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ON GENERALIZED DEGENERATE TYPE 2 EULER POLYNOMIALS

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ABSTRACT. Recently, Kim-Kim studied the generalized degenerate Euler-Genocchi polynomials. In this paper we define the generalized degenerate type 2 Euler polynomials as a degenerate version of the Euler-Genocchi polynomials. In addition, we define their higher-order version, namely the generalized degenerate type 2 Euler-Genocchi polynomials of order α. The aim of this paper is to investigate some properties and identities involving those polynomials the generalizes falling factorials, the degenerate type 2 Euler polynomials of order α, the degenerate Stirling numbers of the second kind and the alternating degenerate power sum of integers.

1. INTRODUCTION

Masjed-Jamei et. al.[1] studied the new type Euler polynomials and their properties, and also we studied the type 2 degenerate cosine-Euler and the type 2 degenerate sine-Euler polynomials in [11]. Recently, Kim et.al.[11] studied the generalized degenerate Euler-Genocchi polynomials and investigate interesting properties of them. In this paper, we introduce the generalized degenerate type 2 Euler polynomials as a degenerate version of the type 2 Euler-Genocchi polynomials and are motivated by the paper [11]. In addition, we introduce their higher-order version namely the generalized degenerate type 2 Euler-Genocchi polynomials of order α, as a degenerate version of the generalized type 2 Euler-Genocchi polynomials of order α. The aim of this paper is to study certain properties and identities involving those polynomials. The outline of this paper as follows. In section 1, we recall the degenerate exponential function, the degenerate Euler polynomials, the degenerate type 2 Euler polynomials of order α and the degenerate type 2 Genocchi polynomials of order α. The section 2 is the main result of this paper and we introduce the generalized degenerate type 2 Euler-Genocchi polynomials $l_{n,\lambda}^{(r)}(x)$, as a generalization of both the degenerate type 2 Euler polynomials and the type 2 degenerate Genocchi polynomials.
As is known, the type 2 Euler polynomials $E_n^*(x)(n \geq 0)$ and the type 2 Genocchi polynomials $G_n^*(x)(n \geq 0)$ are respectively defined by

$$\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}e^{xt}=\sum_{n=0}^{\infty}E_n^*(x)\frac{t^n}{n!},\quad (\text{see } [5,10,11,13]), \tag{1}$$

and

$$\frac{2t}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}e^{xt}=\sum_{n=0}^{\infty}G_n^*(x)\frac{t^n}{n!},\quad (\text{see } [5,6,12,14]). \tag{2}$$

When $x=0, E_n^*=E_n^*(0)$ (or $G_n^*=Gn^*(x)$) are called type 2 Euler (or type 2 Genocchi) numbers.

Key words and phrases. generalized degenerate type 2 Euler polynomials, the generalized degenerate type 2 Euler-Genocchi polynomials of order α, the degenerate Stirling numbers of the second kind.

For any nonzero $\lambda \in \mathbb{R}$, the degenerate exponential functions are defined as

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}} \quad e_\lambda(t) = e_\lambda^1(t) = (1+\lambda t)^{\frac{1}{\lambda}}, \quad \text{(see [10,15])}. \tag{3}$$

From (3), we easily get

$$e_\lambda^x(t) = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \quad \text{(see [2,15])}, \tag{4}$$

where $(x)_{0,\lambda} = 1, (x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda), \quad (n \geq 1)$.
Kim-Kim considered the degenerate type 2 Euler polynomials given by

$$\frac{2}{e_\lambda(t)+e_\lambda^{-1}(t)} e_\lambda^x(t) = \sum_{n=0}^{\infty} \mathscr{E}_{n,\lambda}(x) \frac{t^n}{n!}, \quad \text{(see [11,12,13])}. \tag{5}$$

When $x = 0, \mathscr{E}_{n,\lambda} = \mathscr{E}_{n,\lambda}(0)$ are called the degenerate type 2 Euler numbers.

