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A NOTE ON r-TRUNCATED DEGENERATE SPECIAL POLYNOMIALS

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ABSTRACT. Recently, Kim-Kim introduced truncated degenerate bell polynomials and studied their properties. Also, Kim-Kim-Kim introduced λ-analogue Stirling numbers of second kind and investigated some identities and properties of those numbers. The aim of this paper, we consider r-truncated version of degenerate bell polynomials and λ-analogue Stirling numbers of second kind and investigate some properties and identities of them.

1. INTRODUCTION

Carlitz [1] initiated the exploration of degenerate Bernoulli and Euler polynomials, which are degenerate versions of the ordinary Bernoulli and Euler polynomials In recent years, many researchers have been done truncated degenerate Bell polynomials and λ-analogue r-truncated Stirling numbers of second kind have yielded many useful arithmetical and combinatorial results.

For any $\lambda \in \mathbb{R}$ with $\lambda \neq 0$,the degenerate exponential functions are defined by

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}, e_\lambda(t) = e_\lambda^1 = (1+\lambda t)^{\frac{1}{\lambda}}. \tag{1}$$

Thus, we get

$$e_\lambda^x(t) = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \quad (\text{see}[1-9]), \tag{2}$$

where

$$(x)_{0,\lambda} = 1,\ (x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda),\ (n \geq 1).$$

In[1], Carlitz introduced the degenerate Bernoulli polynomials given by

$$\frac{t}{e_\lambda(t)-1} e_\lambda^x(t) = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!}. \tag{3}$$

Note that

$$\lim_{\lambda \to 0} \beta_{n,\lambda} = B_n(x).$$

where $B_n(x)$ are the ordinary Bernoulli polynomials given by

$$\frac{t}{e^t-1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad (\text{see}[1,2]). \tag{4}$$

From (3), we get

$$\beta_{0,\lambda}(x) = 1,\ \beta_{n,\lambda} = \sum_{l=0}^{n} \binom{n}{l} (x)_{n-l,\lambda} \beta_{l,\lambda}. \tag{5}$$

In [9,15,17], the degenerate Bell polynomials are defined by generating function to be

$$e^{x(e_\lambda(t)-1)} = \sum_{n=0}^{\infty} \mathrm{Bel}_{n,\lambda}(x) \frac{t^n}{n!}. \tag{6}$$

The degenerate Stirling numbers of the first kind are defined by

$$(x)_n=\sum_{k=0}^{n}S_{1,\lambda}(n,k)(x)_{k,\lambda},\quad (\text{see}[7,10,13]), \tag{7}$$

and

$$\frac{1}{k!}(\log_\lambda(1+t))^k=\sum_{n=k}^{\infty}S_{1,\lambda}(n,k)\frac{t^n}{n!},\quad (\text{see}[7,10,13]). \tag{8}$$

The degenerate Stirling numbers of the second kind are defined by

$$(x)_{n,\lambda}=\sum_{i=0}^{n}S_{2,\lambda}(n,i)(x)_i,\quad (n\geq 0). \tag{9}$$

and

$$\frac{1}{k!}(e_\lambda(t)-1)^k=\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!},\quad (\text{see}[7,10,13]). \tag{10}$$

Now, we consider r-truncated degenerate Stirling numbers of the second kind $S_{2,\lambda}^{[r]}(n,kr+k)$ given by

$$\frac{1}{k!}(e_\lambda(t)-\sum_{j=0}^{r-1}\frac{t^j}{j!}(1)_{j,\lambda})^k=\sum_{n=kr}^{\infty}S_{2,\lambda}^{[r]}(n,kr)\frac{t^n}{n!},\quad (\text{see}[3,4,8,9,13]). \tag{11}$$

In particular, when $r=1$, we have

$$\frac{1}{k!}(e_\lambda(t)-1)^k=\sum_{n=k}^{\infty}S_{2,\lambda}^{[1]}(n,k)\frac{t^n}{n!},\quad (\text{see}[3,4,8,9,13]). \tag{12}$$

Thus, by (11) and (12),

$$S_{2,\lambda}^{[1]}(n,k)=S_{2,\lambda}(n,k).$$

The λ-analogues of Stirling numbers of the first kind are introduced as

$$(x)_{n,\lambda}=\sum_{k=0}^{n}S_{1,\lambda}^{*}(n,k)x^k,\quad (\text{see}[10]). \tag{13}$$

As the inversion of (13), the λ-analogues of Stirling numbers of the second kind are given by

$$x^n=\sum_{k=0}^{n}S_{2,\lambda}^{*}(n,k)(x)_{k,\lambda},\quad (\text{see}[10]). \tag{14}$$

2. r-TRUNCATED DEGENERATE BELL POLYNOMIALS

In the view of some motivation of degenerate Bell polynomials, we introduce r-truncated bell polynomials which are given by generating function to be

$$e^{x(e_\lambda(t)-\sum_{j=0}^{r-1}\frac{t^j}{j!}(1)_{j,\lambda})}=\sum_{n=0}^{\infty}\mathrm{Bel}_{n,\lambda}^{[r]}(x)\frac{t^n}{n!}. \tag{15}$$

