline{\delta}| && (6)\\ |\overline{\delta}| &\leq \deg(\delta)H(\delta) && (7)\\ H(\delta) &\leq (2|\overline{\delta}|)^{\deg(\delta)} && (8)\\ \log H(\beta+\delta) &\leq c_1 \log H && (9)\\ \log H(\beta\delta) &\leq c_2 \log H, && (10) \end{aligned}$$

where $H := \max(H(\beta), H(\delta), 2)$ and c_1, c_2 are two computable numbers depending only on d.

Let $\alpha_1, \dots, \alpha_n$ represent non-zero algebraic numbers. For each $j \in \{1, \dots, n\}$, let H_j be a real number ≥ 3 such that $H(\alpha_j) \leq H_j$. Let d, Ω, Ω' be the numbers defined by:

$$d = [\mathbb{Q}(\alpha_1, \dots, \alpha_n) : \mathbb{Q}], \tag{11}$$

$$\Omega = \textstyle\prod_{j=1}^{n} \log H_j, \qquad \Omega' = \displaystyle\prod_{j=1}^{n-1} \log H_j. \tag{12}$$

Theorem 5 (Baker[16] (see [15], p. 30))**.** *Let $B \geq 2$ be a real number. There exist computable absolute constants c and c' such that the inequalities*

$$0 < \big|1 - \prod_{i=1}^{n} \alpha_i^{b_i}\big| < \exp\Big(- (cnd)^{c'n} \Omega \log \Omega' \log B \Big) \tag{13}$$

have no solution in rational integers $b_1, \dots, b_n$ of absolute values not exceeding B.

2. Main results

2.1. Nullpotent sequence in the group $\mathbb{Z}$.

We first present a detailed proof for the following result stated in [13], p.488.

Theorem 6. *Let P be a finite set of primes and let (a_n) be a sequence of positive integers. Assume that $a_n = b_n d_n$, where b_n is composed exclusively of primes belonging to P and the positive integers d_n satisfy $d_n = O(b_n^{\varepsilon})$ for every $\varepsilon > 0$ as $n \to +\infty$. Then the sequence (a_n) is nullpotent.*

Proof. Set b, k integers ≥ 1 and $(\varepsilon_1, \dots, \varepsilon_k) \in \{-1, 1\}^k$. Let us show that the equality

$$\sum_{i=1}^{k} \varepsilon_i a_{n_i} = b \tag{14}$$

has only a finite number of solutions $(a_{n_1}, \dots, a_{n_k})$ such that

$$\sum_{j \in J} \varepsilon_j a_{n_j} \neq 0 \qquad \text{for every nonempty subset } J \text{ of } \{1, \dots, k\}.$$

By dividing the two members of (14) by $\gcd(b, \varepsilon_1 a_{n_1}, \dots, \varepsilon_k a_{n_k})$, we can assume $\gcd(b, \varepsilon_1 a_{n_1}, \dots, \varepsilon_k a_{n_k}) = 1$.
We shall apply Theorem 3 with:

$$S_0 = P,\ x_0 = -b,\ x_j = \varepsilon_j a_{n_j} \qquad (j = 1, \dots, k).$$

We therefore need to verify that the condition (d) is satisfied. Let $\varepsilon \in]0, 1/k[$. By hypothesis, there exists a real number $r_\varepsilon > 0$ such that $d_n \le r_\varepsilon b_n^\varepsilon$ for all $n \in \mathbb{N}$.
The condition (d) is satisfied for $d = \varepsilon k$ since

$$\begin{aligned} \prod_{i=0}^{k} |x_i| \prod_{p \in S_0} |x_i|_p &\le |x_0| \prod_{i=1}^{k} r_\varepsilon b_{n_i}^\varepsilon \\ &\le |x_0| r_\varepsilon^k \big(\max_{1 \le i \le k} b_{n_i} \big)^{\varepsilon k} \\ &\le c\big(\max_{1 \le i \le k} |x_i| \big)^d \end{aligned}$$

where $c = |x_0| r_\varepsilon^k$.
Thus Theorem 6 is obtained as a consequence of Theorem 3. □

Example. Let $u_1, \dots, u_k$ be integers greater than 1 and $T_1, \dots, T_k$ be polynomials with integer coefficients. Let $(\varphi_1(n)), \dots, (\varphi_k(n))$ be strictly increasing sequences of positive integers. Then the sequence $(\sum_{j=1}^k T_j(n) u_j^{\varphi_j(n)})$ is nullpotent.