For any nonzero $\alpha \in \mathbb{C}$, the degenerate type 2 Euler polynomials of order α are defined by

$$\left(\frac{2}{e_\lambda(t)+e_\lambda^{-1}(t)} \right)^\alpha e_\lambda^x(t) = \sum_{n=0}^{\infty} \mathscr{E}_{n,\lambda}^{(\alpha)}(x) \frac{t^n}{n!}, \quad \text{(see [3,12,14])}. \tag{6}$$

When $x = 0$, $\mathscr{E}_{n,\lambda}^{(\alpha)} = \mathscr{E}_{n,\lambda}^{(\alpha)}(0)$ are called the degenerate type 2 Euler numbers of order α.
Recently, the degenerate type 2 Genocchi polynomials of order α are defined by

$$\left(\frac{2t}{e_\lambda(t)+e_\lambda^{-1}(t)} \right)^\alpha e_\lambda^x(t) = \sum_{n=0}^{\infty} g_{n,\lambda}^{(\alpha)}(x) \frac{t^n}{n!}. \quad \text{(see [6,12,14])}. \tag{7}$$

When $x = 0$, $g_{n,\lambda}^{(\alpha)} = g_{n,\lambda}^{(\alpha)}(0)$ are called the degenerate type 2 Genocchi numbers of order α.
In particular $\alpha = 1$, $g_{n,\lambda}(x) = g_{n,\lambda}^{(1)}(x)$ are called the degenerate type 2 Genocchi polynomials.

For $n \geq 0$, the degenerate Stirling numbers of the second kind are introduced by Kim-Kim as

$$(x)_{n,\lambda} = \sum_{k=0}^{n} S_{2,\lambda}(n,k)(x)_k, \quad \text{(see [7,8,9])}, \tag{8}$$

where $(x)_0 = 1, \quad (x)_n = x(x-1)\cdots(x-n+1), \quad (n \geq 1)$.

For $k \geq 0$, the incomplete Bell polynomials are defined by

$$\frac{1}{k!} \left(\sum_{i=1}^{\infty} x_i \frac{t^i}{i!} \right)^k = \sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \cdots x_{n-k+1}) \frac{t^n}{n!}, \quad \text{(see [4,7,16])}, \tag{9}$$

where

$$B_{n,k}(x_1, x_2, \cdots, x_{n-k+1}) = \sum_{l_1+\cdots+(n-k+1)l_{n-k+1}=n} \frac{n!}{l_1! l_2! \cdots l_{n-k+1}!} \left(\frac{x_1}{1!} \right)^{l_1} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!} \right)^{l_{n-k+1}}.$$

2. The generalized degenerate type 2 Euler-Genocchi numbers and polynomials

For $r \in \mathbb{Z}$ with $r \geq 0$, we consider the generalized degenerate type 2 Euler-Genocchi polynomials given by

$$\frac{2t^r}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^x(t)=\sum_{n=0}^{\infty} l_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}. \tag{10}$$

Note that $l_{0,\lambda}^{(r)}=l_{1,\lambda}^{(r)}(x)=\cdots=l_{r-1,\lambda}^{(r)}=0$.

When $x=0$, $l_{n,\lambda}^{(r)}=l_{n,\lambda}^{(r)}(0)$ are called the generalized degenerate type 2 Euler-Genocchi numbers and note that $l_{n,\lambda}^{(0)}=\mathscr{E}_{n,\lambda}(x)$, $l_{n,\lambda}^{(1)}(x)=g_{n,\lambda}(x), \quad (n\geq 0)$.

