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ON THE MEAN FINSLER p-LAPLACIAN

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ABSTRACT. We introduce the mean p-Laplacian ($p > 1$) on a Finsler manifold as average of the p-Laplacian on the associated projective sphere bundle, and we give some general estimates of the first eigenvalue of this Finslerian mean p-Laplacian in term of the conformal class of the Finsler metric. When the Finsler manifold is a Randers manifold or a Berwald space, several interesting estimate results have been obtained.

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1. INTRODUCTION

The p-Laplace operator ($p > 1$), a generalization of the Laplace-Beltrami operator, is an important tool in differential geometry and physics. It has many applications such as modelisation of pseudoplastics ($1 < 2 < p$) and study of non-Newtonian fluids ($p > 2$). Particularly, the study of the first (nonzero) eigenvalue of this operator is an old but active researsh field. In Riemannian geometry, several estimates of the first eigenvalue under geometric assumptions are well known [8, 7, 11].

However, in Finsler geometry, there is no canonical way to introduce the Laplacian. Hence, several authors proposed different extensions of the standard Riemannian Laplacian to the Finsler setting like Bao and Lackey [1], Barthelmé [2] and Shen [10]. It is well known that Shen's Finsler Laplacian is a non-linear operator unlike others. By using the non-linear Laplacian, Q. He and S-T. Yin introduced the p-Laplacian on Finsler manifolds and obtained some estimates on the first eigenvalue and also a regularity's result on its associated eigenfunctions [4, 5]. In some situations, such as for example the study of the dynamic of the manifold, it is more convenient to use linear Finsler Laplacian defined by Barthelmé in [2]. So, it would be interesting to study the p-Laplacian associated to this operator.

The purpose of this paper is to give an extension of the p-Laplace operator to Finsler manifolds based on the definition of the dynamical Finsler Laplacian [2]. Recently, He et al. [6] obtained several characterizations of the dynamical Finsler Laplace operator and proved that it coincide with the average of the Laplacian on the associated projective sphere bundle. Thus, we have a natural extension of this operator. Let (M, F) be a Finsler manifold and HM be the projective shpere bundle endowed with the Sasakian metric G. For $u \in C^2(M)$, the mean p-Laplacian ($p > 1$) of u is defined as

$$\Delta_p^F u(x) = \int_{H_xM} \Delta_p^G \bar{u} \ \beta_x^F,$$

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where Δ_p^G and $\bar{u}$ stand for the p-Laplacian on (HM, G) and the lift of u, respectively. H_xM is the fiber over $x \in M$ while β_x^F is the nornalized volume form on H_xM. Refer to sections 2 and 3 for related notations and precise definitions. We will focus, in this paper, on the study of the first eigenvalue of this operator.

In section 4, through the homogeneous bundle , we define a like conformal volume for Finsler manifolds and obtain an upper bound the the first eigenvalue in term of this quantity. In the last two sections, we are interested to particular class of Finsler manifolds. Firstly, the class of Randers manifolds whose metrics are on the form $F = \sqrt{g} + \beta$ where g is a Riemannian metric and β a 1-form which norm with respect to g is smaller than one. We give relation between the first eigenvalue of the mean p-Laplacian of F and g (see section 5). Thereafter, in section 6, we consider Berwald manifolds and we prove among others results the following

Theorem 1.1. *Let (M, F) be a closed n-dimensinnal Berwald manifold. For any $p \geq 2$, there exists a constant C_p depending only on F and p such that if the Ricci curvature satifies $Ric \geq (n-1)C_pK$ for some constant $K > 0$ then, it holds*

$$\lambda_{1,p}^F \geq \left(\frac{(n-1)K}{p-1} \right)^{p/2},$$

where $\lambda_{1,p}^F$ denotes the first eigenvalue of the mean p-Laplacian on (M, F).

This theorem can be regarded as generalization of [6, Theorem 1.4] and concides with [8, Theorem 3.2] whenever F is Riemannian. The necesary notions such as Ricci curvature of Finsler manifolds are settled down in sections 2 and 3.