From (15), we observe that

$$\sum_{n=0}^{\infty}\mathrm{Bel}_{n,\lambda}^{[r]}(x)\frac{t^n}{n!}=e^{x(e_\lambda(t)-\sum_{j=0}^{r-1}\frac{t^j}{j!}(1)_{j,\lambda})}=\sum_{k=0}^{\infty}x^k\sum_{n=kr}^{\infty}S_{2,\lambda}^{[r]}(n,kr)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}x^kS_{2,\lambda}^{[r]}(n,kr)\right)\frac{t^n}{n!}. \tag{16}$$

Therefore, by comparing the coefficients on both sides (16), we obtain the following theorem.

Theorem 1. *For $n\in\mathbb{N}\cup\{0\}$,we have*

$$\mathrm{Bel}_{n,\lambda}^{[r]}(x)=\sum_{k=0}^{n}x^kS_{2,\lambda}^{[r]}(n,kr).$$

In particular, when r=1, we note that

$$\mathrm{Bel}_{n,\lambda}^{[1]}(x)=\sum_{k=0}^{n}x^kS_{2,\lambda}^{[1]}(n,k)=\sum_{k=0}^{n}x^kS_{2,\lambda}(n,k)=\mathrm{Bel}_{n,\lambda}(x). \tag{17}$$

By (15), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\mathrm{Bel}_{n,\lambda}^{[r]}(x)\frac{t^n}{n!}&=e^{x(e_\lambda(t)-\sum_{j=0}^{r-1}\frac{t^j}{j!}(1)_{j,\lambda})}\\&=e^{x(\sum_{k=r}^{\infty}\frac{t^k}{k!}(1)_{j,\lambda})}\\&=\sum_{l=0}^{\infty}\frac{x^l}{l!}(\sum_{k=r}^{\infty}\frac{t^k}{k!}(1)_{k,\lambda})^l\\&=\sum_{l=0}^{\infty}\frac{x^l}{l!}\underbrace{[(\sum_{k=r}^{\infty}\frac{t^k}{k!}(1)_{k,\lambda})\cdots(\sum_{k=r}^{\infty}\frac{t^k}{k!}(1)_{k,\lambda})]}_{l-\text{times}}\\&=\sum_{l=0}^{\infty}\frac{x^l}{l!}(\sum_{n=rl}^{\infty}\sum_{i_1+\cdots+i_l=n}\binom{n}{i_1\cdots i_l}(1)_{l_1,\lambda}\cdots(1)_{l_l,\lambda})\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}(\sum_{l=0}^{n}\frac{x^l}{l!}\sum_{i_1+\cdots+i_l=n}\binom{n}{i_1\cdots i_l}(1)_{l_1,\lambda}\cdots(1)_{l_l,\lambda})\frac{t^n}{n!}.\end{aligned} \tag{18}$$

Thus by (18), we get the following theorem.

Theorem 2. *For $n\in\mathbb{N}\cup\{0\}$ and $r\geq 0(r\in\mathbb{Z})$, we have*

$$\mathrm{Bel}_{n,\lambda}^{[r]}(x)=\sum_{l=0}^{n}\frac{x^l}{l!}\sum_{i_1+\cdots+i_l=n}\binom{n}{i_1\cdots i_l}(1)_{l_1,\lambda}\cdots(1)_{l_l,\lambda}.$$

From (11), we observe that

(19)

$$\begin{aligned}\sum_{n=kr}^{\infty} S_{2,\lambda}^{[r]}(n,k)\frac{t^n}{n!} &= \frac{1}{k!}(e_\lambda(t) - \sum_{j=0}^{r-1}\frac{t^j}{j!}(1)_{j,\lambda})^k \\ &= \frac{1}{k!}\sum_{m=0}^{k}\binom{k}{m}e_\lambda^{k-m}(t)(-1)^m \sum_{j_1+\cdots+j_m=0}^{r-1}\frac{(1)_{j_1,\lambda}\cdots(1)_{j_m,\lambda}}{j_1!\cdots j_m!}t^{j_1+\cdots+j_m} \\ &= \frac{1}{k!}\sum_{m=0}^{k}\binom{k}{m}\sum_{l=0}^{\infty}\frac{(k-m)_{l,\lambda}}{l!}t^l(-1)^m \sum_{j_1+\cdots+j_m=0}^{r-1}\frac{(1)_{j_1,\lambda}\cdots(1)_{j_m,\lambda}}{j_1!\cdots j_m!}t^{j_1+\cdots+j_m} \\ &= \sum_{n=0}^{\infty}\frac{1}{k!}\sum_{m=0}^{k}\binom{k}{m}(-1)^m \sum_{j_1+\cdots+j_m=0}^{r-1}\frac{n!(1)_{j_1,\lambda}\cdots(1)_{j_m,\lambda}}{j_1!\cdots j_m!}\frac{t^{j_1+\cdots+j_m+l}}{(n-(j_1+\cdots+j_m))!n!}.\end{aligned}$$

Thus, by comparing the coefficients on both sides (19), we get the following theorem.