From (10), we have

$$\begin{aligned}\sum_{n=0}^{\infty} l_{n,\lambda}^{(r)}(x+1)\frac{t^n}{n!}&=\frac{2t^r}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^{x+1}(t)\\&=\frac{2t^r}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^{x}(t)e_\lambda(t)\\&=\left(\sum_{m=0}^{\infty} l_{m,\lambda}^{(r)}(x)\frac{t^m}{m!}\right)\left(\sum_{i=0}^{\infty}(1)_{i,\lambda}\frac{t^i}{i!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}l_{m,\lambda}^{(r)}(x)(1)_{n-m,\lambda}\right)\frac{t^n}{n!}.\end{aligned}\tag{11}$$

Therefore, by comparing the coefficients on both sides of (11), we obtain the following theorem.

Theorem 1. *For $n\geq 0$, we have*

$$l_{n,\lambda}^{(r)}(x+1)=\sum_{m=0}^{n}\binom{n}{m}l_{m,\lambda}^{(r)}(x)(1)_{n-m,\lambda}, \quad (n\geq 0).$$

From (10), we get

$$\begin{aligned}\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}&=e_\lambda^x(t)=\frac{e_\lambda(t)+e_\lambda^{-1}(t)}{2t^r}\sum_{m=0}^{\infty} l_{m,\lambda}^{(r)}(x)\frac{t^m}{m!}\\&=\frac{1}{2t^r}\left(\sum_{m=r}^{\infty} l_{m,\lambda}^{(r)}(x)\frac{t^m}{m!}\right)\left(e_\lambda(t)+e_\lambda^{-1}(t)\right)\\&=\frac{1}{2t^r}\left(\sum_{m=0}^{\infty} l_{m+r,\lambda}^{(r)}(x)\frac{t^{m+r}}{(m+r)!}\right)\left(\sum_{k=0}^{\infty}(1)_{k,\lambda}\frac{t^k}{k!}+\sum_{k=0}^{\infty}(1-e_\lambda(t))^k\right)\\&=\frac{1}{2}\left(\sum_{m=0}^{\infty} l_{m+r,\lambda}^{(r)}(x)\frac{t^{m}}{(m+r)!}\right)\left(\sum_{k=0}^{\infty}(1)_{k,\lambda}\frac{t^k}{k!}+\sum_{k=0}^{\infty}\sum_{i=k}^{\infty}S_{2,\lambda}(i,k)(-1)^k k!\frac{t^i}{i!}\right)\\&=\frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{l_{m+r,\lambda}^{(r)}(x)}{(m+r)!}\frac{(1)_{n-m,\lambda}}{(n-m)!}\right)t^m\\&+\frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{l_{m+r,\lambda}^{(r)}(x)}{(m+r)!}\sum_{k=0}^{n-m}\frac{S_{2,\lambda}(n-m,k)}{(n-m)!}(-1)^k k!\right)t^n\end{aligned}\tag{12}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n+r}{m+r}\frac{l_{m+r,\lambda}^{(r)}(x)}{(n+r)_r}(1)_{n-m,\lambda}\right)\frac{t^n}{n!}$$
$$+\frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{n-m}\binom{n+r}{m+r}\frac{l_{m+r,\lambda}^{(r)}(x)}{(n+r)_r}S_{2,\lambda}(n-m,k)k!\right)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\left\{\frac{1}{2}\sum_{m=0}^{n}\binom{n+r}{m+r}\frac{l_{m+r,\lambda}^{(r)}(x)}{(n+r)_r}\left((1)_{n-m,\lambda}+\sum_{k=0}^{n-m}S_{2,\lambda}(n-m,k)k!\right)\right\}\frac{t^n}{n!}.$$

Therefore, by comparing the coefficients on both sides of (2), we obtain the following theorem.