2. On the geometry of the projective sphere bundle

Let M be an n-dimensional smooth manifold. TM and T^*M denote respectively the tangent and the cotangent bundles. A point $(x, y) \in TM$ is such that $y = x^i \frac{\partial}{\partial x^i}$ in the local coordinates (x^i, y^i) on TM. Define the projective sphere bundle as $HM := (TM\backslash\{0\})/*$, where $(x, y) * (x', y')$ if and only if $x = x'$ and $y = \lambda y'$ for some $\lambda > 0$.

Consider on M a Finsler metric, that is a function $F : TM \to \mathbb{R}^+$ satisfying the following :

(i) F is smooth in $TM\backslash\{0\}$;
(ii) for any $(x, y) \in TM$ and every $\lambda > 0$, $F(x, \lambda y) = \lambda F(x, y)$;
(iii) the fundamental quadratic form

$$g := g_{ij}(x, y) dx^i \otimes dx^j, \qquad g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j},$$

is positively definite.

The canonical projection $\pi : HM \to M$ gives rise to a vector and covector bundle π^*TM and π^*T^*M. The Finsler metric F induces two important global sections on $\pi^*T^*M \subset T^*HM$: the Hilbert form $\omega = F_{y^i} dx^i$ and the Cartan form $I = A_{ijk} g^{jk} dx^i$, where (g^{jk}) and A_{ijk} denote respectively the inverse of the matrix (g_{jk}) and the coefficients of the Cartan tensor

$$A = A_{ijk} dx^i \otimes dx^j \otimes dx^k, \quad A_{ijk} = \frac{F}{2} \frac{\partial g_{ij}}{\partial y^k}.$$

Let

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}, \qquad \delta y^i = \frac{1}{F}(dy^i + N_j^i dx^j),$$

where $N_j^i = \gamma_{jk}^i y^k - A_{jk}^i \gamma_{rs}^k y^r y^s$ and $\gamma_{jk}^i = \frac{1}{2} g^{il}\Big(\frac{\partial g_{lk}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\Big)$.

There exists a unique linear connection on π^*TM called the Chern connection whose connection forms are caracterized by

(i) Torsion freeness : $dx^j \wedge \omega_j^i = 0$;

(ii) Almost g-compatibility : $dg_{ij} - g_{ik}\omega_j^k - g_{jk}\omega_i^k = 2A_{ijk}\delta y^k$.

Recall that $\omega_j^i = \Gamma_{jk}^i dx^k$ and $\Gamma_{jk}^i = \Gamma_{kj}^i$. The curvature 2-forms of this connection are given by

$$\Omega_i^j = d\omega_i^j - \omega_i^k \wedge \omega_k^j = \frac{1}{2} R_{ikl}^j dx^k \wedge dx^l + P_{ikl}^j dx^k \wedge \delta y^l,$$

where $R_{ikl}^j = -R_{ilk}^j$ and $P_{ikl}^j = P_{kil}^j$ are the hh-curvature and the hv-curvature respectively.

The flag curvature tensor is given by $R_k^j = g^{il} R_{ikl}^j$, while the Ricci curvature is defined as the trace $Ric(x, y) = \frac{1}{F^2} R_j^j$.

The pull-back of the Sasaki metric from $TM\backslash\{0\}$ to HM is the Riemannian metric defined by

$$G := g_{ij} dx^i \otimes dx^j + \delta_{\alpha\beta}\omega_n^\alpha \otimes \omega_n^\beta = \delta_{ab}\omega^a \otimes \omega^b,$$

for which the collection $\{\omega^a\}$ is an orthonormal coframe on HM. Hence the volume element $d\mu_G$ of HM is given by

$$d\mu_G = \omega^1 \wedge \cdots \wedge \omega^n \wedge \omega^{n+1} \wedge \cdots \wedge \omega^{2n-1} = \Omega dx \wedge d\eta,$$

where $\Omega := det\Big(\frac{g_{ij}}{F}\Big)$, $dx = dx^1 \wedge \cdots \wedge dx^n$ and

$$d\eta := \sum_{i=1}^{n} (-1)^{i-1} y^i dy^1 \wedge \dots \widehat{dy^i} \wedge \cdots \wedge dy^n.$$

One can also define a volume form on M as

$$d\mu_F = \sigma(x)dx, \qquad \sigma(x) = \frac{1}{c_{n-1}} \int_{H_xM} \Omega d\eta,$$

where c_{n-1} denotes the volume of the unit Euclidean $(n-1)$-dimensinal sphere.