Our main objective in the rest of this section is the study of nullpotency of linear recurrence sequences of integers. To illustrate two different approaches for this study, we propose to deal first with binary recurrence sequences.
Let

$$a_{n+2} = u a_{n+1} + v a_n, \qquad (n = 0, 1, \dots) \tag{15}$$

be a binary recurrence sequence such that $(a_0, a_1) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ and $(u, v) \in \mathbb{Z}^* \times \mathbb{Z}$ such that $u^2 + 4v \ne 0$. If α and β are the roots in $\mathbb{C}$ of the polynomial $z^2 - uz - v$, we know that the sequence (a_n) can be expressed in the form:

$$a_n = a\alpha^n + b\beta^n \qquad (n = 0, 1, 2, \dots) \tag{16}$$

with

$$a = \frac{\beta a_0 - a_1}{\beta - \alpha}, \quad b = -\frac{\alpha a_0 - a_1}{\beta - \alpha}.$$

Suppose in the following results that

$$a \ne 0. \tag{17}$$

Theorem 7. *Suppose $|\alpha| \ge |\beta| > 1$. Let $k \ge 1$, $A_1, \dots, A_k$ are given non-zero integers. There exist computable numbers N_k and c_k depending only on the sequence (a_n) and $(A_1, \dots, A_k)$ such that, for all $N_k < n_1 < \dots < n_k$ we have*

$$|\sum_{j=1}^{k} A_j a_{n_j}| \ge |\alpha|^{n_k} \exp\big(- c_k (\log n_k)^k \big). \tag{18}$$

whenever

$$\sum_{j \in J} A_j a_{n_j} \ne 0 \text{ for all nonempty subsets } J \text{ of } \{1, \dots, k\}. \tag{19}$$

Proof. We use an argument similar to the one used by Stewart and Shorey in their proofs concerning lower bounds for $|a_n|$ and $|a_n - a_m|$(see [15], p. 64). We proceed by induction on k.

For $k = 1$, we first write

$$|a_{n_1}| = |a||\alpha|^{n_1}\big|1 + \frac{b}{a}(\frac{\beta}{\alpha})^{n_1}\big|. \tag{20}$$

Theorem 5 shows that the inequality

$$|(\frac{\beta}{\alpha})^{n_1} + 1| \geq \exp(-\frac{c}{2}\log n_1)$$

is satisfied for a suitable constant $c > 0$ and for all sufficiently large integers n_1, and we therefore obtain

$$|a_{n_1}| \geq |\alpha|^{n_1}\exp(-c\log n_1) \qquad (n_1 \geq N_1) \tag{21}$$

as desired.

Now we assume (18) is valid for $k - 1$, and we shall prove it for k. We distinguish two cases.

Case 1. $|\sum_{j=2}^{k} A_j a_{n_j}| \geq 2|A_1 a_{n_1}|$.

Then

$$\begin{aligned} |\sum_{j=1}^{k} A_j a_{n_j}| &\geq |\sum_{j=2}^{k} A_j a_{n_j}| - |A_1 a_{n_1}| \\ &\geq \frac{1}{2}|\sum_{j=2}^{k} A_j a_{n_j}| \\ &\geq \frac{1}{2}|\alpha|^{n_k}\exp\big(-c_k(\log n_k)^{k-1}\big) \quad \text{(by the induction hypothesis)} \\ &\geq |\alpha|^{n_k}\exp\big(-c_k(\log n_k)^{k}\big) \end{aligned} \tag{22}$$

for sufficiently large integer n_1.

Case 2. $|\sum_{j=2}^{k} A_j a_{n_j}| \leq 2|A_1 a_{n_1}|$.

By the induction hypothesis, this case gives

$$|\alpha|^{n_k}\exp\big(-c_k(\log n_k)^{k-1}\big) \leq c|\alpha|^{n_1} \tag{23}$$

which implies for sufficiently large n_k

$$n_k - n_1 \leq c'_k(\log n_k)^k \tag{24}$$

since $|\alpha| > 1$.

