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NOTE ON THE RECIPROCAL OF POWER SERIES ASSOCIATED WITH INCOMPLETE DEGENERATE LAH-BELL POLYNOMIALS

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ABSTRACT. In this paper, we aim to find the reciprocal formal power series associated with the incomplete degenerate Lah-Bell polynomials. For our purposes, we first consider a new type of both degenerate incomplete and complete Lah-Bell polynomials as a multivariate version of a new type of degenerate Lah-Bell polynomials, and derive some identities for them. Second, we derive the reciprocal formal power series associated with our incomplete degenerate Lah-Bell polynomials and give new polynomials associated with these incomplete degenerate Lah-Bell polynomials including an expression for the reciprocal of the degenerate exponential function.

1. INTRODUCTION

Existence problems such as reciprocal numbers, reciprocal (or inverse) matrices, reciprocal (or inverse) functions, etc. are important in mathematics [3, 5, 8, 14, 15]. Shingh showed that the coefficients of reciprocal power series were expressed the finite sum of partial ordinary (or incomplete) Bell polynomials by using Faa di Brunos's formula [14]. In addition, Kim at el. introduced some new polynomials associated with the incomplete degenerate Bell polynomials [7].

In this paper, we divide two parts to study the reciprocal formal power series associated with the incomplete degenerate Lah-Bell polynomials. In the first part, we introduce new types of both degenerate complete and incomplete Lah-Bell polynomials, and derive some relations for them. In the second part, we derive the reciprocal formal power series associated with this new type of incomplete degenerate Lah-Bell polynomials and give new polynomials associated with this new type of incomplete degenerate Lah-Bell polynomials. In particular, we deduce the new polynomials $Z_n^{\lambda}(x_1,x_2,\cdots,x_n)$ associated with the degenerate incomplete Lah-Bell polynomials. As a corollary, we obtain an expression for the reciprocal of the degenerate exponential function $e_{\lambda}(\frac{1}{1-at})$.

As is well known, for any $\lambda \in \mathbb{R}$,

$$e_{\lambda}^{x}(t) = (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}, \quad (\text{see } [2,9-12]),$$

where $(x)_{0,\lambda} = 1$ and $(x)_{n,\lambda} = x(x-\lambda)\cdots\big(x-(n-1)\lambda\big)$, $(n \geq 1)$.

When $\lambda = 1$, $(x)_0 = 1$ and $(x)_n = x(x-1)\cdots\big(x-(n-1)\big)$, $(n \geq 1)$.

T. Kim considered the complete degenerate Bell polynomials given by

$$\exp\left(\sum_{i=1}^{\infty}(1)_{i,\lambda}x_i\frac{t^i}{i!}\right) = \sum_{n=0}^{\infty}B_n^{(\lambda)}(x_1,x_2,\cdots,x_n)\frac{t^n}{n!}, \quad (\text{see } [11]). \tag{1}$$

The incomplete degenerate Bell polynomials are given by

$$\frac{1}{k!}\left(\sum_{i=1}^{\infty}(1)_{i,\lambda}x_i\frac{t^i}{i!}\right)^k = \sum_{n=k}^{n}B_{n,k}^{(\lambda)}(x_1,x_2,\cdots,x_{n-k+1})\frac{t^n}{n!}, \quad (\text{see } [11]), \tag{2}$$

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and $B_{0,0}^{(\lambda)}(x_1,x_2,\cdots,x_{n+1})=1,\ b_{n,0}^{(\lambda)}(x_1,x_2,\cdots,x_{n+1})=0,(n\in\mathbb{N})$.

In particular, $B_{n,k}^{(\lambda)}(1,1,\cdots,1)=Bel_{n,\lambda}$, $(n\geq 0)$,
where $\sum_{n=0}^{\infty}Bel_{n,\lambda}\frac{t^n}{n!}=exp(e_\lambda(t)-1)$, (see [12]).