Theorem 3. *For $n,m,k \in \mathbb{N}\cup\{0\}$,we have*

$$\frac{1}{k!}\sum_{m=0}^{k}\binom{k}{m}(-1)^m \sum_{j_1+\cdots+j_m=0}^{r-1}\frac{n!(1)_{j_1,\lambda}\cdots(1)_{j_{m},\lambda}}{j_1!\cdots j_m!}\frac{1}{n-(j_1+\cdots+j_m)!} = \begin{cases} S_{2,\lambda}^{[r]}(n,k), & (if\, n \geq kr), \\ 1, & (if\, n < kr). \end{cases}$$

From (15), we observe that

(20)

$$\begin{aligned}\sum_{n=0}^{\infty}\mathrm{Bel}_{n,\lambda}^{[r]}(x)\frac{t^n}{n!} &= e^{x(e_\lambda(t)-\sum_{i=0}^{r-1}\frac{(1)_{i,\lambda}}{i!}t^i)} = e^{-x(\sum_{i=0}^{r-1}\frac{(1)_{i,\lambda}}{i!}t^i)}e^{x(e_\lambda(t))} \\ &= \sum_{l=0}^{\infty}(-1)^l x^l\frac{1}{l!}\left[(\sum_{i=0}^{r-1}\frac{(1)_{i,\lambda}}{i!}t^i)\cdots(\sum_{i=0}^{r-1}\frac{(1)_{i,\lambda}}{i!}t^i)\right]^l e^{x(e_\lambda(t))} \\ &= \sum_{l=0}^{\infty}(-1)^l x^l\frac{1}{l!}\left[(\sum_{i_1=0}^{r-1}\frac{(1)_{i_1,\lambda}}{i!}t^{i_1})\cdots(\sum_{i_l=0}^{r-1}\frac{(1)_{i_l,\lambda}}{i!}t_l^i)\right]^l e^{x(e_\lambda(t))} \\ &= \sum_{l=0}^{\infty}(-1)^l x^l\frac{1}{l!}\sum_{m=0}^{l(r-1)}m!(\sum_{i_1+\cdots+i_l=m}\frac{(1)_{i_1,\lambda}\cdots(1)_{i_l,\lambda}}{i_1!\cdots i_l!})\frac{t^m}{m!}e^{x(e_\lambda(t))} \\ &= \sum_{m=0}^{\infty}(m!\sum_{l=0}^{[\frac{m}{r-1}]}(-1)^l x^l\frac{1}{l}\sum_{i_1+\cdots+i_l=m}\frac{(1)_{i_1,\lambda}\cdots(1)_{i_l,\lambda}}{i_1!\cdots i_l!})\frac{t^m}{m!}e^{x(e_\lambda(t))} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}m!\sum_{l=0}^{[\frac{m}{r-1}]}(-1)^l x^l\frac{1}{l!}\sum_{i_1+\cdots+i_l=m}(\frac{(1)_{i_1,\lambda}\cdots(1)_{i_l,\lambda}}{i_1!\cdots i_l!})\frac{1}{m!}\sum_{j=0}^{\infty}x^j(j)_{n-m,\lambda}\frac{1}{(n-m)!})t^{m+k} \\ &= \sum_{n=0}^{\infty}(\sum_{m=0}^{n}\sum_{l=0}^{[\frac{m}{r-1}]}(-1)^l x^l\frac{1}{l!}\sum_{i_1+\cdots+i_l=m}(\frac{(1)_{i_1,\lambda}\cdots(1)_{i_l,\lambda}}{i_1!\cdots i_l!})\frac{1}{m!}\sum_{j=0}^{\infty}x^j(j)_{n-m,\lambda}\frac{1}{(n-m)!})\frac{t^n}{n!}.\end{aligned}$$

Thus, by comparing coefficients on both sides (20), we obtain the following theorem.

Theorem 4. *For $n,m \in \mathbb{N}\cup\{0\}$,we have*

$$\mathrm{Bel}^{[r]}_{n,\lambda}(x) = \sum_{m=0}^{n}\binom{n}{m}\sum_{l=0}^{\frac{m}{r-1}}(-1)^l x^l \frac{1}{l!}\sum_{i_1+\cdots+i_l=m}\binom{m}{i_1\cdots i_l}(1)_{i_1,\lambda}\cdots(1)_{i_l,\lambda}\sum_{j=0}^{\infty}x^j (j)_{n-m,\lambda}.$$

