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ON REDUCTION OF INTERVAL DECISION-MAKING PROBLEMS

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ABSTRACT. We consider the method of partial ordering of real intervals with the help of a numerical indicator and describe the applications of the indicator for the reduction of various classes of interval decision-making problems to deterministic problems of linear and non-linear programming. The properties of reduced problems are established. Illustrating examples are considered.

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KEYWORDS AND PHRASES. Interval indicator, decision-making problem, nonlinear programming problem, linear programming problem.

1. INTRODUCTION

Many mathematical models used in the natural sciences, engineering, social and humanitarian fields can be considered as interval ones. One of the examples is physics, in which the phenomena of the surrounding world are described by dependencies with experimental or approximately calculated data. Other examples are found in computational mathematics, inverse problem theory, optimization methods, operations research, economics, sociology, and other fields of knowledge. The needs of theory and practice stimulated the emergence in the 1950's of a new mathematical discipline - interval analysis. Initially, the objects of study of interval analysis were problems of computational mathematics, and the main efforts were focused on obtaining two-sided (interval) estimates of solutions. Almost at the same time, decision-making problems began to be investigated: extreme, game, controlled. Their specificity is related with the need to use the preference relation. In the «ordinary» extreme problem, the preference relation is naturally given by a real objective function. Of the two points in its domain, in the one of which the function has a smaller (or larger) value is considered preferable. In the case of an interval-valued objective function, in order to select a preferred point, it is necessary to compare two intervals of its values for «less» (or «greater»).

Interval problems can be treated as a parametric family of problems generated by all parameter values in admissible intervals. In the paper [1], the solution of the interval problem is considered to be one «acceptable» solution for the entire family of problems. It is shown that finding an acceptable («universal») solution is reduced to solving a regularized deterministic problem of the same type as the original interval problem.

Another approach employs comparison of intervals. In [2,8,9], the partial order on the set of real intervals is understood in the strong and weak sense. Strong comparison is defined for intervals without common interior points; all other intervals are considered incomparable. A weak comparison allows the intersection of intervals, while the «smaller» interval on the numerical line may be to the right of the «larger» interval.

The problem of partial ordering of intervals can be approached applying the methodology of other mathematical disciplines - probability theory, fuzzy set theory. They use real indicators of binary operations on sets - probabilities of random events, membership functions. By analogy, it is natural to set a partial order relation on the set of real intervals by defining a pairwise comparison of intervals using a numerical indicator of interval inequality [3]. As a result, a formal basis appears for correct mathematical formulations of interval decision-making problems and development of methods for their solution. The present work is devoted to the presentation of these questions.

Thematically, the presentation of the material consists of two related parts. The first part provides the necessary information about the indicator of interval inequality. The second part shows the application of the indicator for the study of interval decision-making problems. The reductions of typical problems to similar deterministic problems are described, in which the target conditions and constraints can be meaningfully interpreted in terms of probability. On this base, the interval nonlinear and linear problems are considered. The approach and obtained results can be applied to the many economic and other models.

2. INTERVAL INEQUALITY INDICATOR

2.1. Comparison of intervals

Consider the space $\mathbb{IR}$ of regular closed real intervals $\boldsymbol{a} = [\underline{a}, \overline{a}]$, $\underline{a} \le \overline{a}$. The *center* and *radius* of the interval $\boldsymbol{a}$ will be denoted

$$a_0 = 0.5(\underline{a} + \overline{a}),\ \Delta a = 0.5(\overline{a} - \underline{a}).$$

Expressing the ends of the interval in terms of the center and the radius, we obtain an equivalent *symmetrical* representation of the interval

$$\boldsymbol{a} = [a_0 - \Delta a, a_0 + \Delta a].$$

An interval $\boldsymbol{a}$ is called *degenerate* if $\Delta a = 0$ and *non-degenerate* if $\Delta a > 0$.