Let the unsigned Lah-number $L(n,k)$ counts the number of partitions of a set with $1,2,\cdots,n$ elements into k ordered blocks with no box left empty.

It is well known that Lah numbers and the generating function of Lah-numbers respectively are given by

$$L(n,k)=\frac{n!}{k!}\binom{n-1}{k-1},\quad \text{and}\quad \frac{1}{k!}\left(\frac{t}{1-t}\right)^k=\sum_{n=k}^{\infty}L(n,k)\frac{t^n}{n!},\qquad \text{(see [3, 6, 9, 13])}.$$

The Lah-Bell polynomials is given by

$$Lb_n=\sum_{k=0}^{\infty}L(n,k)x^k,\quad (n\geq 0),\qquad \text{(see [6])},$$

and the generating function of Lah-Bell polynomials is given by

$$e^{x(\frac{t}{1-t})}=\sum_{n=0}^{\infty}Lb_n(x)\frac{t^n}{n!},\qquad \text{(see [6, 9, 13])}. \tag{3}$$

When x=1, Lb_n are called the Lah-Bell numbers.

Kim-Kim introduced the degenerate Lah-Bell polynomials given by the generation function

$$e_\lambda^x\left(\frac{t}{1-t}\right)=\sum_{n=0}^{\infty}B_{n,\lambda}^L(x)\frac{t^n}{n!},\qquad \text{(see [6])}.$$

On the other hand, Kim-Lee introduced another degenerate Lah-Bell polynomials given by the generation function

$$e_\lambda\left(x\frac{t}{1-t}\right)=\sum_{n=0}^{\infty}b_{n,\lambda}^L(x)\frac{t^n}{n!},\qquad \text{(see [9])}.$$

2. Some identities of a new type of complete and incomplete degenerate Lah-Bell polynomials

In this section, we introduce new types of both degenerate complete and incomplete Lah-Bell polynomials, and derive some relations for them.

We define a new type of degenerate Lah-number $L_\lambda(n,k)$ by

$$\frac{1}{k!}\left(\sum_{i=1}^{\infty}(1)_{i,\lambda}t^i\right)^k=\sum_{n=0}^{\infty}L_\lambda(n,k)\frac{t^n}{n!}. \tag{4}$$

When $\lambda\to 0, L_\lambda(n,k)=L(n,k)$.

In view of the Lah-Bell polynomials, we give the degenerate Lah-Bell polynomials $Lb_{n,\lambda}(x)$ by

$$Lb_{n,\lambda}(x)=\sum_{k=0}^{n}L_\lambda(n,k)x^k. \tag{5}$$

From (4) and (5), we get the generating function of the degenerate Lah-Bell polynomials $Lb_{n,\lambda}(x)$ as follows.

$$\sum_{n=0}^{\infty}Lb_{n,\lambda}(x)\frac{t^n}{n!}=\exp\left(x\left(\sum_{i=1}^{\infty}(1)_{i,\lambda}t^i\right)\right). \tag{6}$$

When $\lambda \to 0, Lb_{n,\lambda}(x) = Lb_n(x)$.
When $x = 1$, $Lb_{n,\lambda} = Lb_{n,\lambda}(1)$ are called the degenerate Lah-Bell numbers.
As a multivariate version of a new type of degenerate Lah numbers,
we introduce the incomplete degenerate Lah-Bell polynomials given by

$$\frac{1}{k!}\left(\sum_{m=1}^{\infty}(1)_{m,\lambda}x_m t^m\right)^k = \sum_{n=k}^{\infty} LV_{n,k}^{\lambda}(x_1, x_2, \cdots, x_{n-k+1})\frac{t^n}{n!}, \tag{7}$$

where $LV_{0,0}^{\lambda}(x_1, x_2, \cdots x_{n+1}) = 1$ and $LV_{n,0}^{\lambda}(x_1, x_2, \cdots x_{n+1}) = 0, (n \in \mathbb{N})$.
As another multivariate version of a new type of degenerate Lah-Bell polynomials,
we naturally introduce the complete degenerate Lah-Bell polynomials $LV_n^{\lambda}(x_1, \cdots, x_n)$ given by

$$\exp\left(\sum_{m=1}^{\infty}(1)_{m,\lambda}x_m t^m\right) = \sum_{n=0}^{\infty} LV_n^{\lambda}(x_1, \cdots, x_n)\frac{t^n}{n!}. \tag{8}$$