We close this section by recalling the folloving key lemma

Lemma 2.1. [3] *For any function f on the sphere bundle HM, we have*

$$\int_{H_xM} g^{ij}[F^2 f]_{y^i y^j} \Omega d\eta = 2n \int_{H_xM} f \ \Omega d\eta.$$

3. On the averaged p-Laplace operator

Let (M, F) be a Finsler manifold and u be a smooth function on M. The mean Laplacian of u is defined in [6] by

$$\Delta^F u = \int_{H_xM} \Delta^G(\pi^* u)\varpi d\eta,$$

where $\Delta^G(\pi^* u)$ is the Laplacian of the lift $\pi^* u$ of u on the Riemannian manifold (HM, G) and we set $\varpi := \Omega/Q$, $Q = \int_{H_x M} \Omega d\eta$. Following this idea, one can also define the p-Laplacian on M as the average on fibers of the p-Laplacian on the projective sphere bundle (HM, G).

Definition 3.1. *Let (M, F) be a Finsler manifold and let $p > 1$. We define the mean p-Laplacian $\Delta_p^F u$ of u as*

$$\Delta_p^F u(x) := \int_{H_x M} \Delta_p^G(\pi^* u)\ \varpi d\eta.$$

p-Harmonic functions are naturally defined as functions u satisfying $\Delta_p^F u = 0$.

Remark 3.2. (i) *The averaged p-Laplacian Δ_p^F coincides with the standard p-Laplacian when the metric F is Riemannian.*

(ii) *For $p = 2$, it was shown in* [6] *that the mean Laplacian $\Delta^F := \Delta_2^F$ coincides with the dynamical Finsler Laplace operator introduced by Barthelmé in his PhD thesis* [2]. *It is an elliptic operator and its symbol is given by*

$$s_x^F(\xi_1, \xi_2) = \int_{H_x M} \langle \nabla^G(\pi^* f), \nabla^G(\pi^* g)\rangle\ \varpi d\eta,$$

where $f, g : M \to \mathbb{R}$ satisfy $f(x) = 0 = g(x)$, $\xi_1 = df_x$ and $\xi_2 = dg_x$, and $\nabla^G(\pi^* f)$ denotes the gradient of $\pi^* f$ with respect to the Riemannian metric G.

Let h be the dual metric of the symbol. Then, its inverse coefficients are given by

$$h^{ij} := \int_{H_x M} g^{ij} \varpi d\eta,$$

so we call it the mean Riemannian metric associated with the Finsler metric F.

Define the mean S-curvature of h as

$$S_h(X) := X^i \frac{\partial \tau}{\partial x^i},$$

where $\tau = \ln \frac{\sqrt{det(h_{ij})}}{\sigma}$. It was proved that for any smooth function u on M, the mean Finsler Laplacian satifies $\Delta^F u = \Delta^h u - S_h(\nabla u)$, where $\Delta^h u$ and ∇u denote respectively the Laplacian and the gradient of u with respect to the Riemannian metric h.

In this point of view, one can also define a weighted p-Laplace type operator associated with the mean Laplacian, as

$$^{\omega}\Delta_p^F = \Delta_p^h - |\nabla u|_h^{p-2} S_h(\nabla u),$$

where Δ_p^h is the p-Laplacian associated with h. Unfortunetely, these two generalizations of p-Laplacian do not coincide whether $p \neq 2$. However, we do not pursue the study of the weigthed mean p-Laplacian on this paper.

Let (M, F) be a closed Finsler manifold. Consider on the set $C^\infty(M)$ of smooth functions on M the following norm :

$$|u|_p := \left(\int_M |u|^p\ d\mu_F + \int_{HM} |\nabla^G(\pi^* u)|_G^p\ d\mu_G \right)^{1/p},\ p > 1.$$

Denote by $W^{1,p}(M)$ the completion of $C^\infty(M)$ with respect to this norm.

An eigenfunction of Δ_p^F is a nonzero function u such that there exists a real number λ such that $\Delta_p^F u + \lambda |u|^{p-2}u = 0$. The real λ is then called an eigenvalue of Δ_p^F on M. It is evident that zero is an eigenvalue of Δ_p^F with constant function as associated eigenfunction.