Following [9], for intervals $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{IR}$ we give the definitions of inequality $\boldsymbol{a} \le \boldsymbol{b}$ in the «strong», «weak» and «central» senses:

$$\begin{gathered} \boldsymbol{a} \le \boldsymbol{b} \Leftrightarrow \big((\forall a \in \boldsymbol{a})(\forall b \in \boldsymbol{b})(a \le b)\big), \\ \boldsymbol{a} \le \boldsymbol{b} \Leftrightarrow \big((\exists a \in \boldsymbol{a})(\exists b \in \boldsymbol{b})(a \le b)\big), \\ \boldsymbol{a} \le \boldsymbol{b} \Leftrightarrow (a_0 \le b_0). \end{gathered} \tag{1.1}$$

A strong definition means that on the real axis the interval $\boldsymbol{a}$ is located to the left of the interval $\boldsymbol{b}$ and can have no more than one common point $\overline{a} = \underline{b}$ with it. The weak definition includes the strong one as a special case and allows intervals to intersect. If in that the intervals have a common point $\underline{a} = \overline{b}$, then $\boldsymbol{a}$ is to the right of $\boldsymbol{b}$. The central comparison of intervals is made by their centers regardless of the radii.

Let $\boldsymbol{a}, \boldsymbol{b}$ be some intervals from $\mathbb{IR}$. It is natural to take as an indicator of interval inequality a set of points $(a, b) \in \boldsymbol{a} \times \boldsymbol{b}$ with coordinates $a \le b$, correlated by some extent with the entire rectangle $\boldsymbol{a} \times \boldsymbol{b}$. If we consider the points $a \in \boldsymbol{a}$, $b \in \boldsymbol{b}$ as independent uniformly distributed random variables and understand the area as the measure of a flat set, then we naturally come to a geometric interpretation of the probability p of the event $a \le b$. Unfortunately, it is not convenient to take p as an indicator of inequality $\boldsymbol{a} \le \boldsymbol{b}$ because of the non-linear dependence of the probability on the centers and ends of the intervals $\boldsymbol{a}, \boldsymbol{b}$. We will dwell on this issue in more detail later. For now, we only note that the application of a probabilistic indicator to linear decision-making models would lead to more complex nonlinear models and cast doubt on the very idea of reducing interval problems to non-interval problems.

Let us use other considerations. Consider non-degenerate intersecting intervals ***a***, ***b*** on the real axis (Fig. 1.1).

Fig. 1.1 The inequality $a \le b$ holds for all points $a \in [\underline{a}, \underline{b}]$, $b \in [\overline{a}, \overline{b}]$

As can be seen from the Fig. 1.1, the points $a \in \boldsymbol{a}$ and $b \in \boldsymbol{b}$ satisfying the inequality $a \le b$ are in the intervals $[\underline{a}, \underline{b}]$ and $[\overline{a}, \overline{b}]$, respectively. The share of the total length of these intervals in the total length of intervals ***a***, ***b*** is equal to the absolute value of the ratio

$$\frac{(\underline{b} - \underline{a}) + (\overline{b} - \overline{a})}{2(\Delta a + \Delta b)} = \frac{b_0 - a_0}{\Delta a + \Delta b}. \tag{1.2}$$

Taking into account the above considerations and formula (1.2), we accept the real number

$$R(\boldsymbol{a} \le \boldsymbol{b}) = \frac{b_0 - a_0}{\Delta a + \Delta b} \tag{1.3}$$

as the *indicator* R *of the interval inequality* $\boldsymbol{a} \le \boldsymbol{b}$.

Formula (1.3) embraces the definition (1.1) of the interval inequality in the strong, weak, and central senses, if respectively

$$R(\boldsymbol{a} \le \boldsymbol{b}) \ge 1, R(\boldsymbol{a} \le \boldsymbol{b}) \ge -1, R(\boldsymbol{a} \le \boldsymbol{b}) \ge 0.$$

If $R(\boldsymbol{a} \le \boldsymbol{b}) > 1$ then the interval ***a*** is located to the left of the interval ***b*** and has no common points with it, if $R(\boldsymbol{a} \le \boldsymbol{b}) < -1$ then the pattern is mirror symmetric − the interval ***b*** is located to the left of the interval ***a*** and does not intersect with it.

With an additional agreement, the indicator R can be extended on degenerate intervals. Indeed, we can apply formula (1.3) to intervals $\boldsymbol{a}_\varepsilon = [a-\varepsilon, a+\varepsilon]$, $\boldsymbol{b}_\varepsilon = [b-\varepsilon, b+\varepsilon]$ of small radius $\varepsilon > 0$. We get

$$R(\boldsymbol{a}_\varepsilon \le \boldsymbol{b}_\varepsilon) = (b-a)/(2\varepsilon).$$

Hence, in the limit as $\varepsilon \to 0$ we find

$$\begin{aligned} R(a \le b) &= +\infty \Leftrightarrow a < b; \\ R(a \le b) &= 0 \Leftrightarrow a = b; \\ R(a \le b) &= -\infty \Leftrightarrow a > b. \end{aligned} \tag{1.4}$$

Correspondences (1.4) determine the indicator (1.3) on degenerate intervals if the interval of its values $(-\infty, +\infty)$ is supplemented with symbols $-\infty$, $+\infty$.