Write now

$$0 < |\sum_{j=1}^{k} A_j a_{n_j}| = |A + B|, \tag{25}$$

where

$$A := a\sum_{j=1}^{k} A_j\alpha^{n_j}, \qquad B := b\sum_{j=1}^{k} A_j\beta^{n_j}.$$

Suppose for example that $A \neq 0$. We see that

$$\frac{B}{A} = \frac{b}{a}\Big(\frac{\beta}{\alpha}\Big)^{n_1}\frac{A_1 + A_2\beta^{n_2-n_1} + \cdots + A_k\beta^{n_k-n_1}}{A_1 + A_2\alpha^{n_2-n_1} + \cdots + A_k\alpha^{n_k-n_1}}.$$

Applying Theorem 5 with:

$$d \leq 2,\ \ n = 3,\ B = n_1,\ \ \log H_1 \leq c_0,\ \ \log H_2 \leq c_0,$$

$$\text{and } \log H_3 \leq c_k(n_k - n_1) \leq c_k''(\log n_k)^k,$$

we get

$$|1 + \frac{B}{A}| \geq \exp\big(-c_k''(\log n_k)^k\big),$$

and so by (25) we may conclude

$$|\sum_{j=1}^{k} A_j a_{n_j}| \geq |A| \exp\big(-c_k(\log n_k)^k\big). \tag{26}$$

Similarly, we prove by induction on k that:

$$|A| \geq |\alpha|^{n_k}\exp\big(-c_k(\log n_k)^k\big).$$

□

Theorem 8. *The binary recurrence sequence* (a_n) *is nullpotent if and only if* α *and* β *are not roots of unity.*

Proof. It is easy to verify that if one of α, β is a root of unity, then (a_n) cannot be nullpotent. Suppose now that the numbers α and β are not roots of unity and $|\beta| \geq |\alpha|$. We distinguish the cases: $|\alpha| < 1$, $|\alpha| = 1$ and $|\alpha| > 1$.
In the first case we have

$$a_{n+1} - \beta a_n = b(\alpha - \beta)\alpha^n;$$

this gives $\beta \in \mathbb{R} \setminus \mathbb{Q}$ (by using (17)) and

$$N_\beta(a_n) := \|\beta a_n\| \to 0. \tag{27}$$

If $|\alpha| = 1$, the roots of the polynomial $z^2 - uz - v$ are α and $\beta = \overline{\alpha}$ with modulus $-v = 1$. So $u = \pm 1$ (since $\Delta = u^2 + 4v < 0$ and u integer). This implies that α is a root of unity, which is a contradiction.
If $1 < |\alpha| \leq |\beta|$, by Theorem 7, we see that the equation (1) cannot have an infinite number of solutions. This completes the proof of the theorem 8. □

More generally, let us now consider a linear recurrence sequence $(a_n)_{n\geq 0}$ in $\mathbb{Z}$ of order $k \geq 1$:

$$a_{n+k} = c_1 a_{n+k-1} + c_2 a_{n+k-2} + \cdots + c_k a_n \qquad (n \geq k) \tag{28}$$

with constant coefficients $c_i \in \mathbb{Z}, c_k \neq 0$, and initial values $a_0, \ldots, a_{k-1}$. By definition, the order k is the smallest positive integer satisfying (28).

It is well known (see, for example, [15], Theorem C.1, p.33) that the linear recurrence sequence (a_n) can be expressed as a polynomial exponential sum:

$$a_n = \sum_{\ell=1}^{L_0} P_\ell(n)\alpha_\ell^n \qquad (n \geq 0) \tag{29}$$

where the P_j are polynomials with complex coefficients and the α_ℓ are the roots of the characteristic polynomial:

$$Q(t) = t^k - c_1 t^{k-1} - c_2 t^{k-2} + \cdots - c_k.$$

We present a proof of the following result conjectured by I.Z. Ruzsa [13], p.488.

Theorem 9. *A linear recurrence sequence is nullpotent in $\mathbb{Z}$ unless there is a root of unity among the roots of the characteristic polynomial.*

The proof we present is based on the following result of Laurent. Let us recall some basic notations and terminology related to this context.
Let $\mathbb{K}$ be a field of characteristic 0 and let d be a positive integer. For $d-$tuples $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in (\mathbb{K}^*)^d$ and $\boldsymbol{m} = (m_1, \ldots, m_d) \in \mathbb{Z}^d$, we use the notation: $\boldsymbol{\alpha}^{\boldsymbol{m}} := \alpha_1^{m_1} \ldots \alpha_d^{m_d}$.
Let $f_1, \ldots, f_r$ be polynomials in d variables with coefficients in $\mathbb{K}$ and $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_r \in (\mathbb{K}^*)^d$.