Theorem 1. *For $n \geq 0$, we have*

$$\begin{aligned} &LV_{n,k}^{\lambda}(1, 1, \cdots, 1) = L_{\lambda}(n, k) \\ &\quad \text{and} \quad LV_n^{\lambda}(x, \cdots, x) = Lb_{n,\lambda}(x). \end{aligned} \tag{9}$$

In addition, when $x = 1$, $LV_n^{\lambda}(1, 1, \cdots, 1) = Lb_n^{\lambda}$.

Proof. By (4), we observe that

$$\sum_{n=0}^{\infty} LV_n^{\lambda}(x, \cdots, x)\frac{t^n}{n!} = \exp\left(x\sum_{m=1}^{\infty}(1)_{m,\lambda}t^m\right) = \sum_{n=0}^{\infty} Lb_{n,\lambda}(x)\frac{t^n}{n!}. \tag{10}$$

Therefore, from (10), we have

$$LV_n^{\lambda}(x, \cdots, x) = Lb_{n,\lambda}(x).$$

By (7), we observe that

$$\sum_{n=k}^{\infty} LV_{n,k}^{\lambda}(1, 1, \cdots, 1)\frac{t^n}{n!} = \frac{1}{k!}\left(\sum_{m=1}^{\infty}(1)_{m,\lambda}t^m\right)^k = \sum_{n=k}^{\infty} L_{\lambda}(n, k)\frac{t^n}{n!}. \tag{11}$$

Therefore, from (11), we have

$$LV_{n,k}^{\lambda}(1, 1, \cdots, 1) = L_{\lambda}(n, k).$$

□

Theorem 2. *For $n \geq 0$, we have*

$$LV_n^{\lambda}(x_1, x_2, \cdots, x_n) = \sum_{k=0}^{n} LV_{n,k}^{\lambda}(x_1, x_2, \cdots, x_{n-k+1}) \quad (n \geq 0). \tag{12}$$

Proof. We observe that

$$
\begin{aligned}
\sum_{n=0}^{\infty} LV_n^{\lambda}(x_1,x_2,\cdots,x_n)\frac{t^n}{n!} &= \exp\left(\sum_{m=1}^{\infty}(1)_{m,\lambda}x_m t^m\right)\\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{m=1}^{\infty}(1)_{m,\lambda}x_m t^m\right)^k\\
&= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} LV_{n,k}^{\lambda}(x_1,x_2,\cdots,x_{n-k+1})\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} LV_{n,k}^{\lambda}(x_1,x_2,\cdots,x_{n-k+1})\right)\frac{t^n}{n!}.
\end{aligned}
\tag{13}
$$

Therefore, from (13), we have the result. □

Theorem 3. *For $n \geq k \geq 1$, we have*

$$
LV_{n,k}^{\lambda}(x_1,x_2,\cdots,x_{n-k+1}) = \sum_{l=0}^{n-k} h(n-1)_h(1)_{h+1,\lambda}x_{h+1}Lb_{n-h-1,k-1}^{\lambda}(x_1,\cdots,x_{n-h-k+1}).
$$