Consider on $W^{1,p}(M)$ the p-energy functional which is defined by

$$\mathcal{E}_p(u) := \frac{\int_{HM} |d\pi^* u|_G^p \, d\mu_G}{\int_{HM} |\pi^* u|^p \, d\mu_G}.$$

The first (nonzero) eigenvalue $\lambda_{1,p}^F$ of Δ_p^F is defined as the minimal critical value of the p-energy functional :

$$\lambda_{1,p}^F = \inf_{u \in \mathcal{H}^{1,p}} \mathcal{E}_p(u),$$

where $\mathcal{H}^{1,p}$ denotes the set of functions $u \in W^{1,p}(M)$ satisfying

$$\int_M u|u|^{p-2} \, d\mu_F = 0.$$

Obviously, if $\lambda_{1,p}^G$ denotes the first eigenvalue of the p-Laplacian on the Riemannian manifold (HM, G) then we have $\lambda_{1,p}^G \le \lambda_{1,p}^F$.

4. Some general estimates of the first eigenvalue

Let (M, F) be a smooth n-dimensional closed manifold. The conformal class of F is the set $[F]$ of Finsler metrics of the form $e^\kappa F$ where $\kappa \in C^\infty(M)$. Let $\bar{F} \in [F]$. Let h and $\bar{h}$ be the mean metrics associated to F and $\bar{F}$ respectively. From a simple computation it holds:

$$\bar{h} = e^{-2\kappa} h \qquad \textit{and} \qquad d\mu_{\bar{F}} = e^{n\kappa} d\mu_F \ ,$$

which means that $\bar{h} \in [h]$.

Proposition 4.1. *Let M be a smooth closed manifold of dimension n endowed to a Finsler metric F. Then, for any $p \in (1, 2]$, there exists a constant $K = K(n, p, [F])$ depending only on p, n and the conformal class $[F]$ of F such that*

$$\lambda_{1,p}^F \le \frac{K}{Vol(M, F)^{\frac{p}{n}}}, \tag{4.1}$$

where $Vol(M, F)$ is the volume of M with respect to $d\mu_F$.

Proof. For any $p \in (1, 2]$ and $u \in C^\infty(M)$, we have

$$\frac{\int_{HM} |d\pi^* u|_G^p \, d\mu_G}{\int_{HM} |\pi^* u|^p \, d\mu_F} \le \frac{\int_M |\nabla u|_h^p \, d\mu_F}{\int_M |u|^p \, d\mu_F} \le \frac{\sup_M e^{-\tau}}{\inf_M e^{-\tau}} \frac{\int_M |\nabla u|_h^p \, d\mu_h}{\int_M |u|^p \, d\mu_h}.$$

Then $\lambda_{1,p}^F \le C \lambda_{1,p}^h$, where $C = \frac{\sup_M e^{-\tau}}{\inf_M e^{-\tau}}$. There exists a constant C_1 which depends only on n, p and the conformal class h satisfying $\lambda_{1,p}^h Vol(M, h)^{\frac{p}{n}} \le C_1$ ([7, Theorem 3.2]). Hence $\lambda_{1,p}^F Vol(M, F)^{\frac{p}{n}} \le CC_1 (\sup_M e^{-\tau})^{p/n}$. Now, as we have seen ealier, if F and $\bar{F}$ are conformal related then the associated mean metrics are also in the same conformal class, so $CC_1(\sup_M e^{-\tau})^{p/n}$ depends only on n, p and $[F]$. $\square$

Remark 4.2. *In* [7, Theorem 3.2]*, the dependence of the constant* C_1 *on the conformal class of the Riemannian metric comes from the* n*-conformal volume of the compact Riemannian manifold* (M, g) *which is defined as*

$$V_m^c(M, [g]) := \inf_{\phi \in I_m(M,[g])} \sup_{\gamma \in G_m} Vol(M, (\gamma \circ \phi)^* can),$$

where can denotes the canonical Riemannian metric on the m*-dimensional sphere* $\mathbb{S}^m$*,*

$$G_m := \{\gamma \in Diff(\mathbb{S}^m) | \ \gamma^* can \in [can]\},$$

the group of conformal diffeomorphism of $(\mathbb{S}^m, can)$ *and*

$$I_m(M, [g]) := \{\phi : M \to \mathbb{S}^m | \ \phi^* can \in [g])\},$$

the set of conformal immersion from (M, g) *to* $(\mathbb{S}^m, can)$*.*

Let F be a Finsler metric on a smooth manifold M and G is the Sasaki metric induced by F on the projective sphere bundle HM. For $m \in \mathbb{N}$, we set

$$I_m(M, F) := \{\varphi : M \to \mathbb{S}^m | \ (\varphi \circ \pi)^* can \in [G]\}.$$