Consider the following equation:

$$\sum_{j=1}^{r} f_j(\boldsymbol{m})\boldsymbol{\alpha}_j^{\boldsymbol{m}} = 0 \qquad \text{with } \boldsymbol{m} \in \mathbb{Z}^d \text{ unknown.} \tag{30}$$

A solution $\boldsymbol{m} \in \mathbb{Z}^d$ is called *non-degenerate* if there is no non-empty proper subset I of $\{1, \ldots, r\}$ such that $\sum_{i \in I} f_i(\boldsymbol{m})\boldsymbol{\alpha}_i^{\boldsymbol{m}} = 0$.
Define

$$\mathcal{M} := \{\boldsymbol{m} \in \mathbb{Z}^n : \boldsymbol{\alpha}_1^{\boldsymbol{m}} = \cdots = \boldsymbol{\alpha}_r^{\boldsymbol{m}}\}$$

with $\mathcal{M} = \{0\}$ in the particular case $r = 1$.
We shall use the following theorem of Laurent.

Theorem 10 (Laurent[7, 8], see [3], p. 327)**.** *If $\mathcal{M} = \{0\}$ then the equation (30) has only finitely many non-degenerate solutions.*

Proof of the Theorem 9.
Assume there exist an integer $b' \neq 0$, an integer $k' \geq 1$, a k'-tuple of non-zero integers $(A_1, \ldots, A_{k'})$ such that for all integers $n \geq 1$ we have

$$b' = \sum_{i=1}^{k'} A_i a_{\varphi_i(n)} \tag{31}$$

where $\varphi_i(n) \to +\infty$ for $i = 1, \ldots, k'$, and with the property that all representations (31) are primitive. Then, by using the expression (29) of a_n, the representation (31) becomes

$$b' = \sum_{i=1}^{k'} \sum_{\ell=1}^{L_0} A_i P_\ell(\varphi_i(n))\alpha_\ell^{\varphi_i(n)} \qquad (n = 1, 2, \ldots). \tag{32}$$

For every $\mathcal{I} \subset \{1,\dots,k'\}$ and $\mathcal{L} \subset \{1,\dots,L_0\}$, we consider the subset $\mathcal{A}_{\mathcal{I},\mathcal{L}}$ of $\mathbb{N}$ whose elements n satisfy

$$b' = \sum_{i\in\mathcal{I}}\sum_{\ell\in\mathcal{L}} A_i P_\ell(\varphi_i(n))\alpha_\ell^{\varphi_i(n)} \tag{33}$$

with

$$\sum_{i\in\mathcal{I}'}\sum_{\ell\in\mathcal{L}'} A_i P_\ell(\varphi_i(n))\alpha_\ell^{\varphi_i(n)} \neq 0 \tag{34}$$

for all non-empty subset $\mathcal{I}'$ of $\mathcal{I}$ and all non-empty subset $\mathcal{L}'$ of $\mathcal{L}$.
By (32), we have $\mathbb{N} = \cup_{\mathcal{I},\mathcal{L}}\mathcal{A}_{\mathcal{I},\mathcal{L}}$, thus, there exist $\mathcal{I}$ and $\mathcal{L}$ such that the set $\mathcal{A}_{\mathcal{I},\mathcal{L}}$ is infinite.
For simplicity of notations and without loss of generality, we may suppose that the index set $\mathcal{I}$ is of the form $\{1,\dots,k\}$ and the index set $\mathcal{L}$ is of the form $\{1,\dots,L\}$. Hence we can write

$$b' = \sum_{i=1}^{k}\sum_{\ell=1}^{L} A_i P_\ell(\varphi_i(n))\beta_\ell^{\varphi_i(n)} \qquad (n\in\mathcal{A}_{\mathcal{I},\mathcal{L}}) \tag{35}$$

with

$$\sum_{i\in\mathcal{I}'}\sum_{\ell\in\mathcal{L}'} A_i P_\ell(\varphi_i(n))\beta_\ell^{\varphi_i(n)} \neq 0 \tag{36}$$

for all non-empty subset $\mathcal{I}'$ of $\{1,\dots,k\}$ and all non-empty subset $\mathcal{L}'$ of $\{1,\dots,L\}$.