Proof. For $n,k\in\mathbb{Z}$ with $n\geq k\geq 1$, differentiating on both sides of (7) with respect to t,

$$
\begin{aligned}
&\sum_{n=k}^{\infty} LV_{n,k}^{\lambda}(x_1,x_2,\cdots,x_{n-k+1})\frac{t^{n-1}}{(n-1)!}\\
&= \frac{1}{(k-1)!}\left(\sum_{m=1}^{\infty}(1)_{m,\lambda}x_m t^m\right)^{k-1}\left(\sum_{h=1}^{\infty}(1)_{h,\lambda}x_h\, h!\, t^{h-1}\right)\\
&= \sum_{m=k-1}^{\infty} LV_{m,k-1}^{\lambda}(x_1,x_2,\cdots,x_{m-k+2})\frac{t^m}{m!}\sum_{h=1}^{\infty}(1)_{h,\lambda}x_h\, h!\,\frac{t^{h-1}}{(h-1)!}\\
&= \sum_{n=k}^{\infty}\sum_{h=1}^{n-k+1}\binom{n-1}{h-1}(1)_{h,\lambda}x_h\, h!\, LV_{n-h,k-1}^{\lambda}(x_1,\cdots,x_{n-h-k+2})\frac{t^{n-1}}{(n-1)!}\\
&= \sum_{n=k}^{\infty}\sum_{h=0}^{n-k} h(n-1)_h(1)_{h+1,\lambda}x_{h+1}LV_{n-h-1,k-1}^{\lambda}(x_1,\cdots,x_{n-h-k+1})\frac{t^{n-1}}{(n-1)!}.
\end{aligned}
\tag{14}
$$

By comparing with coefficients of (14), we have the desired result. □

Theorem 4. *For $n \geq 0$, we have*

$$
LV_{n+1}^{\lambda}(x_1,x_2,\cdots,x_{n+1}) = \sum_{m=0}^{n}(n)_m(m+1)(1)_{m+1,\lambda}x_{m+1}LV_{n-m}^{\lambda}(x_1,x_2,\cdots,x_{n-m}).
$$

Proof. Differentiating on both sides of (8) with respect to t, the left side of (8) is

$$
\begin{aligned}
&\sum_{m=1}^{\infty}(1)_{m+1,\lambda}x_m m t^{m-1}\exp\left(\sum_{m=1}^{\infty}(1)_{m,\lambda}x_m t^m\right)\\
&= \sum_{m=0}^{\infty}(1)_{m+1,\lambda}x_{m+1}(m+1)!\frac{t^m}{m!}\sum_{l=0}^{\infty}LV_l^{\lambda}(x_1,x_2,\cdots,x_l)\frac{t^l}{l!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}(n)_m(m+1)(1)_{m+1,\lambda}x_{m+1}LV_{n-m}^{\lambda}(x_1,x_2,\cdots,x_{n-m})\right)\frac{t^n}{n!}.
\end{aligned}
\tag{15}
$$

On the other hand, the right side of (8) is

$$\sum_{n=1}^{\infty} LV_n^{\lambda}(x_1,x_2,\cdots,x_n)\frac{t^{n-1}}{(n-1)!}=\sum_{n=0}^{\infty} LV_{n+1}^{\lambda}(x_1,x_2,\cdots,x_{n+1})\frac{t^n}{n!}. \tag{16}$$

By comparing with coefficients of (15) and (16), we have what we want. □

From (1), (2), (8) and (7), we have the following explicit formulas.

Theorem 5. *For $n\geq 0$, we have*

$$\begin{aligned} LV_n^{\lambda}(x_1,x_2,\cdots,x_n) &= B_n^{(\lambda)}(1!x_1,2!x_2,\cdots,n!x_n) \\ &= \sum_{l_1+2l_2+\cdots+nl_n=n}\frac{n!}{l_1!l_2!\cdots l_n!}\Big((1)_{1,\lambda}x_1\Big)^{l_1}\Big((1)_{2,\lambda}x_2\Big)^{l_2}\cdots\Big((1)_{n,\lambda}x_n\Big)^{l_n}, \end{aligned} \tag{17}$$

