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# CONTINUITY CONDITIONS FOR LOCALLY BOUNDED ENDOMORPHISMS OF LINEAR LIE GROUPS

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Abstract. We prove a criterion for the continuity of a locally bounded endomorphism of a linear Lie group.

## § 1. Introduction

As is well known, a nontrivial commutative Lie group has discontinuous characters, which can be constructed using the Hamel $\mathbb{Q}$-basis in $\mathbb{R}$ (regarded as a vector space over $\mathbb{Q}$). Certainly, every nonperfect connected Lie group $G$ also has discontinuous characters that are discontinuous characters of the quotient group of $G$ by its commutator subgroup $G'$.

The objective of this note is to give a criterion for the continuity of a (not necessarily continuous) locally bounded endomorphism of a (not necessarily perfect) linear Lie group and discuss an example showing that the continuity of such an endomorphism on the center of the group is not a sufficient condition for the continuity of the endomorphism in question, unlike several situations of this kind (see, e.g., [1]).

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## § 2. Preliminaries

Recall that the following assertions hold for a linear connected Lie group $G$ with the radical $R$ and a Levi subgroup $L$: $L$, $[G,R]$, and $G'=[G,G]$ are closed subgroups of $G$ (moreover, $[G,R]$ is simply connected), the mapping $\gamma(r,l)\mapsto rl$ for every $l\in L$ and $r\in R$ is a covering mapping of $R\times_t L$ onto $G$ and $G'$ is isomorphic to the semidirect product $[G,R]\times_t L$, where $t[l](r)=lrl^{-1}$ for every $l\in L$ and $r\in R$, $[G,R]\cap L$ is the identity subgroup of $G$, and the center of $L$ is finite (see [8], Chap. 3, Exercise 41).

Recall a general automatic continuity theorem for locally bounded finite-dimensional representations of Lie groups [2]; this theorem corrects the corresponding assertions in [3]–[5].

**Theorem.** *Let $G$ be a connected Lie group, let $G'$ be the commutator subgroup of $G$, let $R$ be the radical of $G$, let $S$ be a Levi subgroup of $G$, and let $\pi$ be a locally bounded finite-dimensional representation of $G$. The restriction $\pi|_{G'}$ of the representation $\pi$ to $G'$ is a continuous representation of $G'$ in the intrinsic topology and a continuous representation of $G'\cap R$ in the topology induced by the original topology of the group $G$. The restriction of the representation $\pi$ to $G'$ is continuous in the topology induced by the original topology of the group $G$ if and only if the restriction of $\pi$ to the center $Z_S$ of $S$ is continuous in the topology induced by the original topology of $G$.*

The following theorem holds (see Theorem 2.10.1 of [6]).

**Theorem 1.** *Let $G$ be a Lie group, and let $\mathfrak{g}$ be its Lie algebra. Then the exponential mapping is analytic. Further, it is an analytic diffeomorphism on an open neighborhood of the origin of $\mathfrak{g}$. More generally, let $\mathfrak{g}$ be the direct sum of linear subspaces) $\mathfrak{h}_1,\dots,\mathfrak{h}_s$ $(s\ge])$; then there are open neighborhoods $B_i$ of $0$ in $\mathfrak{h}_i$ $(1\le i\le s)$ and $U$ of $1$ in $G$ such that the mapping*

$$\psi\colon (Z_1,\dots,Z_s)\to \exp Z_1\cdots\exp Z_s$$

*is an analytic diffeomorphism of $B_1\times\cdots\times B_s$ onto $U$.*

## § 3. Main Result

**Theorem 2.** *Let $G$ be a linear connected Lie group, let $G'$ be the commutator subgroup of $G$, let $\mathfrak{g}$ be the Lie algebra of $G$, let $\mathfrak{g}'$ be the commutator subalgebra of $\mathfrak{g}$, let $\mathfrak{h}$ be a vector subspace of $\mathfrak{g}$ complementary to $\mathfrak{g}'$, let $\{h_1,\dots,h_k\}$ be a basis of $\mathfrak{h}$, let the one-dimensional vector subspaces $\mathfrak{h}_1,\dots,$ $\mathfrak{h}_k$ be spanned by $h_1,\dots,h_k$, respectively.*

*A locally bounded endomorphism $\pi$ of $G$ is continuous if and only if the composed mapping of $\mathbb{R}$ to $G$ given by $\mathbb{R} \ni t \mapsto \pi(\exp(th_i))$, $i = 1, \dots, k$, is continuous.*

*Proof.* Obviously, if $\pi$ is continuous, then the composed mapping of $\mathbb{R}$ to $G$ given by $\mathbb{R} \ni t \mapsto \pi(\exp(th_i))$, $i = 1, \dots, k$, is continuous. This proves the "only if" part of the theorem. Therefore, it remains to prove the converse assertion.

Let us prove the "if" part. Let the corresponding conditions of the theorem be satisfied for some vector subspace $\mathfrak{h}$ of $\mathfrak{g}$ complementary to $\mathfrak{g}'$ and for some basis $\{h_1, \dots, h_k\}$ of $\mathfrak{h}$.

Since $G$ is assumed to be linear and locally bounded, it follows that the endomorphism $\pi$ can be regarded as a locally bounded finite-dimensional linear representation of $G$. By Theorem 1.3.2 in [4] corrected in [2], the restriction of the representation $\pi$ to $G'$ is continuous (since $G'$ is closed, it follows that the intrinsic topology of the Lie group $G'$ coincides with the topology of $G'$ induced by the topology of $G$). This implies that the restriction of the endomorphism $\pi$ to $G'$ is continuous.

On the other hand, by the very assumption of the "if" part of the theorem, the composition mapping of $\mathbb{R}$ to $G$ given by $\mathbb{R} \ni t \mapsto \pi(\exp(th_i))$, $i = 1, \dots, k$, is continuous.

Hence, by Theorem 2.10.1 of [6], the endomorphism $\pi$ is separately continuous on some neighborhood of the identity element $e$ of $G$ with respect to the subgroups $G'$, $H_1, \dots, H_k$.

Since the corresponding mapping defined by the restriction of the product of the corresponding exponential mappings on the subgroups to a small neighborhood $U$ of $e$ is an analytic diffeomorphism of $(G' \times H_1 \times \dots \times H_k) \cap U$ onto some neighborhood of $G$ by Theorem 2.10.1 of [6], it follows from the Namioka theorem [7] that the representation $\pi$ has a point of joint continuity in $U$ and, therefore, is continuous.

This completes the proof of Theorem 2.

## § 4. Discussion

Some continuity conditions for locally bounded finite-dimensional representations and automorphisms of Lie groups are related to the equivalence of the continuity of the corresponding mapping to the corresponding property of the restriction of the mapping to the center of the group (see, in particular, [8–10]). Let us use the example given in [1] to show that, for the case under consideration, such an equivalence fails to hold even for solvable linear Lie groups.

**Example.** Consider the Heisenberg group

$$H = \left\{ h(a,b,c) = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \qquad a,b,c \in \mathbb{R} \right\}.$$

Let $\rho$ be the tautological representation of $H$ ($\rho(h) = h$, $h \in H$), and let $\alpha$ and $\beta$ be discontinuous characters of the subgroups $A = \{h(a,b,c) \in H, b = c = 0\}$ and $B = \{h(a,b,c) \in H, a = c = 0\}$, respectively. Then the discontinuous representation $\pi = \alpha\beta\rho$ of $H$ is continuous on the center $C = \{h(a,b,c) \in H, a = b = 0\}$.

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