Let for $i = 0,\dots,k-1$ and $\ell = 1,\dots,L$

$$\boldsymbol{\alpha}_{iL+\ell} = (1,\dots,1,\alpha_\ell,1,\dots,1) \in \mathbb{C}^{kL}$$

where α_ℓ is the $(iL+\ell)^{\text{th}}$ component for $\alpha_{iL+\ell}$. Let also

$$\boldsymbol{\alpha}_{kL+1} = (1,\dots,1).$$

We see that, for all $\boldsymbol{m} = (m_1,\dots,m_{kL}) \in \mathbb{Z}^{kL}$,

$$\boldsymbol{\alpha}_{iL+\ell}^{\boldsymbol{m}} = \alpha_\ell^{m_{iL+\ell}} \qquad (i=0,\dots,k-1, \ell=1,\dots,L). \tag{37}$$

Put

$$f_{iL+\ell}(\boldsymbol{m}) = A_{i+1}P_\ell(m_{iL+\ell}) \qquad (i=0,\dots,k-1,\ell=1,\dots,L), \tag{38}$$

$$f_{kL+1}(\boldsymbol{m}) = -b'. \tag{39}$$

For $n\in\mathcal{A}_{\mathcal{I},\mathcal{L}}$, consider $\boldsymbol{m} = \boldsymbol{m}(n) = (m_1(n),\dots,m_{kL}(n)) \in \mathbb{N}^{kL}$ such that

$$m_{iL+\ell} = m_{iL+\ell}(n) := \varphi_{i+1}(n) \qquad (i=0,\dots,k-1,\ell=1,\dots,L). \tag{40}$$

We have, for every $n\in\mathcal{A}_{\mathcal{I},\mathcal{L}}$,

$$\begin{aligned}\sum_{s=1}^{kL} f_s(\boldsymbol{m})\boldsymbol{\alpha}_s^{\boldsymbol{m}} &= \sum_{i=0}^{k-1}\sum_{\ell=1}^{L} f_{iL+\ell}(\boldsymbol{m})\boldsymbol{\alpha}_{iL+\ell}^{\boldsymbol{m}}\\ &= \sum_{i=0}^{k-1}\sum_{\ell=1}^{L} A_{i+1}P_\ell(\varphi_{i+1}(n))\alpha_\ell^{\varphi_{i+1}(n)}\\ &= b'.\end{aligned}$$

We thus obtain that the equation

$$\sum_{s=1}^{kL+1} f_s(\boldsymbol{m})\boldsymbol{\alpha}_s^{\boldsymbol{m}} = 0 \tag{41}$$

has infinitely many non-degenerate solutions $\boldsymbol{m}$.

On the other hand, with the notation of Theorem 10, recalling that

$$\begin{aligned} \mathcal{M} &:= \{\boldsymbol{m} \in \mathbb{Z}^{kL} : \boldsymbol{\alpha}_1^{\boldsymbol{m}} = \cdots = \boldsymbol{\alpha}_{kL+1}^{\boldsymbol{m}}\} \\ &= \{\boldsymbol{m} \in \mathbb{Z}^{kL} : \boldsymbol{\alpha}_1^{\boldsymbol{m}} = \cdots = \boldsymbol{\alpha}_{kL}^{\boldsymbol{m}} = 1\} \end{aligned}$$

and let us verify that $\mathcal{M} = \{0\}$.
Let $\boldsymbol{m} = (m_1, \ldots, m_{kL}) \in \mathbb{Z}^{kL}$. Since

$$\boldsymbol{\alpha}_{iL+\ell}^{\boldsymbol{m}} = \alpha_\ell^{m_{iL+\ell}}$$

and none of α_i are roots of unity, we see that

$$\boldsymbol{\alpha}_{iL+\ell}^{\boldsymbol{m}} = 1 \Rightarrow m_{iL+\ell} = 0 \qquad (i = 0, \ldots, k-1; \ell = 1, \ldots, L);$$

which gives

$$\mathcal{M} = \{0\}.$$

Hence, by Theorem 10, the equation (41) has only finitely many non-degenerate solutions, which leads to a contradiction. This concludes the proof.