2.2. Indicator properties

For intervals $\boldsymbol{a}, \boldsymbol{b}$ from IR represented in symmetrical form

$$\boldsymbol{a} = [a_0 - \Delta a, a_0 + \Delta a],\ \boldsymbol{b} = [b_0 - \Delta b, b_0 + \Delta b],$$

operations of classical interval arithmetic (addition, multiplication by a real number α) are performed according to the rules

$$\boldsymbol{a} + \boldsymbol{b} = [a_0 + b_0 - \Delta a - \Delta b,\ a_0 + b_0 + \Delta a + \Delta b],$$

$$\alpha\boldsymbol{a} = [\alpha a_0 - |\alpha|\Delta a,\ \alpha a_0 + |\alpha|\Delta a].$$

Using these operations, it is easy to establish [3] the main properties of the indicator which follow from definition (1.3).

- Multiplying the interval inequality by a positive number does not change the inequality indicator; multiplying inequality by a negative number reverses the sign of the indicator.
- Inequality indicator is antisymmetric: $R(\boldsymbol{a} \le \boldsymbol{b}) = -R(\boldsymbol{b} \le \boldsymbol{a})$.
- Intervals $\boldsymbol{a}$, $\boldsymbol{b}$ with equal centers satisfy opposite inequalities $\boldsymbol{a} \le \boldsymbol{b}, \boldsymbol{b} \le \boldsymbol{a}$ with zero indicator.

- When adding interval inequalities $\boldsymbol{a} \le \boldsymbol{b}$ and $\boldsymbol{c} \le \boldsymbol{d}$ with equal indicators, an inequality $\boldsymbol{a}+\boldsymbol{c} \le \boldsymbol{b}+\boldsymbol{d}$ with the same indicator is obtained.
- If pairs of intervals ***a***, ***b*** and ***c***, ***d*** have equal sums of radii then the inequality indicator $\boldsymbol{a}+\boldsymbol{c} \le \boldsymbol{b}+\boldsymbol{d}$ is equal to the arithmetic mean of the inequality indicators $\boldsymbol{a} \le \boldsymbol{b}$ and $\boldsymbol{c} \le \boldsymbol{d}$.
- For intervals $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$ with pairwise equal positive sums of radii, the equality

$$R(\boldsymbol{a} \le \boldsymbol{c}) = R(\boldsymbol{a} \le \boldsymbol{b}) + R(\boldsymbol{b} \le \boldsymbol{c})$$

holds true.

2.3. Relation of indicator with probability

We consider non-degenerate intervals ***a***, ***b*** from $\mathbb{IR}$ as sets of realization of independent uniformly distributed random variables *a*, *b*. Let us establish a relation between the probability p of a random event $a \le b$ and the indicator r of the inequality $\boldsymbol{a} \le \boldsymbol{b}$.

We distinguish three main cases of the location of the rectangle $\boldsymbol{a} \times \boldsymbol{b}$ relative to the half-plane $x \le y$ (Fig. 1.2) in the Cartesian coordinate system *x*, *y*. Let us analyze in more detail the case (a) when the coordinates of the vertices of the rectangle $\boldsymbol{a} \times \boldsymbol{b}$ satisfy the conditions

$$a_0 - \Delta a \le b_0 + \Delta b,\ a_0 - \Delta a > b_0 - \Delta b,$$

$$a_0 + \Delta a > b_0 - \Delta b,\ a_0 + \Delta a > b_0 + \Delta b.$$

From here we find

$$-(\Delta a + \Delta b) \le b_0 - a_0 < -|\Delta a - \Delta b|$$

or taking into account formula (1.3)