Definition 4.3. *Let* $m \in \mathbb{N}$*. The* m*-mean conformal volume of* (M, F) *is defined by*

$$V_m(M, F) := \inf_{\varphi \in I_m(M,F)} \sup_{\gamma \in G_m} Vol(HM, (\gamma \circ \varphi \circ \pi)^* can), \tag{4.2}$$

where $Vol(HM, (\gamma \circ \varphi \circ \pi)^* can)$ *denotes the volume of* HM *with respect to the metric* $(\gamma \circ \varphi \circ \pi)^* can$*.*

We use the convention that $V_m(M, F) = \infty$ if $I_m(M, F)$ is empty.

Theorem 4.4. *Let* (M, F) *be an* n*-dimensional closed Finsler manifold and let* $1 < p \leq 2n - 1$*. Then, for any* $m \in \mathbb{N}$*, there exixts a constant* $C = C(n, p, m)$ *depending only on* n*,* p *and* m *such that*

$$\lambda_{1,p}^F \leq C \left(\frac{V_m(M, F)}{Vol(M, F)} \right)^{\frac{p}{2n-1}}. \tag{4.3}$$

Proof. Let $\varphi \in I_m(M, F)$ and let $p > 1$. Then $\varphi \circ \pi : (HM, G) \to (\mathbb{S}^m, can)$ is a conformal immersion and its level sets are measure zero in (HM, G). By [7, Lemma 2.6], there exists $\gamma \in G_m$ such that the projection functions ψ_i $(1 \leq i \leq m + 1)$ of $\psi = \gamma \circ \varphi \circ \pi$ on $\mathbb{R}^{m+1}$ satisfy

$$\int_{HM} |\psi_i|^{p-2} \psi_i \ d\mu_G = 0.$$

Since the function ψ depends only on the base points $x \in M$, we obtain, for all $1 \leq i \leq m+1$, $\int_M |\phi_i|^{p-2} \phi_i \ d\mu_F = 0$, where $\phi_i = (\gamma \circ \varphi)_i$. Hence, by the variational characterization of the first eigenvalue, we have for any $1 \leq i \leq m + 1$,

$$\lambda_{1,p}^F \leq \mathcal{E}_p(\phi_i) = \frac{\int_{HM} |d\psi_i|^p \ d\mu_G}{\int_{HM} |\psi_i|^p \ d\mu_G} \leq \frac{\int_{HM} \sum_i |d\psi_i|^p \ d\mu_G}{\int_{HM} \sum_i |\psi_i|^p \ d\mu_G}.$$

A straightforward estimate gives

$$\lambda_{1,p}^F \leq \frac{(m+1)^{|1-p/2|}}{Vol(HM, G)} \int_{HM} |d\psi|^p \ d\mu_G,$$

where $|d\psi|$ denotes the Hilbert-Schmidt norm of $d\psi$ (see [7, Lemma 2.7]); and applying Holder inequality, we have

$$\lambda_{1,p}^F \leq \frac{(m+1)^{|1-p/2|}}{Vol(HM,G)^{p/(2n-1)}} \left(\int_{HM} |d\psi|^{2n-1}\, d\mu_G \right)^{p/(2n-1)}.$$

Moreover, it is well known that $\psi^* can = \frac{|d\psi|^2}{2n-1} G$ since ψ is a conformal immersion. Therefore,

$$\begin{aligned} \int_{HM} |d\psi|^{2n-1}\, d\mu_G &= (2n-1)^{(2m-1)/2} Vol(HM, \psi^* can) \\ &\leq (2n-1)^{(2m-1)/2} \sup_{\gamma \in G_m} Vol(HM, \psi^* can), \end{aligned}$$

and

$$\lambda_{1,p}^F \leq \frac{(m+1)^{|1-p/2|}(2n-1)^{p/2}}{Vol(HM,G)^{p/(2n-1)}} \left(\sup_{\gamma \in G_m} Vol(HM, \psi^* can) \right)^{p/(2n-1)}.$$