and

$$\begin{aligned} LV_{n,k}^{\lambda}(x_1,x_2,\cdots,x_{n-k+1}) &= B_{n,k}^{(\lambda)}(1!x_1,2!x_2,\cdots,(n-k+1)!x_{n-k+1}) \\ &= \sum_{\substack{l_1+l_2+\cdots+l_{n-k+1}=k \\ l_1+2l_2+\cdots+(n-k+1)l_{n-k+1}=n}}\frac{n!}{l_1!l_2!\cdots l_{n-k+1}!} \\ &\quad\times\Big((1)_{1,\lambda}x_1\Big)^{l_1}\Big((1)_{2,\lambda}x_2\Big)^{l_2}\cdots\Big((1)_{n-k+1,\lambda}x_{n-k+1}\Big)^{l_{n-k+1}}, \end{aligned} \tag{18}$$

and $LV_{0,0}^{\lambda}(x_1,x_2,\cdots x_{n+1})=1,\quad LV_{n,0}^{\lambda}(x_1,x_2,\cdots x_{n+1})=0.$

3. The reciprocal of power series associated with incomplete degenerate Lah-Bell polynomials

In this section, we derive the reciprocal formal power series associated with this new type of incomplete degenerate Lah-Bell polynomials and give new polynomials associated with this new type of incomplete degenerate Lah-Bell polynomials.

Theorem 6. *For $n,k\in\mathbb{Z}$ with $n\geq k\geq 1$, we have*

$$\begin{aligned} &LV_{n,k}^{\lambda}(x_1,x_2,\cdots,x_{n-k+1}) \\ &\quad=\frac{1}{k}\sum_{h=0}^{n-k}\binom{n}{h+1}(h+1)!(1)_{h+1,\lambda}x_{h+1}LV_{n-h-1,k-1}^{\lambda}(x_1,\cdots,x_{n-h-k+1}). \end{aligned}$$

Proof. From (7), we obtain

$$\begin{aligned} &\sum_{n=k}^{\infty} LV_{n,k}^{\lambda}(x_1,x_2,\cdots,x_{n-k+1})\frac{t^n}{n!} \\ &\quad=\frac{1}{k}\frac{1}{(k-1)!}\Big(\sum_{m=1}^{\infty}(1)_{m,\lambda}x_mt^m\Big)^{k-1}\Big(\sum_{m=1}^{\infty}(1)_{m,\lambda}x_mt^m\Big) \\ &\quad=\frac{1}{k}\sum_{h=k-1}^{\infty} LV_{h,k-1}^{\lambda}(x_1,x_2,\cdots,x_{h-k+2})\frac{t^h}{h!}\sum_{m=1}^{\infty}(1)_{m,\lambda}x_m m!\frac{t^m}{m!} \\ &\quad=\frac{1}{k}\sum_{n=k}^{\infty}\sum_{h=k-1}^{n-1}\binom{n}{h}(n-h)!(1)_{n-h,\lambda}x_{n-h}LV_{h,k-1}^{\lambda}(x_1,\cdots,x_{h-k+2})\frac{t^n}{n!}. \end{aligned} \tag{19}$$

By comparing coefficients both side of (19),

$$LV_{n,k}^{\lambda}(x_1,x_2,\cdots,x_{n-k+1}) = \frac{1}{k}\sum_{h=k-1}^{n-1}\binom{n}{h}(n-h)!(1)_{n-h,\lambda}x_{n-h}LV_{h,k-1}^{\lambda}(x_1,\cdots,x_{h-k+2}) \tag{20}$$

Since $\binom{n}{h+1} = \binom{n}{n-h-1}$, by replacing h by $n-h-1$ in (20), we get

$$Lb_{n,k}^{\lambda}(x_1,x_2,\cdots,x_{n-k+1}) = \frac{1}{k}\sum_{h=0}^{n-k}\binom{n}{h+1}(h+1)!(1)_{h+1,\lambda}x_{h+1}Lb_{n-h-1,k-1}^{\lambda}(x_1,\cdots,x_{n-h-k+1}).$$