$$-1 \le r < -\frac{|\Delta a - \Delta b|}{\Delta a + \Delta b}.$$

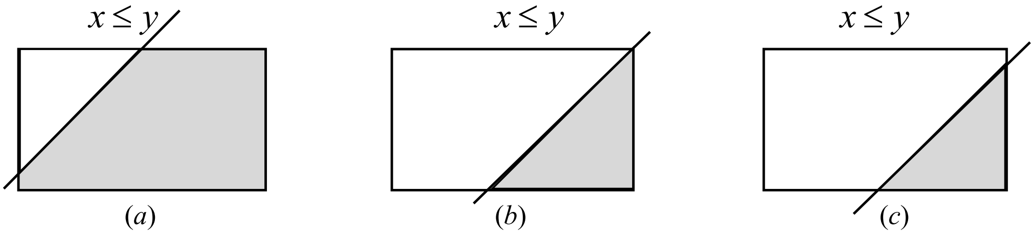

Fig. 1.2 Half plane $x \le y$ of x, y coordinate plane contains k vertices of rectangle $\boldsymbol{a} \times \boldsymbol{b}$:
(a) $k=1$, (b) $2 \le k \le 3$, (c) $3 \le k \le 4$

From geometric considerations, the probability p is equal to the ratio of the area of the light triangle in Fig. 1.2 (a) to the area of a rectangle $\boldsymbol{a} \times \boldsymbol{b}$:

$$p = \frac{(b_0 + \Delta b_0 - a_0 + \Delta a)^2}{8\Delta a \Delta b} = \frac{(\Delta a + \Delta b)^2}{8\Delta a \Delta b}(1+r)^2.$$

Fulfilling similar calculations for cases (b) and (c), we obtain [3] the desired relationship between the probability p and indicator r:

$$\begin{aligned} &p = \alpha(1+r)^2, -1 \le r \le -r_1, \\ &p = 0.5(1+\beta r), |r| \le r_1, \\ &p = 1 - \alpha(1-r)^2, r_1 < r \le 1, \end{aligned} \tag{1.5}$$

$$\alpha = \frac{(\Delta a + \Delta b)^2}{8\Delta a \Delta b},\ \beta = 1 + \frac{\Delta a}{\Delta b},\ r_1 = \frac{|\Delta a - \Delta b|}{\Delta a + \Delta b}. \tag{1.6}$$

The last formulas define a continuous strictly increasing function on the segment $-1 \le r \le 1$ (Fig. 1.3). We extend it by continuity to the entire real axis, setting

$$p = 0,\ r < -1;\ \ p = 1,\ r > 1. \tag{1.7}$$

Denote the resulting function by $\pi(r)$. On the interval $-1 < r < 1$, the function $\pi(r)$ has a single-valued inverse function $r = \pi^{-1}(p)$.

Formulas (1.5) − (1.7) give a certain probabilistic interpretation to the relationships in which the indicator of interval inequality participates. For example, if the inequality $R(\boldsymbol{a} \le \boldsymbol{b}) \ge r$ holds for some real r then the probability of an event $a \le b$ for independent random variables a, b uniformly distributed over intervals $\boldsymbol{a}, \boldsymbol{b}$ is not less than $\pi(r)$. For brevity, we agree in such cases to say that *the interval inequality* $\boldsymbol{a} \le \boldsymbol{b}$ *is satisfied with a probability not less than* $\pi(r)$ *and write it in the form* $P(\boldsymbol{a} \le \boldsymbol{b}) \ge \pi(r)$.

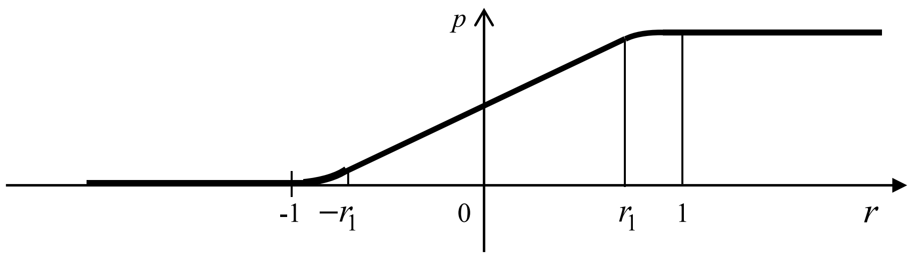

Fig. 1.3 Graph of a function $p = \pi(r)$ given by formulas (1.5) – (1.7)

Example 1.1

Let the intervals $\boldsymbol{a}, \boldsymbol{b}$ have equal radii: $\Delta a = \Delta b$. By formulas (1.5), (1.6) we find

$$p = 0.5(1+r)^2, -1 \le r < 0;$$
$$p = 1 - 0.5(1-r)^2, 0 \le r \le 1. \tag{1.8}$$