Taking the infimum over all $\varphi \in I_m(M,F)$ and using $c_{n-1} Vol(M,F) = Vol(HM,G)$, we obtain (4.3). □

5. First eigenvalue estimates on Randers manifolds

Let $F = \alpha + \beta := \sqrt{a} + \beta$ be a Randers metric on a smooth manifold M, where a is a Riemannian metric on M and β a 1-form with norm $|\beta|$ (w.r.t. the metric a) less than 1. In the sequel we put $b = \sup\{|\beta|(x) : \ x \in M\}$. In the following, we etablish some relations between the first eigenvalue of the mean p-laplacian Δ_p^F and the first eigenvalue of the p-Laplace operator associated with the Riemannian metric a. Let us recall first some properties of Randers metrics.

Lemma 5.1. [9] *Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M. The determinant and the inverse of the fundamental tensor are given respectively by*

$$det(g_{ij}) = \left(\frac{F}{\alpha}\right)^{n+1} det(a_{ij}),$$

$$g^{ij} = \frac{\alpha}{F} a^{ij} + \left(\frac{\beta}{F} + \frac{\alpha}{F}|\beta|^2\right) \frac{y^i}{F}\frac{y^j}{F} - \frac{\alpha}{F}\left(a^{ik} b_k \frac{y^j}{F} + a^{jk} b_k \frac{y^i}{F}\right).$$

Lemma 5.2. *Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M. Then,*

$$\int_{H_x M} \frac{y^i}{F}\, \varpi d\eta = 0. \tag{5.1}$$

Particularly,

$$\int_{H_x M} \frac{\alpha}{F}\, \varpi d\eta = 1.$$

Proof. Let $f = \frac{y^k}{F}$. A simple computation gives

$$g^{ij}[F^2 f]_{y^i y^j} = g^{ij} F_{y^i y^j} y^k + 2 g^{ik} F_{y^i}.$$

Further,

$$\begin{aligned} F^2 g^{ij} F_{y^i y^j} y^k &= F g^{ij} y^k (g_{ij} - F_{y^i} F_{y^j}) \\ &= nFy^k - g^{ij} y^k F_{y^i} y^s g_{sj} \\ &= nFy^k - y^k y^i F_{y^i} = (n-1)Fy^k. \end{aligned}$$

Then, from lemma 2.1, one obtains

$$(n+1)\int_{H_x M} f\ \Omega d\eta = 2n \int_{H_x M} f\ \Omega d\eta,$$

which implies (5.1).

The last equation follows from (5.1) and $\alpha = F - \beta$.

□

From lemmas 5.1 and 5.2 , we obtain the following result :

Proposition 5.3. *Let $F = \alpha + \beta$ be a Randers metric on M and let h be the associated mean metric. Then, for any smooth function u on M, we have*

$$\frac{1}{c}|\nabla u|_a^2 \leq |\nabla u|_h^2 \leq c|\nabla u|_a^2 \ \ \textit{with}\ \ c = \frac{1+b}{1-b}\ . \tag{5.2}$$

Proof. Let $u \in C^\infty(M)$ and $u_i = \frac{\partial u}{\partial x^i}$. We have $|\nabla u|_a^2 = a^{ij}u_i u_j$ and $|\nabla u|_h^2 = h^{ij}u_i u_j$. By lemma 5.1, we have

$$\begin{aligned} g^{ij}u_i u_j &= \frac{\alpha}{F} a^{ij} u_i u_j + \left(\frac{\beta}{F} + \frac{\alpha}{F}|\beta|^2\right)\frac{y^i}{F}\frac{y^j}{F} u_i u_j \\ &- \frac{\alpha}{F}\left(a^{ik} b_k \frac{y^j}{F} + a^{jk} b_k \frac{y^i}{F}\right) u_i u_j \\ &= \frac{\alpha}{F}|\nabla u|_a^2 + \left(\frac{\beta}{F} + \frac{\alpha}{F}|\beta|^2\right)\left(\frac{y^i u_i}{F}\right)^2 - \frac{2\alpha}{F^2}\left(a^{ik} y^j u_i u_j b_k\right) \\ &= \frac{\alpha}{F}\left[|\nabla u|_a^2 + \left(\frac{\beta}{F} + \frac{\alpha}{F}|\beta|^2\right)\frac{a(y,\nabla u)^2}{\alpha F} - \frac{2}{F} a(y,\nabla u) a(b^\#, \nabla u)\right], \end{aligned}$$

where $\beta^\#$ and ∇u denote respectively the dual vector of β and the gradient of u with respect to the metric a.

