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A STUDY ON RECURRENCE RELATIONS FOR BINARY SEQUENCE MATRICES

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Abstract. This paper compares and contrasts some properties of binary sequences with matrices and associated recurrence relations in order to stimulate some enrichment exercises and pattern puzzles by developing some new properties of well-known sequences.

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1. Introduction

Binary numbers are reasonably well known as in Table 1 in which

$B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\breve{B}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ $\breve{B}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$	$B_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$	$\breve{B}_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Table 1. Binary matrices (arrays)

so that more generally

$$B_n + \breve{B}_n = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}_{n \times n.}$$

These binary matrices may also be generated by the first order recurrence relation

$$B_{n+1} = \begin{bmatrix} 0_{2^n \times 1} & B_n \\ 1_{2^n \times 1} & B_n \end{bmatrix}, n>1$$

with initial term

$$B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

In [9], these were related to Gray Codes and recurrence relations. Here we pursue some other recurrences with somewhat similar matrix connections.

2. Good Sequences

Austin and Guy [2] defined a 'good' sequence as an ordered set of zeros and ones in which each 'one' in it has a neighbouring 'one'; that is, if there is a 'one' in the set, then it has a 'one' next to it. a_n is then defined as the number of good sequences of length n (Table 2).

n	'Good' Sequences of length n	a_n
1	0	1
2	00,11	2
3	000,011,110,111	4
4	0000,0011,0110,1100,0111,1110,1111	7
5	00000,00011,00110,01100,11000,00111,01110,11100,01111,11110,11011,11111	12

Table 2. 'Good' sequences

The 'good' numbers, a_n, can be shown to satisfy the fourth order recurrence relation:

$$a_n = a_{n-1} + a_{n-2} + a_{n-4}. \tag{1}$$

What about investigating 'better' sequences, $\{b_n\}$ in which the requirement is that each 'one' (if present) be accompanied by two other 'ones'; that is, in blocks of length three? (This is actually using the mathematical principle of exclusion-inclusion.) Or, as Austin and Guy considered the binary sequences of length n in which the 'ones' occur only in blocks of length at least k. The elements of this are designated by $a_n^{(k)}$, so that the 'good' sequence is $\left\{a_n^{(2)}\right\}$, and we again gradually begin to see how notation becomes a tool of thought rather than an artificial burden [5]. Can you justify that

$$a_{k+n}^{(k)} = 1 + \frac{1}{2}(n+1)(n+2), \quad 0 \leq n \leq k? \tag{2}$$

These are the central polygonal numbers and a solution to the "lazy caterer's problem" [4, 6, 8].

3. Sburlati Sequences

Sburlati [7] used a recursive sequence $\{k_n\}$ defined as a repunit (Beiler [3]) by

$$k_n = \frac{1}{3}(4^n - 1). \tag{3}$$

It satisfies the second order homogeneous linear recurrence relation

$$k_n = 5k_{n-1} - 4k_{n-2}, \;\; n \geqslant 2, \; k_1 = 1, \; k_2 = 5, \tag{4}$$

which is a generalization of the well-known Fibonacci recurrence relation.

$$F_n = F_{n-1} + F_{n-2}, \;\; n \geqslant 2, \; F_1 = 1, \; F_2 = 1, \tag{5}$$

As a means of comparison, the first ten elements of the Fibonacci, Lucas and Pell sequences as well as those from Sburlati are printed in Table 3.

n	Fibonacci	Lucas	Pell	Sburlati
1	1	1	1	1
2	1	3	2	5
3	2	4	5	21
4	3	7	12	85
5	5	11	29	341
6	8	18	70	1365
7	13	29	169	5461
8	21	47	408	21845
9	34	76	985	87381
10	55	123	2378	1398101

TABLE 3. Some second order sequences

Representations of numbers in the last column of Table 3 are disguised in Table 4:

1	1
1 0 1	1 4
1 0 1 0 1	1 4 16
1 0 1 0 1 0 1	1 4 16 64
1 0 1 0 1 0 1 0 1	1 4 16 64 256

TABLE 4. A curious binary triangle and a 'decimal relative'

Let $M = [m_{ij}]$ be the $n \times n$ matrix, defined by

$$m_{ij} = \begin{cases} 4^{j-1}, & i \geq j \\ 0, & i < j \end{cases} .$$

That is,

$$M = \begin{bmatrix} 1 & & & & & \\ 1 & 4 & & & & \\ 1 & 4 & 16 & & & \\ 1 & 4 & 16 & 64 & & \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ 1 & 4 & 16 & \cdots & 4^{n-2} & 4^{n-1} \end{bmatrix}.$$

Then we get the n^{th} element of the Sburlati sequence as follows:

$$k_n = \sum_{i=1}^{n} m_{i,n}$$

What we have called the Sburlati sequence is A002450 in [10] where 'numbral arithmetic' is introduced and further developed in A048888 of [10]. Now consider the following matrix:

$$M_1 = \begin{bmatrix} 4^{n-1} & 4^{n-2} & 4^{n-3} & \cdots & 4 & 1 \\ 4^{n-2} & 4^{n-3} & \cdots & 4 & 1 & 0 \\ 4^{n-3} & \cdots & 4 & 1 & 0 & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots & \vdots \\ 4 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Then we have

$$M_1 M = \begin{bmatrix} k_n & k_n - k_1 & k_n - k_2 & \cdots & \cdots & 4^{n-1} = k_n - k_{n-1} \\ k_{n-1} & k_{n-1} - k_1 & \cdots & \cdots & 4^{n-2} = k_{n-1} - k_{n-2} & 0 \\ k_{n-2} & \cdots & \cdots & \iddots & 0 & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots & \vdots \\ k_2 & 4 = k_2 - k_1 & 0 & \cdots & 0 & 0 \\ k_1 = 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Assume that the matrix D is defined by

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 0 \\ 1 & 4 & 16 \end{bmatrix}.$$

By an inductive argument, we can then obtain

$$(D)^n = \begin{bmatrix} 1 & 0 & 0 \\ k_n & 4^n & 0 \\ d & k_n 4^n = k_{2n} - k_n & 4^{2n} \end{bmatrix},$$

for $n > 2$, where

$$d = \begin{cases} 4\left[(k_{2n-2} + k_{2n-4} + \cdots + k_{n-1}) - (k_{n-2} + k_{n-4} + \cdots + k_1)\right] + 1, & \text{if } n \text{ is odd} \\ 4\left[(k_{2n-2} + k_{2n-4} + \cdots + k_n) - (k_{n-3} + k_{n-5} + \cdots + k_1)\right] + 1, & \text{if } n \text{ is even.} \end{cases}$$

$\sum$				−1	1				−1
			−1	1	−1	1			−5
		−1	1	−1	1	−1	1		−21
	−1	1	−1	1	−1	1	−1	1	−85
				⋮					⋮

Table 5. Additive relationship

4. Concluding Comments

The next challenge for the interested reader is to develop triangles associated with the other sequences, including 'better sequences', and to play with other combinations of these sequences, such as the 2-Fibonacci sequences [1], either directly or by treating the triangles as upper and lower triangular matrices, and by adding along the diagonals of the triangles, including Pascal's triangle, as well as across rows and down columns (partial sums), or expressing the general terms in numbral arithmetic.

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