From (15), we observe that

(21)

$$\begin{aligned}
\sum_{n=0}^{\infty} Bel^{[r]}_{n,\lambda}(x)\frac{t^n}{n!} &= e^{x(e_\lambda(t))-\sum_{i=0}^{r-1}\frac{(1)_{i,\lambda}}{i!}t^i}\\
&= e^{x(e_\lambda(t)-1+1-\sum_{i=0}^{r-1}\frac{(1)_{i,\lambda}}{i!}t^i)}\\
&= e^{x(e_\lambda(t)-1)}e^{-x(\sum_{i=1}^{r-1}\frac{(1)_{i,\lambda}}{i!}t^i)}\\
&= e^{x(e_\lambda-1)}\sum_{l=0}^{\infty}\frac{(-1)^l x^l}{l!}(\sum_{i=1}^{r-1}\frac{(1)_{i,\lambda}}{i!}t^i)^l\\
&= \sum_{k=0} Bel_{k,\lambda}(x)\frac{t^k}{k!}\sum_{l=0}^{\infty}\frac{(-1)^l x^l}{l!}(\sum_{i=1}^{r-1}\frac{(1)_{i,\lambda}}{i!}t^i)^l\\
&= \sum_{k=0}^{\infty} Bel_{k,\lambda}(x)\frac{t^k}{k!}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}x^l\sum_{m=1}^{l(r-1)}\sum_{i_1+\cdots+i_l=m}\frac{(1)_{i_1,\lambda}\cdots(1)_{i_l,\lambda}}{i_1!\cdots i_l!}t^{i_1+\cdots+i_l}\\
&= \sum_{k=0}^{\infty} Bel_{k,\lambda}(x)\frac{t^k}{k!}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}x^l(\sum_{m=1}^{l(r-1)}m!\sum_{i_1+\cdots i_l=m}\frac{(1)_{i_1,\lambda}\cdots(1)_{i_l,\lambda}}{i_1!\cdots i_l!}\frac{t^m}{m!})\\
&= \sum_{k=0}^{\infty} Bel_{k,\lambda}(x)\frac{t^k}{k!}\sum_{m=1}^{\infty}m!\sum_{l=0}^{[\frac{m}{r-1}]}\frac{(-1)^l x^l}{l!}\sum_{i_1+\cdots+i_l=m}\frac{(1)_{i_1,\lambda}\cdots(1)_{i_l,\lambda}}{i_1!\cdots i_l!}\frac{t^m}{m!}\\
&= \sum_{n=0}^{\infty} n!\sum_{m=1}^{n} Bel_{k,\lambda}(x)\frac{1}{k!(n-k)!}m!\sum_{l=0}^{[\frac{m}{r-1}]}\frac{(-1)^l x^l}{l!}\sum_{i_1+\cdots+i_l=m}\frac{(1)_{i_1,\lambda}\cdots(1)_{i_l,\lambda}}{i_1!\cdots i_l!}\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\sum_{m=1}^{n}\binom{n}{k}Bel_{k,\lambda}(x)\sum_{l=0}^{[\frac{m}{r-1}]}\frac{(-1)^l x^l}{l!}\sum_{i_1+\cdots+i_l=m}\binom{m}{i_1\cdots i_l}(1)_{i_1,\lambda}\cdots(1)_{i_l,\lambda}\frac{t^n}{n!}.
\end{aligned}$$

Thus, by comparing coefficients on both sides (21), we obtain the following theorem.

Theorem 5. *For $n \in \mathbb{N}\cup\{0\}$,we have*

$$\mathrm{Bel}^{[r]}_{n,\lambda}(x) = \sum_{m=1}^{n}\binom{n}{k}Bel_{k,\lambda}(x)\sum_{l=0}^{[\frac{m}{r-1}]}\frac{(-1)^l x^l}{l!}\sum_{i_1+\cdots+i_l=m}\binom{m}{i_1\cdots i_l}(1)_{i_1,\lambda}\cdots(1)_{i_l,\lambda}.$$

3. λ-ANALOGUE r-TRUNCATED STIRLING NUMBER OF THE SECOND KIND

For $r \in \mathbb{N}$, we consider the λ-analogue r-truncated Stirling number of the second kind given by

(22) $$\frac{1}{\lambda^k}\frac{1}{k!}(e^{\lambda t}-\sum_{l=0}^{r-1}\frac{\lambda^l t^l}{l!})^k = \sum_{n=kr}^{\infty}S^{[r]}_{2,\lambda}(n,kr)\frac{t^n}{n!}.$$

Note that

$$S^{[1]}_{2,\lambda}(n,k) = S^{*}_{2,\lambda}(n,k).$$

From (22), we have

$$\frac{1}{\lambda^k}\frac{1}{k!}(e^{\lambda t}-\sum_{l=0}^{r-1}\frac{\lambda^l t^l}{l!})^k=\frac{1}{\lambda^k}\frac{1}{k!}(\sum_{l=r}^{\infty}\frac{\lambda^l t^l}{l!})^k \tag{23}$$

$$=\frac{1}{\lambda^k}\frac{1}{k!}\sum_{n=kr}^{\infty}(\sum_{l_1+l_2+\cdots+l_k=n}\frac{n!}{l_1!l_2!\cdots l_k!}\lambda^n)\frac{t^n}{n!}.$$

Therefore, by comparing the coefficients on the both sides of (22) and (23), we obtain the following theorem.