□

Theorem 7. *Assume* $\sum_{l=0}^{\infty}(1)_{l,\lambda}a_l t^l \sum_{m=0}^{\infty}\frac{(1)_{m,\lambda}}{m!}b_m t^m = 1$ *with* $a_0 = 1$. *Them we have*

$$(1)_{0,\lambda}b_0 = 1, \quad (1)_{n,\lambda}b_n = \sum_{k=1}^{n}(-1)^k k!\ LV_{n,k}^{\lambda}(a_1,\cdots,a_{n-k+1}), \quad (n \geq 1). \tag{21}$$

Proof. We observe that

$$\sum_{h=0}^{\infty}(1)_{h,\lambda}a_h t^h \sum_{m=0}^{\infty}\frac{(1)_{m,\lambda}}{m!}b_m t^m = \sum_{n=0}^{\infty}\sum_{h=0}^{n}\binom{n}{h}h!\ (1)_{h,\lambda}a_h(1)_{n-h,\lambda}b_{n-h}\frac{t^n}{n!} = 1. \tag{22}$$

From (22), we have

$$(1)_{0,\lambda}b_0 = 1, \quad \sum_{h=0}^{n}\binom{n}{h}h!(1)_{h,\lambda}a_h(1)_{n-h,\lambda}b_{n-h} = 0, \quad (n \geq 1). \tag{23}$$

From (23), we get

$$(1)_{n,\lambda}b_n = -\sum_{h=1}^{n}\binom{n}{h}h!(1)_{h,\lambda}a_h(1)_{n-h,\lambda}b_{n-h}. \tag{24}$$

By using induction on $n \geq 1$ to proof (21), when $n = 1$,

$$(1)_{1,\lambda}b_1 = -(1)_{1,\lambda}a_1 = \sum_{k=1}^{1}LV_{1,1}^{\lambda}(a_1)(-1)^k k!.$$

Assume $n \geq 1$ and (21) hold for all positive integers smaller than n.

Combining by (21) and Theorem 3.1, we obtain

$$\begin{aligned}
(1)_{n,\lambda}b_n &= -\sum_{h=1}^{n}\binom{n}{h}h!(1)_{h,\lambda}a_h(1)_{n-h,\lambda}b_{n-h} \\
&= -n!(1)_{n,\lambda}a_n - \sum_{h=1}^{n-1}\binom{n}{h}h!(1)_{h,\lambda}a_h\sum_{k=1}^{n-h}LV_{n-h,k}^{\lambda}(a_1,\cdots,a_{n-h-k+1})(-1)^k k! \\
&= -n!(1)_{n,\lambda}a_n - \sum_{k=1}^{n-h}(-1)^k k!\sum_{h=1}^{n-k}\binom{n}{h}h!(1)_{h,\lambda}a_h LV_{n-h,k}^{\lambda}(a_1,\cdots,a_{n-h-k+1}) \\
&= -n!(1)_{n,\lambda}a_n - \sum_{k=2}^{n}(-1)^{k-1}(k-1)!\sum_{h=1}^{n-k+1}\binom{n}{h}h!(1)_{h,\lambda}a_h LV_{n-h,k-1}^{\lambda}(a_1,\cdots,a_{n-h-k+2}) \\
&= -n!(1)_{n,\lambda}a_n - \sum_{k=2}^{n}(-1)^{k-1}(k-1)!\sum_{h=0}^{n-k}\binom{n}{h+1}(h+1)! \\
&\qquad\qquad \times (1)_{h+1,\lambda}a_{h+1}LV_{n-h-1,k-1}^{\lambda}(a_1,\cdots,a_{n-h-k+1}) \\
&= -\sum_{k=1}^{n}(-1)^{k-1}(k-1)!\sum_{h=0}^{n-k}\binom{n}{h+1}(h+1)!(1)_{h+1,\lambda}a_{h+1}LV_{n-h-1,k-1}^{\lambda}(a_1,\cdots,a_{n-h-k+1}) \\
&= \sum_{k=1}^{n}(-1)^k k!\frac{1}{k}\sum_{h=0}^{n-k}\binom{n}{h+1}(h+1)!(1)_{h+1,\lambda}a_{h+1}LV_{n-h-1,k-1}^{\lambda}(a_1,\cdots,a_{n-h-k+1}) \\
&= \sum_{k=1}^{n}(-1)^k k! LV_{n-h-1,k-1}^{\lambda}(a_1,\cdots,a_{n-h-k+1}).
\end{aligned}$$