Theorem 2. *For $n \geq 0$, we have*

$$(x)_{n,\lambda} = \frac{1}{2(n+r)_r}\sum_{m=0}^{n}\binom{n+r}{m+r}l_{m+r,\lambda}^{(r)}(x)\left((1)_{n-m,\lambda}+\sum_{k=0}^{n-m}S_{2,\lambda}(n-m,k)k!\right).$$

For $m \in \mathbb{N}$ with $m \equiv 1(mod2)$, we have

$$\sum_{n=0}^{\infty} l_{n,\lambda}^{(r)}(x)\frac{t^n}{n!} = \frac{2t^r}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^x(t)\times\frac{e_\lambda(t)}{e_\lambda(t)}$$
$$= \frac{2t^r}{e_\lambda^2(t)+1}e_\lambda^{x+1}(t) = \frac{2t^r}{1+e_\lambda^{2m}(t)}\sum_{k=0}^{m-1}(-1)^k e_\lambda^{2k+x+1}(t)$$
$$(13) \qquad = \frac{2(mt)^r}{1+e_{\frac{\lambda}{m}}^2(mt)}\frac{1}{m^r}\sum_{k=0}^{m-1}(-1)^k e_{\frac{\lambda}{m}}^{\frac{2k+x+1}{m}}(mt)$$
$$= \frac{1}{m^r}\sum_{k=0}^{m-1}(-1)^k\sum_{n=0}^{\infty} l_{n,\frac{\lambda}{m}}^{(r)}\left(\frac{2k+x+1}{m}-1\right)m^n\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty} m^{n-r}\sum_{k=0}^{m-1}(-1)^k l_{n,\frac{\lambda}{m}}^{(r)}\left(\frac{2k+x+1}{m}-1\right)\frac{t^n}{n!}.$$

Therefore, by comparing the coefficients on both sides of (13), we obtain the following theorem.

Theorem 3. *For $m \in \mathbb{N}$ with $m \equiv 1(mod2)$, we have*

$$l_{n,\lambda}^{(r)}(x) = m^{n-r}\sum_{k=0}^{m-1}(-1)^k l_{n,\frac{\lambda}{m}}^{(r)}\left(\frac{2k+x+1}{m}-1\right).$$

For nonzero $\alpha \in \mathbb{C}$, and $r \in \mathbb{Z}$ with $r \geq 0$, we consider the generalized degenerate type 2 Euler Genocchi polynomials of order α which are given by

$$(14) \qquad t^r\left(\frac{2}{e_\lambda(t)+e_\lambda^{-1}(t)}\right)^\alpha e_\lambda^x(t) = \sum_{n=0}^{\infty} l_{n,\lambda}^{(r,\alpha)}(x)\frac{t^n}{n!}.$$

When $x = 0$, $l_{n,\lambda}^{(r,\alpha)} = l_{n,\lambda}^{(r,\alpha)}(0)$ are called the generalized degenerate type 2 Euler-Genocchi numbers of order α.

From (14), we note that

$$\begin{aligned}\sum_{n=0}^{\infty} l_{n,\lambda}^{(r,\alpha)}(x)\frac{t^{n!}}{n!} &= t^r\left(\frac{2}{e_\lambda(t)+e_\lambda^{-1}(t)}\right)^{\alpha} e_\lambda^x(t)\\ &= \left(\sum_{m=0}^{\infty} l_{m,\lambda}^{(r,\alpha)}\frac{t^{m!}}{m!}\right)\left(\sum_{k=0}^{\infty}(x)_{k,\lambda}\frac{t^k}{k!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m} l_{m,\lambda}^{(r,\alpha)}(x)_{n-m,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{15}$$

Therefore, by comparing the coefficients on both sides of (15), we obtain the following theorem.

Theorem 4. *For $n \geq 0$, we have*

$$\begin{aligned}l_{n,\lambda}^{(r,\alpha)}(x) &= \sum_{m=0}^{n}\binom{n}{m} l_{m,\lambda}^{(r,\alpha)}(x)_{n-m,\lambda}\\ &= \sum_{m=0}^{n}\binom{n}{m} l_{n-m,\lambda}^{(r,\alpha)}(x)_{m,\lambda}.\end{aligned}$$