Theorem 6. *For $n,k\geq 0$ with $n\geq kr$, we have*

$$S_{2,\lambda}^{[r]}(n,k)=\frac{1}{\lambda^k}\frac{1}{k!}\sum_{l_1+l_2+\cdots l_k=n}\frac{n!}{l_1!l_2!\cdots l_k!}\lambda^n$$

$$=\frac{1}{\lambda^k}\frac{1}{k!}\sum_{l_1+l_2+\cdots l_k=n}\binom{n}{l_1\cdots l_k}\lambda^n.$$

From (22), we have

(24)

$$\frac{1}{\lambda^k}\frac{1}{k!}(e^{\lambda t}-\sum_{l=0}^{r-1}\frac{\lambda^l t^l}{l!})^k=\frac{1}{\lambda^k}\frac{1}{k!}(e^{\lambda t}-1-\sum_{l=1}^{r-1}\frac{\lambda^l t^l}{l!})^k$$

$$=\frac{1}{\lambda^k}\frac{1}{k!}\sum_{m=0}^{k}\binom{k}{m}(e^{\lambda t}-1)^m(-1)^{k-m}(\sum_{l=1}^{r-1}\frac{\lambda^l t^l}{l!})^{k-m}$$

$$=\frac{1}{\lambda^k}\frac{1}{k!}\sum_{m=0}^{k}\frac{k!}{m!(k-m)!}(e^{\lambda t}-1)^m(-1)^{k-m}(\sum_{l=1}^{r-1}\frac{\lambda^l t^l}{l!})^{k-m}$$

$$=\frac{1}{\lambda^k}\sum_{m=0}^{k}\frac{1}{(k-m)!}\frac{1}{m!}(e^{\lambda t}-1)^m(-1)^{k-m}(\sum_{l=1}^{r-1}\frac{\lambda^l t^l}{l!})^{k-m}$$

$$=\frac{1}{\lambda^k}\sum_{m=0}^{k}\frac{1}{(k-m)!}\sum_{i=m}^{\infty}S_2(i,m)\frac{(\lambda t)^i}{i!}(-1)^{k-m}(\sum_{l=1}^{r-1}\frac{\lambda^l t^l}{l!})^{k-m}$$

$$=\frac{1}{\lambda^k}\sum_{m=0}^{k}\frac{1}{(k-m)!}\sum_{i=m}^{\infty}S_2(i,m)\frac{(\lambda t)^i}{i!}(-1)^{k-m}\sum_{j=k-m}^{(r-1)(k-m)}\sum_{l_1+\cdots l_{k-m}=j}\frac{\lambda^j t^j}{l_1!\cdots l_{k-m}!}$$

$$=\frac{1}{\lambda^k}\sum_{m=0}^{k}\frac{1}{(k-m)!}\sum_{i=m}^{\infty}S_2(i,m)\frac{(\lambda t^i)}{i!}(-1)^{k-m}\sum_{j=k-m}^{\infty}\sum_{l_1\cdots l_{r-1}=j}\frac{\lambda^j j!}{l_1!\cdots l_{k-m}!}\frac{t^j}{j!}$$

$$=\frac{1}{\lambda^k}\sum\frac{1}{(k-m)!}(-1)^{k-m}\sum_{n=k}^{\infty}(\sum_{j=k-m}^{n}\sum_{l_1+\cdots+l_{k-m}=j}\frac{j!}{l_1!\cdots l_{k-n}!}\lambda^n S_2(n-j,m)\binom{n}{j})\frac{t^n}{n!}$$

$$=\sum_{n=k}^{\infty}(\sum_{m=0}^{k}\sum_{j=k-m}^{[\frac{n}{(r-1)(k-m)}]}\sum_{l_1+\cdots+l_{k-m}=j}\frac{(-1)^{k-m}\binom{n}{j}}{(k-m)!}\lambda^{n-k}\binom{j}{l_1\cdots l_{k-n}}S_2(n-j,m))\frac{t^n}{n!}.$$

Theorem 7. *For $n,k \geq 0$, we have*

$$\sum_{m=0}^{k}\sum_{j=k-m}^{[\frac{n}{(r-1)(k-m)}]}\sum_{l_1+\cdots+l_{k-m}=j}\binom{j}{l_1\cdots l_{k-n}}\frac{(-1)^{k-m}\binom{n}{j}\lambda^{n-k}}{(k-m)!}S_2(n-j,n)=\begin{cases}S_{2,\lambda}^{[r]}(n,k), & (\text{if } n\geq kr),\\ 0, & (\text{if } 0\leq n<kr).\end{cases}$$

