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Partition energy of some lexicographic product of two graphs

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Abstract

In this paper, we consider some lexicographic product of two graphs G and H of the form $G_m(H_n)$ and determine its m-partition energy. Also, we determine the partition energy with respect to their m-complement and $m(i)$-complement.

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1 Introduction

The energy of a graph was introduced by I. Gutman [10] as the sum of absolute values of all graph eigenvalues. In connection with graph energy, eigenvalues of several matrices are studied in literature, see [12]. Recently, E. Sampathkumar and M. A. Sriraj in [16] have introduced L-matrix (also called partition matrix) of $G = (V, E)$ of order n with respect to a partition $P_k = \{V_1, V_2, \ldots, V_k\}$ of the vertex

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set V. It is a unique square symmetric matrix $P_k(G) = [a_{ij}]$ with zero diagonal, whose entries a_{ij} are defined as follows:

$$a_{ij} = \begin{cases} 2 & \text{if } v_i \text{ and } v_j \text{ are adjacent where } v_i,\, v_j \in V_r, \\ -1 & \text{if } v_i \text{ and } v_j \text{ are non-adjacent where } v_i,\, v_j \in V_r, \\ 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent between the sets} \\ & V_r \text{ and } V_s \text{ for } r \neq s \text{ where } v_i \in V_r \text{ and } v_j \in V_s, \\ 0 & \text{otherwise} \end{cases}$$

This L-matrix (partition matrix) determines the partition of the vertex set of G uniquely. The eigenvalues of this matrix is called partition eigenvalues of G.
The partition of V into independent sets $V_1, V_2, \ldots, V_k$ leads to vertex coloring of graph G. Note that if k is the chromatic number of G, then k-partition energy and color energy introduced by C. Adiga et al. in [2] are same. Further k-partition energy of a graph G denoted by $E_{P_k}(G)$ is defined as the sum of the absolute values of k-partition eigenvalues of G in [17], where eigenvalues of $P_k(G)$ are k-partition eigenvalues of G. If the vertex set of a graph G of order n is partitioned into n sets, then the partition energy coincides with the usual energy of a graph. So partition energy may be considered as a generalization of energy of a graph introduced by I. Gutman in [10]. For more details on various graph energies, see [1], [6], [7], [9], [12].

Recently, in [13], [17], [18] the authors have studied the k-partition energy of some class of graphs. N. Akgunes et al., have studied some properties on lexicographic product and tensor product of graphs obtained by monogenic semigroups in [3] and [4] respectively. In literature, various graph products have been introduced and their properties are very much studied, see [11].

Motivated by this, we determine m-partition energy of lexicographic product of two graphs $C_m(K_n), K_m(C_n), C_m(C_n), S_m(K_n)(where\ S_m = K_{m-1,1}), K_{m\times 2}(K_n)$, $K_{m,n}(K_n)$, and their generalized complements.

Definition 1.1. *[11] The lexicographic product of two graphs G_m and H_n is formed by taking m copies of H_n and joining any two vertices (u, v) and (x, y) if, and only if, either u is adjacent to x in G_m or $u = x$ and v is adjacent to y. It is represented by $G_m[H_n]$ and is also called as composition of graphs.*

The generalized complements of a graph are defined as follows:

Definition 1.2. *[14] Let G be a graph and $P_k = \{v_1, v_2, \ldots, v_k\}$ be a partition of its vertex set V. Then the k-complement of a graph G is obtained as follows: For all V_i and V_j in P_k, $i \neq j$ remove the edges between V_i and V_j and add the edges between the vertices of V_i and V_j which are not in G and is denoted by $\overline{(G)}_k$. The matrix of k-complement is obtained from L-matrix $P_k(G)$ as follows: In $P_k(G)$ interchange 1 and 0 in the non-principal diagonal entries. The matrix thus obtained is the matrix of $(\overline{G}_k)$ and is denoted by $P_k(\overline{(G)}_k)$.*

Definition 1.3. *[15] Let G be a graph and $P_k = \{v_1, v_2, \ldots, v_k\}$ be a partition of its vertex set V. Then $k(i)$ complement of G is obtained as follows: For each vertex set V_r in P_k, remove the edges of G joining the vertices within V_r and add the edges of $\overline{G}$(complement of G) joining the vertices of V_r and is denoted by $\overline{(G)_{k(i)}}$. The matrix of $k(i)$-complement is obtained by interchanging 2 and -1 in the matrix $P_k(G)$ and is denoted by $P_k(\overline{(G)_{k(i)}})$.*

Definition 1.4. *[5] The spectrum of graph G is the list of distinct eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_r$, with their multiplicities $m_1, m_2, \ldots, m_r$ and we write it as*

$$Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ m_1 & m_2 & \cdots & m_r \end{pmatrix}.$$

Now we state the lemmas and definitions which are used for computation of spectrum of the partition matrices.