Let $\alpha = -m(m \in \mathbb{N})$ and by (6), we get

$$\begin{aligned}\sum_{n=0}^{\infty} l_{n,\lambda}^{(r,-m)}(x)\frac{t^n}{n!} &= t^r\left(\frac{2}{e_\lambda(t)+e_\lambda^{-1}(t)}\right)^{-m} e_\lambda^x(t) = \frac{t^r}{2^m}(e_\lambda+e_\lambda^{-1}(t))^m e_\lambda^x(t)\\ &= \frac{t^r}{2^m}e_\lambda^x(t)\sum_{l=0}^{m}\binom{m}{l}e_\lambda^{m-l}(t)e_\lambda^{-l}(t) = \frac{t^r}{2^m}e_\lambda^x(t)\sum_{l=0}^{m}\binom{m}{l}e_\lambda^{m-2l}(t)\\ &= \frac{t^r}{2^m}\sum_{l=0}^{m}\binom{m}{l}e_\lambda^{m-2l+x}(t) = \frac{t^r}{2^m}\sum_{l=0}^{m}\binom{m}{l}\sum_{n=0}^{\infty}(m-2l+x)_{n,\lambda}\frac{t^n}{n!}\\ &= \sum_{n=r}^{\infty}\sum_{l=0}^{m}\frac{1}{2^m}\binom{m}{l}(m-2l+x)_{n-r,\lambda}\frac{n!}{(n-r)!}\frac{t^n}{n!}\\ &= \sum_{n=r}^{\infty}\frac{1}{2^m}\sum_{l=0}^{m}\binom{m}{l}(m-2l+x)_{n-r,\lambda}(n)_r\frac{t^n}{n!}.\end{aligned} \tag{16}$$

Therefore, by comparing the coefficients on both sides of (16), we obtain following theorem.

Theorem 5. *For $m \in \mathbb{N}$, we have*

$$l_{n,\lambda}^{(r,-m)}(x) = \frac{(n)_r}{2^m}\sum_{l=0}^{m}\binom{m}{l}(m-2l+x)_{n-r,\lambda},$$

when $x = 0$, we have

$$l_{n,\lambda}^{(r,-m)} = \frac{(n)_r}{2^m}\sum_{l=0}^{m}\binom{m}{l}(m-2l)_{n-r,\lambda}.$$

From (14) and (16), we have

$$
\begin{aligned}
l_{n,\lambda}^{(r,-m)}(x) &= \sum_{m=0}^{n} \binom{n}{m} l_{n-m,\lambda}^{(r,-m)}(x)_{m,\lambda} \\
&= \sum_{m=0}^{n} \binom{n}{m} \frac{(n)_r}{2^m} \sum_{l=0}^{m} \binom{m}{l} (m-2l)_{n-r,\lambda}(x)_{m,\lambda} \\
&= \frac{1}{2^m} \sum_{m=0}^{n} \sum_{l=0}^{m} \binom{n}{m} \binom{m}{l} (n-m+2l)_{r,\lambda}(x)_{m,\lambda}(m-2l)_{n-m-r}.
\end{aligned} \tag{17}
$$

By (17), we get

$$
\begin{aligned}
\sum_{n=0}^{\infty} l_{n,\lambda}^{(r,\alpha)}(x) \frac{t^n}{n!} &= t^r \left(\frac{2}{e_\lambda(t) + e_\lambda^{-1}(t)} \right)^{\alpha} e_\lambda^x(t) = t^r \sum_{n=0}^{\infty} E_{n,\lambda}^{(\alpha)}(x) \frac{t^n}{n!} \\
&= t^r \sum_{n=r}^{\infty} E_{n-r,\lambda}^{(\alpha)}(x) \frac{t^{n-r}}{n!} = \sum_{n=r}^{\infty} E_{n-r,\lambda}^{(\alpha)}(x) \frac{t^n}{(n-r)!} \frac{n!}{n!} \\
&= \sum_{n=r}^{\infty} E_{n-r,\lambda}^{(\alpha)}(x)(n)_r \frac{t^n}{n!}.
\end{aligned} \tag{18}
$$