From (22), we note that

(25)

$$\begin{aligned}\frac{1}{\lambda^k}\frac{1}{k!}(e^{\lambda t}-\sum_{l=0}^{r-1}\frac{\lambda^l t^l}{l!})^k &= \frac{1}{\lambda^k}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}e^{\lambda jt}(-1)^{k-j}(\sum_{l=0}^{n-1}\frac{\lambda^l t^l}{l!})^{k-j}\\
&= \frac{1}{\lambda^k}\frac{1}{k!}\sum_{j=0}^{k}\frac{k!}{j!(k-j)!}\sum_{m=0}^{\infty}\frac{\lambda^m j^m t^m}{m!}(-1)^{k-j}\sum_{i=0}^{\infty}\sum_{l_1+\cdots+l_{k-j}=i}\frac{\lambda^i i!}{l_1!\cdots l_{k-j}!}\frac{t^i}{i!}\\
&= \frac{1}{\lambda^k}\frac{1}{k!}\frac{k!}{j!(k-j)!}(-1)^{k-j}\sum_{m=0}^{\infty}\lambda^m j^m\frac{t^m}{m!}\sum_{i=0}^{\infty}\sum_{l_1+\cdots+l_{k-j}=i}\binom{i}{l_1\cdots l_{k-j}}\frac{\lambda^i t^i}{i!}\\
&= \frac{1}{\lambda^k}\frac{1}{k!}\frac{k!}{j!(k-j)!}\sum_{n=0}^{\infty}\sum_{i=0}^{[\frac{n}{(r-1)(k-j)}]}\binom{n}{i}\binom{i}{l_1\cdots l_{k-j}}\lambda^n j^{n-i}\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{k}\sum_{i=0}^{[\frac{n}{(r-1)(k-j)}]}\binom{k}{j}\binom{n}{i}\left(\binom{i}{l_1\cdots l_{k-j}}\right)\right)\frac{t^n}{n!}.\end{aligned}$$

Therefore, we get the following theorem.

Theorem 8. *For $n,k \geq 0$,we have*

$$S_{2,\lambda}^{[r]}(n,kr)=\sum_{j=0}^{k}\sum_{i=0}^{[\frac{n}{(r-1)(k-j)}]}\binom{k}{j}\binom{n}{i}\binom{i}{l_1\cdots l_{k-j}}.$$

Using the binomial expansion to (22), we have

(26)

$$\begin{aligned}\frac{1}{\lambda^k}\frac{1}{k!}(e^{\lambda t}-\sum_{l=0}^{n-1}\frac{\lambda^l t^l}{l!})^k &= \frac{1}{\lambda^k}\frac{1}{k!}\sum_{m=0}^{k}\binom{k}{m}e^{\lambda t(k-m)}(-1)^m(\sum_{l=0}^{r-1}\frac{\lambda^l t^l}{l!})^m\\
&= \frac{1}{\lambda^k}\frac{1}{k!}\sum_{m=0}^{k}\binom{k}{m}e^{\lambda t(k-m)}(-1)^m(\sum_{l=0}^{r-1}\frac{\lambda^l t^l}{l!})^m\\
&= \frac{1}{\lambda^k}\frac{1}{k!}\sum_{m=0}^{k}\binom{k}{m}\sum_{j=0}^{\infty}(k-m)^j\lambda^j\frac{t^j}{j!}(-1)^m\sum_{l_1\cdots l_m=0}^{n-1}\frac{\lambda^{l_1+\cdots+l_m}}{l_1!\cdots l_m!}t^{l_1+\cdots+l_m}\\
&= \sum_{n=0}^{\infty}(\frac{1}{\lambda^k}\frac{1}{k!}\sum_{m=0}^{k}\binom{k}{m}(-1)^m\sum_{l_1\cdots l_m=0}^{n-1}\frac{n!\lambda^n(k-m)^{n-(l_1+\cdots+l_m)}}{l_1!\cdots l_m!(n-(l_1+\cdots+l_m))!})\frac{t^n}{n!}.\end{aligned}$$

Therefore, by comparing the coefficients on the both sides of (26),we obtain the following theorem.

Theorem 9. *For $n,k \geq 0$, we have*

$$\frac{1}{\lambda^k}\frac{1}{k!}\sum_{m=0}^{k}\binom{k}{m}=\begin{cases}S_{2,\lambda}^{[r]}(n,kr), & (\text{if } n\geq kr),\\ 0, & (\text{if } 0\leq n<kr).\end{cases}$$

Now, we observe k=1 in (22)

$$(e^{\lambda t}-\sum_{l=0}^{r-1}\frac{\lambda^l t^l}{l!})=\lambda\sum_{n=r}^{\infty}S_{2,\lambda}^{[r]}(n,r)\frac{t^n}{n!}$$
$$=\lambda t^r\sum_{n=0}^{\infty}S_{2,\lambda}^{[r]}(n+r,r)\frac{n!}{(n+r)!}\frac{t^n}{n!}. \tag{27}$$