Moreover, we have following estimates

- $\beta(y) \leq |b^\#|_a |y|_a \leq b\alpha$ hence $\frac{1}{1+b} \leq \frac{\alpha}{F} \leq \frac{1}{1-b}$;
- $\left|\frac{\beta}{F} + \frac{\alpha}{F}|\beta|^2\right| \leq |\beta| \leq b$.

So, since $g^{ij}u_i u_j \geq 0$, conbining these estimates with Cauchy-Schawrtz inequality provides

$$\begin{aligned} g^{ij}u_i u_j &\leq \frac{1}{1-b}\left[|\nabla u|_a^2 + \left(\frac{\beta}{F} + \frac{\alpha}{F}|\beta|^2\right)\frac{a(y,\nabla u)^2}{\alpha F} - \frac{2}{F} a(y,\nabla u) a(b^\#, \nabla u)\right] \\ &\leq \frac{1}{1-b}\left(1 + b\frac{\alpha}{F}\right)|\nabla u|_a^2 - \frac{2}{(1-b)} a(b^\#, \nabla u)\frac{a(y,\nabla u)}{F} \end{aligned}$$

and

$$\begin{aligned} g^{ij}u_iu_j &\geq \frac{1}{1+b}\left[|\nabla u|_a^2 + \left(\frac{\beta}{F}+\frac{\alpha}{F}|\beta|^2\right)\frac{a(y,\nabla u)^2}{\alpha F} - \frac{2}{F}a(y,\nabla u)a(b^{\#},\nabla u)\right] \\ &\geq \frac{1}{1+b}\left(1-b\frac{\alpha}{F}\right)|\nabla u|_a^2 - \frac{2}{(1+b)}a(b^{\#},\nabla u)\frac{a(y,\nabla u)}{F} \end{aligned}$$

Integrating these inequalities and using lemma 5.2 completes the proof. □

Now, we can state our comparison result in this section.

Theorem 5.4. *Let $F=\alpha+\beta$ be a Randers metric on a closed manifold M. Then we have*

$$\begin{aligned} \lambda^F_{1,p} &\leq \left(\frac{1+b}{1-b}\right)^{\frac{p}{2}}\lambda^a_{1,p}, \qquad 1<p\leq 2, \\ \lambda^F_{1,p} &\geq \left(\frac{1-b}{1+b}\right)^{\frac{p}{2}}\lambda^a_{1,p}, \qquad p\geq 2, \end{aligned}$$

where $\lambda^a_{1,p}$ denotes the first eigenvalue of the p-laplacian on (M,a).

Particularly, for $p=2$, we have

$$\frac{1-b}{1+b}\lambda^a_{1,2} \leq \lambda^F_{1,2} \leq \frac{1+b}{1-b}\lambda^a_{1,2}.$$

Proof. Let h be the mean metric associated with F and let u be a smooth function on M. We have

$$|\nabla u|_h^2 = \int_{H_xM}|\nabla\pi^*u|_G^2\varpi d\eta.$$

Then,

$$|\nabla u|_h^p \leq \int_{H_xM}|\nabla\pi^*u|_G^p\varpi d\eta, \qquad p\geq 2,$$

and

$$|\nabla u|_h^p \geq \int_{H_xM}|\nabla\pi^*u|_G^p\varpi d\eta, \qquad 1<p\leq 2.$$

Hence, from Proposition 5.3, we have

$$\left(\frac{1-b}{1+b}\right)^{\frac{p}{2}}|\nabla u|_a^p \leq \int_{H_xM}|\nabla\pi^*u|_G^p\varpi d\eta, \qquad p\geq 2,$$

and

$$\left(\frac{1+b}{1-b}\right)^{\frac{p}{2}}|\nabla u|_a^p \geq \int_{H_xM}|\nabla\pi^*u|_G^p\varpi d\eta, \qquad 1<p\leq 2.$$

However, it is well known that for Randers metrics, the volume form $d\mu_F$ coincides with the Riemannian volume $d\mu_a$ of the metric a. That completes the proof. □

6. First eigenvalue estimates on Berwald spaces

A Finsler metric is called of Berwald type if the Chern connection coefficients Γ^i_{jk} in natural coordinates have no y dependence. He et al. obtained the following interesting result.