Theorem 6. *For $n, r \geq 0$ with $n \geq r$, we have*

$$
l_{n,\lambda}^{(r,\alpha)}(x) = (n)_r E_{n-r,\lambda}^{(\alpha)}(x).
$$

In particular for $x = 0$, we get

$$
l_{n,\lambda}^{(r,\alpha)} = (n)_r E_{n-r,\lambda}^{(\alpha)}. \tag{19}
$$

By (15) and (19), we get

$$
\begin{aligned}
l_{n,\lambda}^{(r,\alpha)}(x) &= \sum_{m=0}^{n} \binom{n}{m} l_{n-m,\lambda}^{(r,\alpha)}(x)_{m,\lambda} \\
&= \sum_{m=0}^{n-r} \binom{n}{m} l_{n-m}^{(r,\alpha)}(x)_{m,\lambda} = \sum_{m=0}^{n-r} \binom{n}{m} (n-m)_r E_{n-m-r,\lambda}^{(\alpha)}(x)_{m,\lambda} \\
&= (n)_r \sum_{m=0}^{n-r} \binom{n-r}{m} E_{n-m-r,\lambda}^{(\alpha)}(x)_{m,\lambda} \\
&= (n)_r (x)_{n-r,\lambda} + (n)_r \sum_{m=0}^{n-r-1} \binom{n-r}{m} E_{n-m-r,\lambda}^{(\alpha)}(x)_{m,\lambda}.
\end{aligned} \tag{20}
$$

Therefore, by(20), we obtain the following theorem.

Theorem 7. *For any nonzero $\alpha \in \mathbb{C}$ and $n, r \geq 0$ with $n \geq r$, we have*

$$
l_{n,\lambda}^{(r,\alpha)}(x) = (n)_r (x)_{n-r,\lambda} + (n)_r \sum_{m=0}^{n-r-1} \binom{n-r}{m} E_{n-m-r,\lambda}^{(\alpha)}(x)_{m,\lambda}.
$$

Now, we consider the degenerate higher-order type 2 Euler numbers and note that

$$
\begin{aligned}
\sum_{n=0}^{\infty} E_{n,\lambda}^{(\alpha)} \frac{t^n}{n!} &= \left(\frac{2}{e_\lambda(t)+e_\lambda^{-1}(t)}\right)^{\alpha} = \left(\frac{1}{2}\left(e_\lambda(t)+e_\lambda^{-1}(t)\right)\right)^{-\alpha} \\
&= \left(\frac{1}{2}\left(e_\lambda^2(t)+1\right)\right)^{-\alpha} = \left(\frac{1}{2}\left(1+\sum_{i=0}^{\infty}(2)_{i,\lambda}\frac{t^i}{i!}\right)\right)^{-\alpha} \\
&= \left(1+\sum_{i=1}^{\infty}\frac{1}{2}(2)_{i,\lambda}\frac{t^i}{i!}\right)^{-\alpha} = \left(1+\sum_{i=1}^{\infty}2^i(1)_{i,\lambda}\frac{t^i}{i}\right)^{-\alpha} \\
&= 1+\sum_{k=1}^{\infty}(-\alpha)_k 2^k \frac{1}{k!}\left(\sum_{i=1}^{\infty}(1)_{i,\lambda}\frac{t^i}{i!}\right)^k \\
&= 1+\sum_{k=1}^{\infty}(-\alpha)_k 2^k \sum_{n=k}^{\infty} B_{n,k}\left((1)_{1,\lambda},(1)_{2,\lambda}\cdots(1)_{n-k+1,\lambda}\right)\frac{t^n}{n!} \\
&= 1+\sum_{n=1}^{\infty}\sum_{k=1}^{n}(-\alpha)_k 2^k B_{n,k}\left((1)_{1,\lambda},(1)_{2,\lambda}\cdots(1)_{n-k+1,\lambda}\right)\frac{t^n}{n!}.
\end{aligned} \tag{21}
$$

Therefore, by comparing the coefficients on both sides of (21), we obtain the following theorem.