Lemma 1.5. *[5] Let $A = \begin{pmatrix} A_0 & A_1 \\ A_1 & A_0 \end{pmatrix}$ be a symmetric 2×2 block matrix. Then the spectrum of A is the union of the spectra of $A_0 + A_1$ and $A_0 - A_1$.*

Lemma 1.6. *[8] A circulant matrix of order l is form*

$$C = circ(g_1, g_2, \ldots, g_l) = \begin{pmatrix} g_1 & g_2 & \cdots & g_l \\ g_l & g_1 & \cdots & g_{l-1} \\ \vdots & \vdots & \ddots & \vdots \\ g_2 & g_3 & \cdots & g_1 \end{pmatrix}.$$

Lemma 1.7. *[8] Let $A_1, A_2, \ldots, A_m$ be square matrices of order n. A block circulant matrix of type (m, n) (of order mn) is an $mn \times nm$ matrix of the form*

$$bcirc(A_1, A_2, \ldots, A_m) = \begin{pmatrix} A_1 & A_2 & \cdots & A_m \\ A_m & A_1 & \cdots & A_{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_3 & \cdots & A_1 \end{pmatrix}.$$

Theorem 1.8. *[8] $A \in BC_{m,n}$ if, and only if, it is of the form $A = (F_m \bigotimes F_n)^* diag(M_1, M_2, \ldots, M_n)(F_m \bigotimes F_n)$, where the M_k are arbitrary square matrices of order n.*

Note:[8] Let n be a fixed integer ≥ 1. Let $\omega = exp(\frac{2\pi i}{n}) = cos\left(\frac{2\pi}{n}\right) + isin\left(\frac{2\pi}{n}\right)$, $i = \sqrt{-1}$. Let $\Omega = (\Omega_n) = diag(1, \omega, \omega^2, ..., \omega^{(n-1)})$. Then $\Omega^k = diag(1, \omega^k, \omega^{2k}, ..., \omega^{(n-1)k})$.

Now we give a brief description of the vertex partition of lexicographic product of two graphs considered throughout the paper. Let G_m and H_n be any two graphs with vertex sets $\{u_1, u_2, \ldots, u_m\}$ and $\{w_1, w_2, \ldots, w_n\}$ respectively. The lexicographic product of two graphs G_m and H_n consists of m copies of H_n. Let the vertices in each copy be $V_i = \{v_{ij}\}$, where $v_{ij} = (u_i, w_j)$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Then the vertex set of the lexicographic product $G_m(H_n)$ is $\bigcup_{i=1}^{m} V_i$. The partition of its vertex set is taken as $P_m = \{V_1, V_2, \ldots, V_m\}$.