Lemma 6.1. [6] *Let (M,F) be a Berwald manifold. Then, the Levi-Civita connection of the mean metric h coincides with the Chern connection of (M,F). Furthermore, the mean S-curvature vanishes.*

Combining this lemma with a result of Valtorta [11, Theorem 1.2] from Riemannian geometry, we obtain the following estimate for Berwald spaces

Proposition 6.2. *Let (M, F) be a closed Berwald manifold with non-negative Ricci curvature. Then, for any $p \geq 2$, the first eigenvalue of the mean p-Laplacian satisfies*

$$\lambda_{1,p}^F \geq (p-1)\left(\frac{\pi_p}{diam_h(M)}\right)^p,$$

where $\pi_p = \frac{2\pi}{p\sin(\pi/p)}$ and $diam_h(M)$ denotes the diameter of the Riemannian manifold (M, h).

Proof. Let $p \geq 2$. For any smooth function u on M, we have

$$\frac{\int_M |\nabla u|_h^p \, d\mu_F}{\int_M |u|^p \, d\mu_F} \leq \frac{\int_{HM} |d\pi^* u|_G^p \, d\mu_G}{\int_{HM} |\pi^* u|^p \, d\mu_F}.$$

Then, $\lambda_{1,p}^h \leq \lambda_{1,p}^F$, where $\lambda_{1,p}^h$ denotes the first eigenvalue of the p-laplacian on (M, h). In the sequel, the quantities with "bar" are related to the mean metric h.

From Lemma 6.1, the curvature tensors of (M, F) and (M, h) are the same : $R_{jkl}^i = \bar{R}_{jkl}^i$. Hence, we have for any $X \in T_x M$,

$$\bar{R}ic(X) = \frac{1}{|X|_h^2}\bar{R}_{jil}^i X^j X^l = \frac{1}{|X|_h^2} R_{jil}^i X^j X^l = \frac{F(X)^2}{|X|_h^2} Ric(X). \tag{6.1}$$

Therefore, since $Ric \geq 0$, we also have $\bar{R}ic \geq 0$, and the result follows from [11, Theorem 1.2]. $\square$

Let us state again our Theorem 1.1.

Theorem 6.3. *Let (M, F) be a closed Berwald manifold of dimension n whose Ricci curvature satisfies, for some positive constant $K > 0$,*

$$Ric \geq (n-1)KC_p,$$

where $C_p := \sup\left\{\frac{\int_M |X|_h^p \, d\mu_F}{\int_M |X|^{p-2}F(X)^2 \, d\mu_F} \mid X \in \Gamma(TM)\right\}$, $p \geq 2$. Then, for any $p \geq 2$, the first eigenvalue of the mean p-laplacian satisfies

$$\lambda_{1,p}^F \geq \left(\frac{(n-1)K}{p-1}\right)^{p/2}.$$

Proof. Let us start by remarking that $C_p < +\infty$ for any $p \geq 2$, since M is compact. Recall also that the Bochner-Weintzenbock formula of Δ^h is given by

$$\frac{1}{2}\Delta^h(|\nabla u|_h^2) = |\nabla du|^2 + Ric^h(\nabla u)|\nabla|_h^2 + \nabla u(\Delta^h u).$$

Using (6.1), we have

$$\begin{aligned} \int_M |\nabla u|_h^p Ric(\nabla u) \, d\mu_F &= \int_M |\nabla u|_h^{p-2} F(\nabla u)^2 Ric(\nabla u) \, d\mu_F \\ &\geq (n-1)KC_p \int_M |\nabla u|_h^{p-2} F(\nabla u)^2 \, d\mu_F \\ &\geq (n-1)K \int_M |\nabla u|_h^p \, d\mu_F \qquad (6.2) \end{aligned}$$

The result is obtained similarly to the proof of [8, Theorem 3.2] using (6.2) and the fact that $\lambda_{1,p}^h \leq \lambda_{1,p}^F$. $\square$

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