The inverse (1.8) function has the form

$$r = (2p)^{1/2} - 1,\ 0 \le p < 0.5;$$
$$r = 1 - (2(1-p))^{1/2},\ 0.5 \le p \le 1. \tag{1.9}$$

Using formula (1.9), one can calculate the indicator r of inequality $\boldsymbol{a} \le \boldsymbol{b}$ by the given probability p of its fulfillment. For example, if $p = 0.82$ the indicator r is equal to

$$r = 1 - (2(1-0.82))^{1/2} = 1 - (0.36)^{1/2} = 1 - 0.6 = 0.4\,.$$

Example 1.2

Consider the case when a non-degenerate interval $\boldsymbol{a}$ contains a degenerate interval $\boldsymbol{b} = b$. Then

$$r = R(\boldsymbol{a} \le b) = \frac{b - a_0}{\Delta a},$$
$$p = P(\boldsymbol{a} \le \boldsymbol{b}) = \frac{b - (a_0 - \Delta a)}{2\Delta a} = 0.5\left(1 + \frac{b - a_0}{\Delta a}\right).$$

Deriving the probability p and the indicator r from the latter formulas, we obtain simple relationship between these quantities:

$$\begin{aligned} p &= 0.5(1+r),\ -1 \le r \le 1; \\ r &= 2p - 1,\ 0 \le p \le 1. \end{aligned} \tag{1.10}$$

They will be required further in the analysis of interval decision-making problems.

3. INTERVAL NONLINEAR PROGRAMMING

3.1. System of interval inequalities

Statement of applied decision-making problems reflects the subjective ideas of the researcher about the phenomenon under study and the relationships of the main factors. The hypotheses used, empirical dependencies and initial data often lead to the uncertainty of mathematical models. An example is the balance equations and inequalities that describe the economic development of a manufacturing enterprise in the long term with predictive (expert) estimates of future production technologies, production costs and resource prices. Another source of uncertainty can be modeling errors or input data errors. In connection with these circumstances, the theoretical and practical interest in systems of interval equations and inequalities is understandable.

Consider a system of m interval inequalities

$$\boldsymbol{f}_i(x) \le 0,\ i = 1,...,m, \tag{2.1}$$

and its vector form

$$\boldsymbol{f}(x) \le 0, \tag{2.2}$$

where $\boldsymbol{f}: \mathrm{R}^n \to \mathrm{IR}^m$ − interval function with components $\boldsymbol{f}_1, ..., \boldsymbol{f}_m: \mathrm{R}^n \to \mathrm{IR}$, 0 − null vector in R^m. Here and below, *vector inequalities* are understood component by component.

Correlate the *i*-th inequality (2.1) with the number r_i from the segment $[-1,1] \subset \mathrm{R}$ and the system of inequalities (2.2) with the vector $r = (r_1, ..., r_m)$ from the cube $[-1,1]^m \subset \mathrm{R}^m$. Let us define a *solution* of the system of interval inequalities (2.2) the vector $x \in \mathrm{R}^n$ satisfying the conditions

$$R(\boldsymbol{f}_i(x) \le 0) \ge r_i,\ i = 1, ..., m,$$

or in an equivalent form using the indicator (1.3) of the interval inequality

$$f_{i0}(x) + r_i \Delta f_i(x) \le 0,\ i = 1, ..., m. \tag{2.3}$$

The set of solutions x of the system of inequalities (2.3) for fixed r is denoted by X_r. We note two properties of the set of solutions.

1. If $X_r \ne \varnothing$ and $x \in X_r$ then in accordance with formula (1.10) each i-th inequality (2.1) is satisfied with probability not less $0.5(1 + r_i)$.

2. If $r, s \in [-1,1]^m, r \le s,$ and $X_s \ne \varnothing$ then $X_r \ne \varnothing$ and $X_r \supset X_s$.

To check the second property we suppose $X_s \ne \varnothing$ and $x \in X_s$, i.e., the inequalities

$$f_{i0}(x) + s_i \Delta f_i(x) \le 0,\ i = 1, ..., m,$$

hold. Then for $r_i \le s_i, \Delta f_i(x) \ge 0, i = 1, ..., m$, from here we obtain

$$f_{i0}(x) + r_i \Delta f_i(x) \le f_{i0}(x) + s_i \Delta f_i(x) \le 0,\ i = 1, ..., m.$$

Hence $x \in X_r, X_r \ne \varnothing$ and due to the arbitrariness of $x \in X_s$ the inclusion $X_r \supset X_s$ is true.