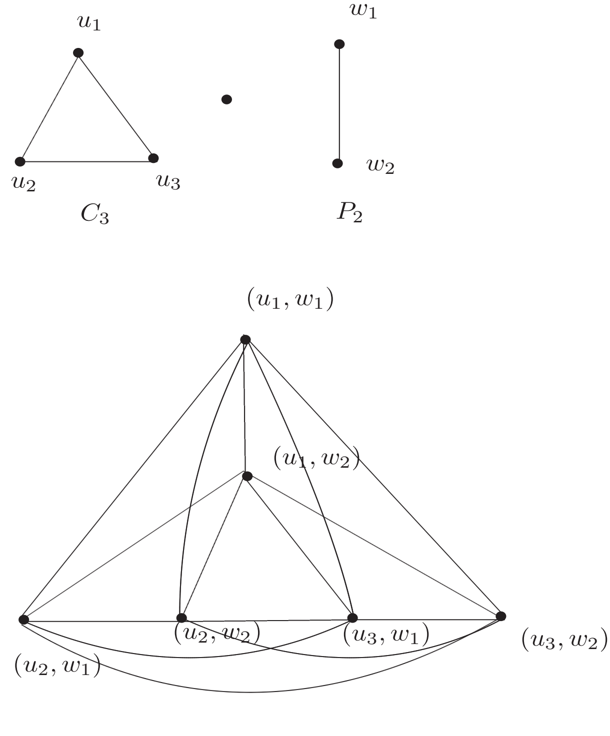

Figure 1

For example in Figure 1 the lexicographic product of C_3 and P_2 that is $C_3(P_2)$ is shown above. Its partition set is given by $P_3 = \{V_1, V_2, V_3\}$, where $V_1 = \{(u_1, w_1), (u_1, w_2)\} = \{v_{11}, v_{12}\}$, $V_2 = \{(u_2, w_1), (u_2, w_2)\} = \{v_{21}, v_{22}\}$ and $V_3 = \{(u_3, w_1), (u_3, w_2)\} = \{v_{31}, v_{32}\}$.

2 Main Results

In this section we determine the m-partition energy of lexicographic product of two graphs $C_m(K_n), K_m(C_n), C_m(C_n), S_m(K_n)(where\ S_m = K_{m-1,1}), K_{m\times 2}(K_n)$, $K_{m,n}(K_n)$, also their generalized complements.

Theorem 2.1. *Let $G = C_m[K_n]$, where C_m is the cycle graph with m vertices and K_n is the complete graph with n vertices. Then the vertex set of G is $\bigcup_{i=1}^{m} V_i$ where $V_i = \{v_{i1}, v_{i2}, \ldots, v_{in}\}$. Let $P_m = \{V_1, V_2, \ldots, V_m\}$ be the partition of the vertex set of G. Then*
$(i) E_{P_m}(G) = 2(mn - m + 2n - 4) + 2\sum_{t=1}^{m-1} \left|(n-1) + n\cos\frac{2t\pi}{m}\right|.$
$(ii) E_{P_m}\overline{(G)}_m = (3mn - 2m - n - 2) + \sum_{t=1}^{m-1} \left|(n-2) - 2n\cos\frac{2t\pi}{m}\right|.$
$(iii) E_{P_m}\overline{(G)}_{m(i)} = (mn - m + n + 1) + \sum_{t=1}^{m-1} \left|(1-n) + 2n\cos\frac{2t\pi}{m}\right|.$

Proof. (i) The matrix $P_m(G)$ is

$$\begin{array}{c|cccccccccccccccccccc}
- & v_{11} & v_{12} & v_{13} & \cdots & v_{1n} & v_{21} & v_{22} & \cdots & v_{2n} & v_{31} & v_{32} & \cdots & v_{3n} & \cdots\cdots & v_{m1} & v_{m2} & \cdots & v_{mn} \\
\hline
v_{11} & 0 & 2 & 2 & \ldots & 2 & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & \ldots\ldots & 1 & 1 & \ldots & 1 \\
v_{12} & 2 & 0 & 2 & \ldots & 2 & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & \ldots\ldots & 1 & 1 & \ldots & 1 \\
v_{13} & 2 & 2 & 0 & \ldots & 2 & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & \ldots\ldots & 1 & 1 & \ldots & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
v_{1n} & 2 & 2 & 2 & \ldots & 0 & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & \ldots\ldots & 1 & 1 & \ldots & 1 \\
v_{21} & 1 & 1 & 1 & \ldots & 1 & 0 & 2 & \ldots & 2 & 1 & 1 & \ldots & 1 & \ldots\ldots & 0 & 0 & \ldots & 0 \\
v_{22} & 1 & 1 & 1 & \ldots & 1 & 2 & 0 & \ldots & 2 & 1 & 1 & \ldots & 1 & \ldots\ldots & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
v_{2n} & 1 & 1 & 1 & \ldots & 1 & 2 & 2 & \ldots & 0 & 1 & 1 & \ldots & 1 & \ldots\ldots & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
v_{m1} & 1 & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots\ldots & 0 & 2 & \ldots & 2 \\
v_{m2} & 1 & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots\ldots & 2 & 0 & \ldots & 2 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
v_{mn} & 1 & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots\ldots & 2 & 2 & \ldots & 0
\end{array}$$