Suppose now that α is an ℓ-th root of unity. It is known (see, for example, [5], Theorem 1.3, p.5) that the subsequence $(a_{\ell n})_n$ of the linear recurrence sequence (a_n) is also a linear recurrence sequence of order at most d, so there exist rational numbers $\lambda_1, \ldots, \lambda_d$ such that

$$1 = \sum_{b=1}^{d} \lambda_b a_{\ell(n+b)} \qquad (n \in \mathbb{N}); \tag{42}$$

where we have used the fact that for given $(c_1, \ldots, c_k) \in \mathbb{Q}^k$ with $c_k \neq 0$, the space $E = E_{(c_1,\ldots,c_k)}$ of linear recurrence sequences in $\mathbb{Q}$ of order at most k and satisfying (28) is a $\mathbb{Q}$-vector space of dimension k.
This shows that the sequence (a_n) is not nullpotent in $\mathbb{Z}$.

2.2. **Ring nullpotent sequences.**

A sequence of elements of an abelian ring A is ring nullpotent in A if there exists a Hausdorff ring topology on A that makes the sequence converging to 0.
The main objective of this section is to study ring nullpotent sequences in $\mathbb{Z}$.

Remarks 2.

1- Let p be prime number. The sequence (p^n) is ring nullpotent in $\mathbb{Z}$ since $|p^n|_p \to 0$ where $|.|_p$ is the p-adic norm on the ring $\mathbb{Z}$.

2- The Fibonacci sequence (u_n) is not ring nullpotent in $\mathbb{Z}$ since:

$$u_{n+1}^2 - u_n u_{n+2} = (-1)^n \qquad (n \geq 0).$$

3- Let α be real number > 0. The sequence (considered in [2]) defined by

$$a_n = \begin{cases} [\alpha^{n/2}]^2 + 1 & \text{if } n = 2.3^k \text{ for some } k \in \mathbb{N}, \\ [\alpha^n] & \text{otherwise} \end{cases}$$

satisfies $\lim_n \frac{a_{n+1}}{a_n} = \alpha$, so, if α is transcendental, the sequence (a_n) is nullpotent in the group $\mathbb{Z}$, but it is not ring nullpotent in $\mathbb{Z}$ because $a_{2.3^k} - a_{3^k}^2 = 1$ for every $k \in \mathbb{N}$.

Notations. For every integer $\ell \geq 1$, we denote by $\mathbb{J}_\ell$ the set of multi-indices $(j_1, \dots, j_\ell)$ such that $1 \leq j_1 \leq \cdots \leq j_\ell$.
Let

$$\mathbb{J} := \cup_{\ell \geq 1} \mathbb{J}_\ell.$$

For every $J = (j_1, \dots, j_\ell) \in \mathbb{J}_\ell$, put

$$J^* := \{j_1, \dots, j_\ell\}, \tag{43}$$

$$a_J := \prod_{i=1}^{\ell} a_{j_i} \tag{44}$$

for a fixed sequence (a_n).
We state our principal result:

Theorem 11. *A sequence of integers $(a_n)_{n\geq 1}$ is ring nullpotent in $\mathbb{Z}$ if and only if: for every $(b, k, L) \in \mathbb{Z}^* \times \mathbb{N} \times \mathbb{N}$ and $(\varepsilon_1, \dots, \varepsilon_k) \in \{-1, 1\}^k$, there is at most a finite number of $J_1, \dots, J_k \in \cup_{1\leq \ell \leq L} \mathbb{J}_\ell$ such that*

$$\sum_{i=1}^{k} \varepsilon_i a_{J_i} = b \tag{45}$$

where

$$\sum_{i \in I} \varepsilon_i a_{J_i} \neq 0 \text{ for all non empty subset } I \text{ of } \{1, \dots, k\}. \tag{46}$$

In fact, under the assumptions of the Theorem 11, the group $(\mathbb{Z}, +)$ can be equipped with a norm N satisfying

$$N(uv) \leq N(u) + N(v) \qquad \text{for all } (u, v) \in \mathbb{Z}^2,$$

such that $N(a_n) \to 0$.

Definition 3. If the integer b is written in the form (45) under the condition (46), we say that the representation (45) is primitive.