3.2. Reduction of the interval nonlinear programming problem

Consider a nonlinear programming problem with a real objective function and interval inequality constraints

$$f_0(x) \to \min,\ \boldsymbol{f}(x) \le 0, \tag{2.4}$$

where $f_0 : \mathrm{R}^n \to \mathrm{R}$ – scalar function, $\boldsymbol{f} : \mathrm{R}^n \to \mathrm{IR}^m$ – vector function with components $\boldsymbol{f}_1, ..., \boldsymbol{f}_m : \mathrm{R}^n \to \mathrm{IR}$. The problem with an interval objective function $\boldsymbol{f}_0(x)$ is reduced to the form (2.4) by replacing the objective condition by minimization of additional unknown limiting the function $\boldsymbol{f}_0(x)$ from above.

Let us give a formalized meaning [6] of the relations (2.4). We consider a vector of indicators $r = (r_1, ..., r_m)$ with a range in cube $[-1,1]^m$. Define as a *solution* and the *set of solutions* of the vector inequality (2.4) the solution *x* and the set of solutions X_r of the vector inequality (2.2). Then, taking into account (2.3), the interval problem (2.4) is reduced to the usual nonlinear programming problem

$$f_0(x) \to \min,\ f_{i0}(x) + r_i \Delta f_i(x) \le 0,\ i = 1, ..., m. \tag{2.5}$$

The parameter vector *r* significantly effects on the solvability and properties of solutions of the problem (2.5). Assume for definiteness that the vector *r* runs through the diagonal of the cube $[-1,1]^m$ as all coordinates increase. Then, in accordance with the conclusions of paragraph 2.1, non-empty sets of solutions X_r «narrow», the minima of the objective function and the probabilities of fulfilling the constraints of the problem (2.5) increase. Thus, by the selection of r it is possible to find a reasonable compromise between the probability of fulfilling the interval constraints and minimum of the objective function.

For $r = 0$ from the family of problems (2.5), show up so-called *central problem*

$$f_0(x) \to \min,\ f_{i0}(x) \le 0,\ i = 1, ..., m. \tag{2.6}$$

Solution of the system of inequalities (2.6) satisfies each interval inequality $\boldsymbol{f}_i(x) \leq 0$ with probability not less 0.5. The set of solutions X_0 contains all sets $X_r, r \in [0.5, 1]^m$. If the ends of the intervals $\boldsymbol{f}_i(x)$ are functions with certain analytic properties, for example, linear, quadratic, convex, then the left-hand side of the inequalities (2.6) have the same analytic properties.

Example 2.1

Consider the problem of *optimal consumer choice* [10]. In essence, we are talking about the purchase by the buyer of two goods in quantity x_1, x_2 at the appropriate price c_1, c_2 in order to maximize the Cobb-Douglas utility function while observing the budget constraint. Mathematically, this is a nonlinear programming problem

$$\begin{gathered} f_0(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha} \to \max, \\ c_1 x_1 + c_2 x_2 \leq b, \; x_1 \geq 0, x_2 \geq 0 \end{gathered} \tag{2.7}$$

with parameters $\alpha, b, c_1, c_2, \alpha > 0$, $\alpha < 1$. Problem (2.7) has a well-known [10] solution

$$\begin{gathered} x_1^{(0)} = \alpha b / c_1, x_2^{(0)} = (1-\alpha) b / c_2, \\ f_0^{(0)} = b \left(\alpha / c_1\right)^{\alpha} \left((1-\alpha)/c_2\right)^{1-\alpha}. \end{gathered} \tag{2.8}$$

Sociological surveys testify [7] about approximate perception the price of purchased goods by buyers. Therefore, we replace the fixed prices c_1, c_2 in the budget constraint by interval ones $\boldsymbol{c}_1, \boldsymbol{c}_2$ and consider the interval interpretation of the problem

$$\begin{gathered} f_0(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha} \to \max, \\ \boldsymbol{c}_1 x_1 + \boldsymbol{c}_2 x_2 \leq b, \; x_1 \geq 0, x_2 \geq 0. \end{gathered} \tag{2.9}$$