The matrix of the graph G is a block circulant matrix of the form

$$\begin{pmatrix} H_1 & H_2 & H_3 & \cdots & H_m \\ H_m & H_1 & H_2 & \cdots & H_{m-1} \\ \vdots & \vdots & \vdots & & \vdots \\ H_2 & H_3 & H_4 & \cdots & H_1 \end{pmatrix}_{mn \times nm},$$

here

$$P(\Omega) = H_1 + H_2\Omega + H_3\Omega^2 + \cdots + H_m\Omega^{m-1},$$

where

$$\Omega = diag(I, \omega I, \omega^2 I, \ldots, \omega^{m-1} I)$$

and $\Omega^k = diag(I, \omega^k I, \omega^{2k} I, \ldots, \omega^{(m-1)k} I)$ and $\omega = e^{\frac{2\pi i}{m}}$. Also $\omega^m = e^{2\pi i} = 1$.
Now $P_m(G) = circ(H_1, H_2, H_3, \ldots, H_m)_{m \times m}$ with

$$H_1 = \begin{pmatrix} 0 & 2 & \cdots & 2 \\ 2 & 0 & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & \cdots & 0 \end{pmatrix}_{n \times n}, H_2 = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}_{n \times n}, H_3 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{n \times n},$$

where $H_2 = H_m$ and $H_3 = H_4 = \cdots = H_{m-1}$. Thus the diagonal form of the matrix is

$$\begin{pmatrix} H_1+2H_2 & 0 & 0 & \cdots & 0 \\ 0 & H_1+H_2\omega+H_2\omega^{m-1} & 0 & \cdots & 0 \\ 0 & 0 & H_1+H_2\omega^2+H_2\omega^{2(m-1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & H_1+H_2\omega^{m-1}+H_2\omega^{(m-1)^2} \end{pmatrix}.$$

Let

$$A = H_1 + 2H_2 = \begin{pmatrix} 2 & 4 & 4 & \cdots & 4 \\ 4 & 2 & 4 & \cdots & 4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 4 & 4 & 4 & \cdots & 2 \end{pmatrix}.$$

Then

$$|\lambda I - A| = \begin{vmatrix} \lambda-2 & -4 & -4 & \cdots & -4 \\ -4 & \lambda-2 & -4 & \cdots & -4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -4 & -4 & -4 & \cdots & \lambda-2 \end{vmatrix}_{n\times n}$$

Using elementary row and column operations

$C\,'_i \to C_i - C_1$, for $i = 2, 3, \ldots, n$
Take $(\lambda+2)^{n-1}$ as common
$R_1' \to R_1 + R_2 + \cdots + R_n$ we get

$$|\lambda I - A| = (\lambda+2)^{n-1}[\lambda - (4n-2)].$$

Let

$$B = H_1 + H_2(\omega + 1/\omega) = \begin{vmatrix} \omega+1/\omega & 2+\omega+1/\omega & \cdots & 2+\omega+1/\omega \\ 2+\omega+1/\omega & \omega+1/\omega & \cdots & 2+\omega+1/\omega \\ \vdots & \vdots & \ddots & \vdots \\ 2+\omega+1/\omega & 2+\omega+1/\omega & \cdots & \omega+1/\omega \end{vmatrix}.$$

Then

$$|\lambda I - B| = \begin{vmatrix} \lambda-(\omega+1/\omega) & -(2+\omega+1/\omega) & \cdots & -(2+\omega+1/\omega) \\ -(2+\omega+1/\omega) & \lambda-(\omega+1/\omega) & \cdots & -(2+\omega+1/\omega) \\ \vdots & \vdots & \ddots & \vdots \\ -(2+\omega+1/\omega) & -(2+\omega+1/\omega) & \cdots & \lambda-(\omega+1/\omega) \end{vmatrix}_{n\times n}$$

Using elementary row and column operations as above, we get

$$|\lambda I - B| = (\lambda+2)^{n-1}\left[\lambda - \left(2(n-1) + 2n\cos\frac{2\pi}{m}\right)\right].$$

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Thus,

$$Spec[C_m[K_n] = \begin{cases} -2 & m(n-1) \text{ times} \\ 4n-2 & \text{once} \\ 2[(n-1)+n\cos\frac{2t\pi}{m}] & t=1,2,3,\dots,(m-1). \end{cases}$$