Proof of the Theorem 11.
$\Rightarrow$) Suppose that for some $b \in \mathbb{Z}^*$, $k \in \mathbb{N}, L \in \mathbb{N}$, the representation (45) is valid with the condition (46) for infinitely many $J_1, \dots, J_k \in \cup_{1\leq \ell \leq L} \mathbb{J}_\ell$. By choosing subsequences if necessary, we can write this in the form:

$$c_1 a_{\varphi_{1,1}(n)} \cdots a_{\varphi_{1,m_1}(n)} + \cdots + c_d a_{\varphi_{d,1}(n)} \cdots a_{\varphi_{d,m_d}(n)} = b' \tag{47}$$

where $b' \neq 0$, $d \in \{1, \dots, k\}$, $c_i \in \mathbb{Z}^*$, $m_i \in \{1, \dots, L\}$, $\varphi_{i,\ell}(n) \to +\infty$ for every $(i, \ell) \in \{1, \dots, d\} \times \{1, \dots, L\}$.
Passage to the limit yields the contradiction: $b = 0$.
$\Leftarrow$) Let $H : \mathbb{Z} \to [0, +\infty[$, be a function. Let $\mathcal{S}$ be the set of all integers s that can be represented in the form $s = \sum_{i=1}^{N} \varepsilon_i a_{J_i}$.

We define the function $N'_H : \mathbb{Z} \to [0, +\infty[$ by: $N'_H(u) = 1$ if $u \notin \mathcal{S}$, and for every $u \in \mathcal{S}$,

$$N'_H(u) := \inf \Big\{ \big(\log(N+1) + M_N \big) \min_{j \in \cup_{1 \le i \le N} J_i^*} H(a_j) : u = \sum_{i=1}^{N} \varepsilon_i a_{J_i},$$
$$N \in \mathbb{N}, \varepsilon_i = \pm 1, J_i \in \mathbb{J}, M_N := \max_{1 \le i \le N} \operatorname{card} J_i^* \Big\}. \quad (48)$$

Now let

$$N_H := \min(N'_H, 1). \quad (49)$$

The reader can easily verify that the function N_H has the following properties:

(i) $N_H(-u) = N_H(u)$ for all $u \in \mathbb{Z}$,
(ii) $N_H(u+v) \le N_H(u) + N_H(v)$ for all $u, v \in \mathbb{Z}$,
(iii) $N_H(uv) \le N_H(u) + N_H(v)$ for all $u, v \in \mathbb{Z}$,
(iv) $N_H(u) \le 1$ for all $u \in \mathbb{Z}$,
(v) $N_H(a_n) \le 2H(a_n)$ for all n.

let $\mathcal{B} = \{b_1, b_2, \dots\}$ denote the countable set of integers $b \in \mathbb{Z}^*$ that can be written in *a primitive* form (45). For each fixed $b \in \mathcal{B}$, let us first define a suitable function $H_b : \mathbb{Z} \to [0, +\infty[$ such that $H_b(a_n) \to 0$.
Put $H_b(0) = 0$ and $H_b(u) = 1$ for all $u \notin \{a_n : n \in \mathbb{N}\} \cup \{0\}$. Set n an integer ≥ 1. We shall define $H_b(a_n)$. We first consider the set

$$E_n := \Big\{ \log(k+1) + \max_{1 \le i \le k} \operatorname{card} J_i^* : b \text{ has the primitive form (45) with}$$
$$n \in \cup_i J_i^*, k \in \mathbb{N}, \varepsilon_i = \pm 1, J_i \in \mathbb{J} \Big\}. \quad (50)$$

If $E_n = \emptyset$, we set $H_b(a_n) = 0$. If $E_n \ne \emptyset$, we set

$$H_b(a_n) := 1/r_n \ \text{ where } r_n := \min E_n.$$

The hypothesis of the Theorem 11 implies that $H_b(a_n) \to 0$ and so $N_{H_b}(a_n) \to 0$.
Furthermore, if $b = \sum_{i=1}^{N} \varepsilon_i a_{J_i}$, we have by letting $M_N := \max_{1 \le i \le N} \operatorname{card} J_i^*$,

$$H_b(a_j) \ge 1 / \big(\log(N+1) + M_N \big),$$

for all $j \in J_i (i = 1, \dots, N)$, and it follows from (48) and (49) that

$$N_{H_b}(b) = 1 \qquad \text{for each } b \in \mathcal{B} = \{b_1, b_2, \dots\}. \quad (51)$$

Now, we consider the function N defined on $\mathbb{Z}$, by

$$N(u) = \sum_{m=1}^{\infty} 2^{-m} N_{H_{b_m}}(|u|).$$

Hence it is clear that N is a norm on $(\mathbb{Z}, +)$ such that

$$N(uv) \le N(u) + N(v) \qquad \text{for all } (u, v) \in \mathbb{Z}^2.$$

Using the fact that the series is uniformly convergent, we see that $N(a_n) \to 0$. □

Remarks 3.