For small $\varepsilon > 0$, we regard the intervals of prices

$$\boldsymbol{c}_1 = [c_1 - \varepsilon c_1, c_1 + \varepsilon c_1], \; \boldsymbol{c}_2 = [c_2 - \varepsilon c_2, c_2 + \varepsilon c_2]$$

for the buyer indistinguishable and acceptable. Requiring the fulfillment of the interval inequality (2.9) with indicator not less than r, $0 \le r \le 1$, we arrive at the reduced problem

$$\begin{aligned} &u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha} \to \max, \\ &c_1 x_1 + c_2 x_2 \le b/(1+\varepsilon r),\ x_1 \ge 0, x_2 \ge 0. \end{aligned} \tag{2.10}$$

Problem (2.10) differs from the original problem (2.7) only in the budget constraint. Replacing the parameter b in formulas (2.8) by $b/(1+\varepsilon r)$, we obtain the solution of the reduced problem

$$\begin{aligned} &x_1^{(r)} = x_1^{(0)}/(1+\varepsilon r),\ x_2^{(r)} = x_2^{(0)}/(1+\varepsilon r), \\ &f_0^{(r)} = f_0^{(0)}/(1+\varepsilon r). \end{aligned} \tag{2.11}$$

Formulas (2.11) define a family of solutions depending on the parameter r. On the interval $0 \le r \le 1$, function $r \to 1/(1+\varepsilon r)$ is positive and strictly decreasing so as r increases the utility of solutions (2.11) and the quantity of purchased goods decrease while the probability of fulfilling the budget constraint increases.

Thus, a buyer has the opportunity to choose at his own discretion from this family any intermediate solutions between highly useful and highly risky $(r = 0)$ and less useful and less risky $(r = 1)$.

Example 2.2

Let $x \to \boldsymbol{a}x^2 + \boldsymbol{b}x + \boldsymbol{c}$ be a quadratic function of the argument $x \in \mathrm{R}$ with interval coefficients $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathrm{IR}$. For certainty, we assume $\boldsymbol{a} \subset (0, \infty)$. We represent the problem of minimizing a quadratic function as an interval nonlinear programming problem

$$\begin{aligned} &f_0(x, y) = y \to \min, \\ &\boldsymbol{f}_1(x, y) = \boldsymbol{a}x^2 + \boldsymbol{b}x + \boldsymbol{c} - y \le 0. \end{aligned} \tag{2.12}$$

Fulfilling the necessary arithmetic operations with the ends of the intervals $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ we find the center and radius of the function $\boldsymbol{f}_1(x, y)$:

$$f_{10}(x,y) = a_0x^2 + b_0x + c_0 - y\,,$$
$$\Delta f_1(x,y) = \Delta a x^2 + \Delta b|x| + \Delta c\,.$$

Using formulas (2.5), we form a reduced parametric problem of nonlinear programming

$$\begin{aligned} &f_0(x,y) = y \to \min, \\ &a_0x^2 + b_0x + c_0 - y + r\left(\Delta a x^2 + \Delta b|x| + \Delta c\right) \le 0 \end{aligned} \tag{2.13}$$

or in the equivalent form

$$y = (a_0 + r\Delta a)x^2 + b_0x + r\Delta b|x| + c_0 + r\Delta c \to \min. \tag{2.14}$$

Solution $x^{(r)}$, $y^{(r)}$ of the problem (2.14) is

$$\begin{aligned} &x^{(r)} = -\frac{b_0 + r\Delta b}{2(a_0 + r\Delta a)},\ y^{(r)} = -\frac{(b_0 + r\Delta b)^2}{4(a_0 + r\Delta a)} + c_0 + r\Delta c\ \left(b_0 + r\Delta b \le 0\right); \\ &x^{(r)} = -\frac{b_0 - r\Delta b}{2(a_0 + r\Delta a)},\ y^{(r)} = -\frac{(b_0 - r\Delta b)^2}{4(a_0 + r\Delta a)} + c_0 + r\Delta c\ \left(b_0 - r\Delta b \ge 0\right). \end{aligned} \tag{2.15}$$

For $r = 0$, problems (2.13) and (2.14) become central − they contain the centers of the intervals $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$. In this case, formulas (2.15) determine the extremum of the «ordinary» square trinomial. A more detailed analysis of quadratic interval functions is available in [5].