Hence,

$$E_{P_m}(G) = 2(mn-m+2n-1)+2\sum_{t=1}^{m-1}\left|(n-1)+n\cos\frac{2t\pi}{m}\right|.$$

(ii) Proceeding with steps similar to proof of (i),

$$Spec[\overline{(C_m[K_n])}_m] = \begin{cases} -2 & m(n-1) \text{ times} \\ mn-n-2 & \text{once} \\ (n-2)-2n\cos\frac{2t\pi}{m} & t=1,2,3,\dots,(m-1). \end{cases}$$

$$E_{P_m}\overline{(G)}_m = (3mn-2m-n-2)+\sum_{t=1}^{m-1}\left|(n-2)-2n\cos\frac{2t\pi}{m}\right|.$$

(iii) Carrying out similar operations we get,

$$E_{P_m}\overline{(G)}_{m(i)} = (mn-m+n+1)+\sum_{t=1}^{m-1}\left|(1-n)+2n\cos\frac{2t\pi}{m}\right|.$$

□

Theorem 2.2. *Let $G = K_m[C_n]$ with vertex set $\{V_1, V_2, \dots, V_m\}$ where $V_i = \{v_{i1}, v_{i2}, \dots, v_{in}\}$. Let $P_m = \{V_1, V_2, \dots, V_m\}$ be the partition of the set of G. Then*
$(i) E_{P_m}(G) = |mn-2n+7|+(m-1)|7-2n|+m\sum_{t=1}^{n-1}\left|1+6\cos\frac{2t\pi}{n}\right|.$
$(ii) E_{P_m}\overline{(G)}_m = m\left[|7-n|+\sum_{t=1}^{n-1}\left|1+6\cos\frac{2t\pi}{n}\right|\right].$
$E_{P_m}\overline{(G)}_m = mE_{P_1}(C_n).$
$(iii) E_{P_m}\overline{(G)}_{m(i)} = |mn+n-8|+(m-1)|n-8|+m\sum_{t=1}^{m-1}\left|-2-6\cos\frac{2t\pi}{n}\right|.$

Proof. (i) The matrix $P_m(G)$ is

$$\begin{pmatrix}
0 & 2 & -1 & -1 & \cdots & 2 & 1 & 1 & 1 & \cdots & 1 & \cdots & 1 & 1 & 1 & \cdots & 1\\
2 & 0 & 2 & -1 & \cdots & -1 & 1 & 1 & 1 & \cdots & 1 & \cdots & 1 & 1 & 1 & \cdots & 1\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots\\
2 & -1 & -1 & -1 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 & \cdots & 1 & 1 & 1 & \cdots & 1\\
1 & 1 & 1 & 1 & \cdots & 1 & 0 & 2 & -1 & \cdots & 2 & \cdots & 1 & 1 & 1 & \cdots & 1\\
1 & 1 & 1 & 1 & \cdots & 1 & 2 & 0 & 2 & \cdots & -1 & \cdots & 1 & 1 & 1 & \cdots & 1\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots\\
1 & 1 & 1 & 1 & \cdots & 1 & 2 & -1 & -1 & \cdots & 0 & \cdots & 1 & 1 & 1 & \cdots & 1\\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots\\
1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & \cdots & 0 & 2 & -1 & \cdots & 2\\
1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & \cdots & 2 & 0 & 2 & \cdots & -1\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots\\
1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & \cdots & 2 & -1 & -1 & \cdots & 0
\end{pmatrix}.$$

The matrix of the graph G is a block circulant matrix of the form

$$\begin{pmatrix}
H_1 & H_2 & H_3 & \cdots & H_m\\
H_m & H_1 & H_2 & \cdots & H_{m-1}\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
H_2 & H_3 & H_4 & \cdots & H_1
\end{pmatrix}_{mn\times nm},$$

where $H_2 = H_3 = \cdots = H_m$ and

$$H_1 = \begin{pmatrix}
0 & 2 & -1 & -1 & \cdots & 2\\
2 & 0 & 2 & -1 & \cdots & -1\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
2 & -1 & -1 & -1 & \cdots & 0
\end{pmatrix}_{n\times n}, H_2 = \begin{pmatrix}
1 & 1 & \cdots & 1\\
1 & 1 & \cdots & 1\\
\vdots & \vdots & \ddots & \vdots\\
1 & 1 & \cdots & 1
\end{pmatrix}_{n\times n}.$$