For k=2, we have

$$(e^{\lambda t}-\sum_{l=0}^{r-1}\frac{\lambda^l t^l}{l!})^2=\lambda^2 t^{2r}(\sum_{j=0}^{\infty}S_{2,\lambda}^{[r]}(j+r,r)\frac{j!}{(j+r)!}\frac{t^j}{j!})(\sum_{l=0}^{\infty}S_{2,\lambda}^{[r]}(l+r,r)\frac{l!}{(l+r)!}\frac{t^l}{l!})$$
$$=\lambda^2 t^{2r}\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}S_{2,\lambda}^{[r]}(j+r,r)S_{2,\lambda}^{[r]}(n-j+r,r)\frac{n!}{(n-j+r)!(j+r)!}\right)\frac{t^n}{n!}. \tag{28}$$

Continuing this process, we have

$$(e^{\lambda t}-\sum_{l=0}^{r-1}\frac{\lambda^l t^l}{l!})=\lambda^k t^{kr}\sum_{n=0}^{\infty}\sum_{j_1+\cdots+j_k=n}\frac{S_{2,\lambda}^{[r]}(j_1+r,r)S_{2,\lambda}^{[r]}(j_2+r,r)\cdots S_{2,\lambda}^{[r]}(j_k+r,r)n!}{(j_1+r)!(j_2+r)\cdots(j_k+r)})\frac{t^n}{n!}. \tag{29}$$

On the other hand,

$$(e^{\lambda t}-\sum_{l=0}^{r-1}\frac{\lambda^l t^l}{l!})^k=\lambda^k k!\sum_{n=kr}^{\infty}S_{2,\lambda}^{[r]}(n,kr)\frac{t^n}{n!}$$
$$=\lambda^k k!\sum_{n=0}^{\infty}S_{2,\lambda}^{[r]}(n+kr,kr)\frac{t^{n+kr}}{(n+kr)!}$$
$$=\lambda^k k! l^{kr}\sum_{n=0}^{\infty}\frac{S_{2,\lambda}^{[r]}(n+kr,kr)n!}{(n+kr)!}\frac{t^n}{n!}. \tag{30}$$

Therefore, comparing the coefficients of both sides of (29) and (30), we obtain the following theorem.

Theorem 10. *For $n,k\geq 0$, we have*

$$S_{2,\lambda}^{[r]}(n+kr,kr)=(n+kr)!\sum_{j_1+\cdots+j_k=m}\frac{S_{2,\lambda}^{[r]}(j_1+r,r)\cdots S_{2,\lambda}^{[r]}(j_k+r,r)}{(j_1+r)!\cdots(j_k+r)!}.$$

Now, we consider the r-truncated Bernoulli polynomials of order k given by

$$\frac{(\lambda^r t^r)^k}{(e^{\lambda t}-\sum_{l=0}^{r-1}\frac{\lambda^l t^l}{l!})^k}e^{xrt}=\sum_{n=0}^{\infty}B_{n,\lambda}^{[r,k]}(x)\frac{t^n}{n!}. \tag{31}$$

When $x=0$, $B_{n,\lambda}^{[r,k]}=B_{n,\lambda}^{[r,k]}(0)$ are called the r-truncated Bernoulli numbers of order k.
Now, we observe that

$$\begin{aligned}\lambda^{kr}t^{kr} &= (e^{\lambda t}-\sum_{l=0}^{r-1}\frac{\lambda^l t^l}{l!})^k\sum_{l=0}^{\infty}B_{l,\lambda}^{[r,k]}\frac{t^l}{l!} \\ &= \lambda^k k!\sum_{j=kr}^{\infty}S_{2,\lambda}^{[r]}(j,kr)\sum_{l=0}^{\infty}B_{0,\lambda}^{[r,k]}\frac{t^l}{l!} \\ &= \lambda^k k!\sum_{n=kr}^{\infty}(\sum_{l=0}^{n-kr}\frac{S_{2,\lambda}^{[r]}(n-l,kr)n!}{(n-l)!l!}B_{l,\lambda}^{[r,k]})\frac{t^n}{n!} \\ &= \sum_{n=kr}^{\infty}\lambda^k k!(\sum_{l=0}^{n-kr}\binom{n}{l}S_{2,\lambda}^{[r]}(n-l,kr)B_{l,\lambda}^{[r,k]})\frac{t^n}{n!}.\end{aligned}\tag{32}$$

Therefore, we obtain the following theorem.