3.3. Interval linear programming

Consider the interval linear programming problem [6, 11]

$$\sum_{j=1}^{n} c_j x_j \to \min, \sum_{j=1}^{n} \boldsymbol{a}_{ij} x_j \le \boldsymbol{b}_i,\ i = 1,...m,\ x_j \ge 0,\ j = 1,...n\,. \tag{2.16}$$

In standard vector-matrix notation, the problem takes the form

$$c'x \to \min,\ \boldsymbol{A}x \le \boldsymbol{b},\ x \ge 0\,,$$

where $\boldsymbol{A} \in \mathrm{IR}^{m\times n}$ − given matrix, $\boldsymbol{b} \in \mathrm{IR}^{m}$, $c \in \mathrm{R}^{n}$ − given vectors, $x \in \mathrm{R}^{n}$ − sought-for vector, $c'x$ − dot product of vectors *c*, *x*.

In the economic interpretation, the non-interval problem (2.16) is to minimize the production costs of *n* types of goods with a known technological matrix *A*, resource vector *b* and cost vector *c*. The interval version of the model (2.16) arises in the approximate calculation or expert evaluation of the matrix *A* and the vectors *b*, *c*. The problem with the interval vector ***c*** can easily be reduced to the form (2.16) by introducing an additional minimized unknown limiting the interval function $\boldsymbol{c}'x$ from above.

Let us take up the reduction of problem (2.16). We introduce the parameter $r \in [-1,1]$. We require that the indicator of fulfilling of each interval inequality (2.16) be at least *r*:

$$R\left(\sum_{j=1}^{n} \boldsymbol{a}_{ij} x_j \le \boldsymbol{b}_i\right) \ge r,\ i = 1,...m. \tag{2.17}$$

Assuming

$$\boldsymbol{a}_{ij} = [a_{ij0} - \Delta a_{ij}, a_{ij0} + \Delta a_{ij}],\ \boldsymbol{b}_i = [b_{i0} - \Delta b_i, b_{i0} + \Delta b_i],$$
$$i = 1,...,m,\ j = 1,...,n,$$

and taking into account the non-negativity of unknowns $x_j, j = 1,...,n$, by the rules of interval arithmetic, we find

$$\sum_{j=1}^{n} \boldsymbol{a}_{ij} x_j = \left[\sum_{j=1}^{n} a_{ij0} x_j - \sum_{j=1}^{n} \Delta a_{ij} x_j,\ \sum_{j=1}^{n} a_{ij0} x_j + \sum_{j=1}^{n} \Delta a_{ij} x_j\right],\ i = 1,...,m. \tag{2.18}$$

Using definition (1.3) of the indicator *R* and formula (2.18), we represent inequalities (2.17) in the form

$$b_{i0} - \sum_{j=1}^{n} a_{ij0} x_j \ge r\left(\sum_{j=1}^{n} \Delta a_{ij} x_j + \Delta b_i\right),\ i = 1,...,m,$$

and then, in vector-matrix form

$$(A_0 + r\Delta A)x \le b_0 - r\Delta b.$$

As a result, we arrive at the reduced linear programming problem

$$c'x \to \min,\ (A_0 + r\Delta A)x \le b_0 - r\Delta b,\ x \ge 0 \tag{2.19}$$

with parameter $r \in [-1,1]$.

As before, we denote the set of solutions x of the system of inequalities (2.19) for a fixed r by X_r. By analogy with Section 2.2, it is easy to check the monotonicity of non-empty sets X_r by inclusion:

$$\Big((-1 \le r_1 \le r_2 \le 1)\,\&\,(X_{r_2} \ne \varnothing)\Big) \Rightarrow \Big((X_{r_1} \ne \varnothing)\,\&\,(X_{r_1} \supset X_{r_2})\Big). \tag{2.20}$$

Let the implication (2.20) be satisfied. Then, from the property of the minimum and the monotonicity of the function $\pi(r)$ of the form (1.5) – (1.7), we have

$$\min_{x \in X_{r1}} c'x \le \min_{x \in X_{r2}} c'x,$$

$$0 \le \pi(r_1) \le \pi(r_2) \le 1.$$

Thus, if the linear programming problem (2.19) is solvable for the values of the parameter $r \in [-1, 1]$, then, as r increases, the minimum of the objective function increases and the lower estimate $\pi(r)$ of the probability of fulfillment of each interval inequality (2.17) increases.

CONCLUSION

In this paper we considered some interval decision-making models. To highlight rationally justified the solution of the problem, a numerical indicator of comparing intervals for «more» or «less» is used. The indicator allows to convert the initial interval model to a deterministic model of the same type with parameters. Due to this, the solutions of the deterministic model can be meaningfully interpreted in terms of the probability of their implementation, which is important for the final choice of the solution.

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