Theorem 8. *For $n \geq 0$, we have*

$$
\begin{aligned}
E_{n,\lambda}^{(\alpha)} &= \sum_{k=1}^{n}(-\alpha)_k 2^k B_{n,k}\left((1)_{1,\lambda},(1)_{2,\lambda}\cdots(1)_{n-k+1,\lambda}\right) \\
&= \sum_{k=1}^{n}(-\alpha)_k 2^k S_{2,\lambda}(n,k).
\end{aligned}
$$

From Theorem 8, we have

$$
\begin{aligned}
l_{n,\lambda}^{(r,\alpha)}(x) &= (n)_r(x)_{n-r,\lambda} + (n)_r \sum_{m=0}^{n-r-1}\binom{n-r}{m} E_{n-m-r,\lambda}^{(\alpha)}(x)_{m,\lambda} \\
&= (n)_r(x)_{n-r,\lambda} + (n)_r \sum_{m=0}^{n-r-1}\binom{n-r}{m}(x)_{m,\lambda}\sum_{j=1}^{n-m-r}(-\alpha)_j 2^j S_{2,\lambda}(n-m-r,j) \\
&= (n)_r(x)_{n-r,\lambda} + (n)_r \sum_{m=0}^{n-r-1}\sum_{j=1}^{n-m-r}\binom{n-r}{m}(-\alpha)_j 2^j S_{2,\lambda}(n-m-r,j)(x)_{m,\lambda}.
\end{aligned} \tag{22}
$$

and

$$
l_{n,\lambda}^{(r,\alpha)} = (n)_r \sum_{j=1}^{n-r}(-\alpha)_j 2^j S_{2,\lambda}(n-r,j), \tag{23}
$$

where, $n, r \in \mathbb{Z}$ with $n > r \geq 0$.
Replacing n by $n+r$ in (23) , we get

$$
l_{n+r,\lambda}^{r,\alpha}(x) = (n+r)_r(x)_{n,\lambda} + (n+r)_r \sum_{m=0}^{n-1}\sum_{j=1}^{n-m}\binom{n}{m}(-\alpha)_j 2^j S_{2,\lambda}(n-m,j)(x)_{m,\lambda}, \quad (n \geq 1). \tag{24}
$$

Thus we have

$$(x)_{n,\lambda}=\frac{1}{(n+r)_r}l^{(r,\alpha)}_{n+r,\lambda}(x)-\sum_{m=0}^{n-1}\sum_{j=1}^{n-m}\binom{n}{m}(-\alpha)_j2^jS_{2,\lambda}(n-m,j)(x)_{m,\lambda},\quad (n\geq 1). \tag{25}$$

Therefore, by (25), we obtain the following theorem.

Theorem 9. *For $n\geq 1$, we have*

$$(x)_{n,\lambda}=\frac{1}{(n+r)_r}l^{(r,\alpha)}_{n+r,\lambda}(x)-\sum_{m=0}^{n-1}\sum_{j=1}^{n-m}\binom{n}{m}(-\alpha)_j2^jS_{2,\lambda}(n-m,j)(x)_{m,\lambda},\quad (n\geq 1).$$

3. Conclusion

Recently, many researchers studied the degenerate version of special polynomials and numbers and Kim-Kim-Kim introduced the generalized degenerate Euler-Genocchi polynomials. The aim of this paper, we considered the generalized degenerate type 2 Euler polynomials as the modified version of the generalized degenerate Euler-Genocchi polynomials. In addition, we define their higher-order version, namely the generalized degenerate type 2 Euler-Genocchi polynomials of order α and investigated some properties and identities involving those polynomials the generalizes falling factorials, the degenerate type 2 Euler polynomials of order α, the degenerate Stirling numbers of the second kind and the alternating degenerate power sum of integers.

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