Thus, we arrive at the desired result.

□

From Theorem 3.2, we have a new type associated with incomplete degenerate Lah-bell polynomials $Z_n^{\lambda}(a_1,a_2,\cdots,a_n)$ given by

$$Z_n^{\lambda}(a_1,a_2,\cdots,a_n) = \sum_{k=1}^{n}(-1)^k k! LV_{n,k}^{\lambda}(a_1,a_2,\cdots,a_{n-k+1}), \quad Z_0^{\lambda}(a_1,a_2,\cdots,a_n)=1. \tag{25}$$

In other words,

$$\sum_{n=0}^{\infty}Z_n^{\lambda}(a_1,a_2,\cdots,a_n)\frac{t^n}{n!} = \frac{1}{1+a_1(1)_{1,\lambda}t+a_2(1)_{2,\lambda}t^2\cdots}.$$

We note that

$$\sum_{n=0}^{\infty}Z_n^{\lambda}(a_1,a_2,\cdots,a_n)\frac{t^n}{n!} = 1+\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(-1)^k k! Lb_{n,k}^{\lambda}(a_1,\cdots,a_{n-k+1})\right)\frac{t^n}{n!}.$$

Corollary 8. *For $n,\ k\in\mathbb{Z}$ with $n\geq k\geq 1$, we have*

$$\sum_{n=0}^{\infty}Z_n^{\lambda}(a,a,\cdots,a)\frac{t^n}{n!} = \frac{1}{e_{\lambda}(\frac{1}{1-at})} = e_{\lambda}^{-1}\left(\frac{1}{1-at}\right).$$

In particular,

$$\sum_{n=0}^{\infty}Z_n^{\lambda}(1,1,\cdots,1)\frac{t^n}{n!} = \sum_{i=0}^{n}\sum_{m=0}^{i}\frac{1}{(m-i)!}(-1)_{m,\lambda}L(n,i).$$

Proof. From Theorem 3.2, we observe that

$$\begin{aligned}
\sum_{n=0}^{\infty} Z_n^{\lambda}(1,1,\cdots,1)\frac{t^n}{n!} &= \frac{1}{e_\lambda(\frac{1}{1-at})} = e_\lambda^{-1}\left(\frac{1}{1-t}\right) \\
&= \sum_{m=0}^{\infty}(-1)_{m,\lambda}\frac{1}{m!}\sum_{i=0}^{m}\binom{m}{i}\left(\frac{t}{1-t}\right)^i \\
&= \sum_{m=0}^{\infty}(-1)_{m,\lambda}\frac{1}{m!}\sum_{i=0}^{m}\binom{m}{i}i!\sum_{n=i}^{\infty}L(n,i)\frac{t^n}{n!} \\
&= \sum_{m=0}^{\infty}\sum_{i=0}^{m}\sum_{n=i}^{\infty}\frac{1}{(m-i)!}(-1)_{m,\lambda}L(n,i)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{m=0}^{i}\frac{1}{(m-i)!}(-1)_{m,\lambda}L(n,i)\frac{t^n}{n!}.
\end{aligned} \tag{26}$$

Therefore, we get the desired result. □

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