Theorem 11. *For $n,k\geq 0$ with $n\geq kr$,we have*

$$\sum_{l=0}^{n-kr}\lambda^k k!\binom{n}{l}S_{2,\lambda}^{[r]}(n-l,kr)B_{l,\lambda}^{[r,k]}=\begin{cases}\lambda^{kr}kr!, & (if\, n=kr)\,,\\ 0, & (if\, n<kr)\,.\end{cases}$$

From (31), we have

$$\begin{aligned}e^{xrt} &= \frac{1}{\lambda^r}\frac{1}{t^r}(e^{\lambda t}-\sum_{l=0}^{r-1}\frac{\lambda^l t^l}{l!})\sum_{l=0}^{\infty}B_{l,\lambda}^{[r,1]}(x)\frac{t^l}{l!} \\ &= \frac{1}{\lambda^r}\frac{1}{t^r}(\sum_{j=r}^{\infty}\frac{\lambda^j t^j}{j!})(\sum_{l=0}^{\infty}B_{l,\lambda}^{[r,1]}(x)\frac{t^l}{l!}) \\ &= (\sum_{j=0}^{\infty}\frac{\lambda^j t^j}{(j+r)!})(\sum_{l=0}^{\infty}B_{l,\lambda}^{[r,1]}(x)\frac{t^l}{l!}) \\ &= \sum_{n=0}^{\infty}(\sum_{j=0}^{n}\binom{n}{j}\frac{\lambda^j j!}{(j+r)!}B_{n-j,\lambda}^{[r,1]}(x))\frac{t^n}{n!}.\end{aligned}\tag{33}$$

On the other hand,

$$e^{xrt}=\sum_{n=0}^{\infty}x^n r^n\frac{t^n}{n!}.\tag{34}$$

Therefore, by comparing coefficients of the both sides of(33) and (34), we obtain the following theorem.

Theorem 12. *For, $n\geq 0$, we have*

$$x^n r^n=\sum_{j=0}^{\infty}\binom{n}{j}\frac{\lambda^j j!}{(j+r)!}B_{n-j,\lambda}^{[r-1]}(x).$$

4. Conclusion

In recent years, we have seen that degenerate versions of some special polynomials and numbers were investigated and some results were obtained by adopting various tools and also truncated degenerate special polynomials and numbers were investigated by means of various tools. In this paper, we considered the r-truncated degenerated bell polynomials and λ-analogue r-truncated Stirling number of the second kind and studied by using generating functions their explicit forms, some identities in connection with several degenerate special polynomials and numbers.

REFERENCES

[1] Carlitz, L.Degenerate Stirling, Bernoulli and Eulerian numbers. Utilitas Math. 15(1979), 51-88.

[2] Kim,D.S.;Kim,T.A note on a new type of degenerate Bernoulli numbers. Russ.J.Math.Phys.27(2020),no.2,227-235.

[3] Kim,H.K.;Back.H.;Lee,D.S. A note on truncated degenerate exponential polynomials.Proc.Jangjeon Math.Soc. 24(2021),no.1,47-61.

[4] Kim,T;Kim,D.S. Some identities on truncated polynomials associated with degenerate Bell polynomials.Russ.J.Math.Phys. 28(2021),no.3,342-355.

[5] Kim,T;Kim,D.S.;Lee,H.;Park,S;Kwon,J. New properties on degenerate Bell polynomials.Complexity 2021(2021),Article ID 7648994.

[6] Kim,D.S.;Kim,T. A note on a new type of degenerate Bernoulli numbers, Russ.J.Math.Phys. 27(2020),no.2,227-235.

[7] Kim,T. A note on degenerate Stirling polynomials of the second kind,Proc.Jangjeon Math.Soc.20(2017),no.3,319-331.

[8] Kim,T;Kim,D.S. Degenerate zero-truncated Poisson random variables, arXiv:1911.13227.

[9] Kim,T et al. Some results on r-truncated degenerate Poisson Random Variables, Fractals, Vol. 30, No. 10 (2022) doi: 10.1142/S0218348X22401922.

[10] Kim.D.S.;Kim H.;Kim T. Some identities on λ-analogues of r-Stirling numbers of the second kind, arXiv: 2205.14805v1.

[11] Kim,T.;Kim,D.S. On some degenerate differential and degenerate difference operators, Russ. J. Math. Phys. 29(2022), no.1, 37-46.

[12] Kim,T.;Kim,D.S.;Jang L.-C.;Lee,H.;Kim,H. Representations of degenerate Hermite polynomials, Adv. in Appl. Math. 139(2022), Paper No. 102359.

[13] Kim,T.;Kim,D. S.;Kim,H. Study on r-truncated degenerate Stirling numbers of the second kind, arXiv: 2206.04402v1.

[14] Kim,T.;Kim,D.S. Degenerate zero-truncated Poisson random variables, Russ. J. Math. Phys. 28(2021), no.1, 66-72.

[15] Kim,T.;Kim,D.S. Some identities on truncated polynomials associated with degenerate Bell polynomials, Russ. J. Math. Phys. 28(2021), no.3, 342-355.

[16] Kim,T.;Kim,D.S.;Kim, H.;Lee, H. Some properties on degenerate Fubini polynomials, Applied Mathematics in Science and Engineering 30(2022) no.1, 235-248.

[17] Kim,T.;Kim,D.S.;Lee,H.;Park,S.;Kwon J. New properties on degenerate Bell polynomials, Complexity 2021(2021), Article ID 7648994.

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