1- The proof of theorem 11 shows that the result can also be established in an arbitrary abelian ring.

2- Arguing exactly as in the proof of Theorem 4, we obtain according to Theorem 11 below that the sequence considered in Theorem 4 is in fact ring nullpotent in $\mathbb{Z}$.

3- Let $a_0 = 1, a_{n+1} = a_n^2 + 1$ $(n \geq 1)$. The sequence (a_n) is not ring nullpotent in $\mathbb{Z}$ even if $\lim \frac{a_{n+1}}{a_n} = +\infty$.

The following result gives a criterion for a sequence to be ring nullpotent in $\mathbb{Z}$.

Theorem 12. *Let (a_n) be a strictly increasing sequence of positive integers such that $\lim_n \dfrac{\log a_{n+1}}{\log a_n} = +\infty$. Then (a_n) is ring nullpotent in $\mathbb{Z}$.*

Proof. Let us argue by contradiction and suppose that there exist c, k, L positive integers and $A_1, \ldots, A_k$ nonzero integers such that c has the primitive representation

$$c = \sum_{i=1}^{k} A_i a_{J_{\varphi_i(n)}} \qquad (n = 1, 2, \ldots) \tag{52}$$

where $J_{\varphi_i(n)} \in \cup_{1 \leq \ell \leq L} \mathbb{J}_\ell$, with the condition: $J_{\varphi_i(n)} \neq J_{\varphi_j(n)}$ when $i \neq j$ and for every $i \in \{1, \ldots, k\}$, $\varphi_i(n) \to +\infty$ $(n \to \infty)$.

For $n \geq 1$, put

$$\begin{aligned} j_n &:= \max_{1 \leq i \leq k} J^*_{\varphi_i(n)} \qquad \big(\text{see notation (43)}\big) \\ \nu_n &:= \max \big\{ \nu \in \mathbb{N} : \exists i \in \{1, \ldots, k\}, a_{j_n}^{\nu} | a_{J_{\varphi_i(n)}} \big\}. \end{aligned}$$

Since the positive integers ν_n are not exceeding L, so by considering a subsequence of (ν_n), we may assume that for all $n \in \mathbb{N}, \nu_n = \nu$.

Then we deduce from (52) that

$$c \geq a_{j_n}^{\nu} \big(1 - \sum_{i \in I} |A_i| \frac{1}{a_{j_n}^{\nu}} a_{J_{\varphi_i(n)}} \big) \tag{53}$$

with $I \subset \{1, \ldots, k\}$ and $a_{j_n}^{\nu} \nmid \prod_{j \in J_{\varphi_i(n)}} a_j$ for all $i \in I$.

But $\lim_n a_{j_n} = +\infty$, and for all $i \in I$

$$\frac{1}{a_{j_n}^{\nu}} a_{J_{\varphi_i(n)}} \leq \frac{a_{j_n - 1}^{L}}{a_{j_n}}$$

which tends to zero with n because by the hypothesis we have $\lim_n \frac{\log a_{n+1}}{\log a_n} = +\infty$. This contradicts (53). □

Example. Let $p_1 = 2 < p_2 = 3 < p_3 = 5 < \ldots$ be the increasing sequence of the prime numbers. Let $\varphi(n) = n^{n!}$ for $n \in \mathbb{N}$. Then the sequence $(p_{\varphi(n)})$ is ring nullpotent in $\mathbb{Z}$.

3. Conclusion

Returning to our initial motivation mentioned in the introduction of this work, we realize that this field of study is in need of even further investigation. The clarification brought from such investigations would greatly aid in the acquisition of useful methods and tools devised to enrich the study of the integers. We also think that it would be interesting to present new proofs of the results obtained in this work by using only elementary methods of number theory.

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