The diagonal form of the matrix is

$$\begin{pmatrix}
H_1 + (m-1)H_2 & 0 & \cdots & 0\\
0 & H_1 + H_2(\omega + \omega^2 + \omega^3 + \cdots + \omega^{m-1}) & \cdots & 0\\
\vdots & \vdots & \ddots & \vdots\\
0 & 0 & \cdots & H_1 + H_2(\omega^{m-1} + \omega^{2(m-1)} + \cdots + \omega^{(m-1)^2})
\end{pmatrix}.$$

$$\text{Let } A = H_1 + (m-1)H_2 = \begin{pmatrix}
(m-1) & (m+1) & (m-2) & \cdots & (m-2) & (m+1)\\
(m+1) & (m-1) & (m+1) & \cdots & (m-2) & (m-2)\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
(m+1) & (m-2) & (m-2) & \cdots & (m+1) & (m-1)
\end{pmatrix}_{n\times n}.$$

Then A is a circulant matrix of order n, the eigenvalues of such matrices are

$$\lambda_t = \sum_{k=1}^{n} a_k e^{\frac{2\pi i t(k-1)}{n}}, \text{where } a_k\text{'s are entries of the first row.}$$

Let $\omega = e^{\frac{2\pi i t}{n}}$ then we have

$$\lambda_t = \sum_{k=1}^{n} a_k \omega^{k-1}$$

As the matrix is of order n, we have $\omega = e^{\frac{2\pi i t}{n}}$, $\omega^n = 1$ and

$$\begin{aligned} \lambda_t &= (m-1) + (m+1)(\omega + \omega^{n-1}) + (m-2)(\omega^2 + \omega^3 + \cdots + \omega^{n-2}) \\ &= 1 + 3(\omega + 1/\omega) \end{aligned}$$

Hence,

$$\lambda_t = \begin{cases} mn - 2n + 7 & \text{if } t = 0 \\ 1 + 6\cos\frac{2t\pi}{n} & \text{if } 1 \le t \le n-1. \end{cases}.$$

$$\text{Let} \qquad \begin{aligned} B &= H_1 + H_2(\omega + \omega^2 + \cdots + \omega^{n-1}) \\ &= H_1 - H_2 \\ &= \begin{pmatrix} -1 & 1 & -2 & \cdots & -2 & 1 \\ 1 & -1 & 1 & \cdots & -2 & -2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & -2 & -2 & \cdots & 1 & -1 \end{pmatrix}_{n \times n}. \end{aligned}$$

Then B is a circulant matrix of order n. Hence,

$$\lambda_l = \begin{cases} 7 - 2n & \text{if } l = 0 \\ 1 + 6\cos\frac{2l\pi}{n} & \text{if } 1 \le l \le n-1. \end{cases}$$

$$Spec[K_m[C_n]] = \begin{cases} mn - 2n + 7 & \text{once} \\ 1 + 6\cos\frac{2t\pi}{n},\ 1 \le t \le n-1 & \text{once} \\ 7 - 2n & (m-1) \text{ times} \\ 1 + 6\cos\frac{2l\pi}{n},\ 1 \le l \le n-1 & (m-1) \text{ times.} \end{cases}$$

Thus,

$$E_{P_m}(G) = |mn - 2n + 7| + (m-1)|7 - 2n| + m\sum_{t=1}^{n-1}\left|1 + 6\cos\frac{2t\pi}{n}\right|.$$

□

The proof of (ii) and (iii) follows with similar discussions as above.

Observation 2.3. *The m-partition energy of* $\overline{K_m(G)_m} = mE_{P_1}(G)$

Theorem 2.4. *Let $G = C_m[C_n]$ with vertex set $\{V_1, V_2, \ldots, V_m\}$ where $V_i = \{v_{i1}, v_{i2}, \ldots, v_{in}\}$. Let $P_m = \{V_1, V_2, \ldots, V_m\}$ be the partition of the set of G. Then*
(i) $E_{P_m}(G) = |n+7| + \sum_{p=1}^{m-1} \left|(7-n) + 2n\cos\frac{2p\pi}{n}\right| + m\sum_{t=1}^{n-1}\left|1 + 6\cos\frac{2t\pi}{n}\right|$.
(ii) $E_{P_m}\overline{(G)_m} = |m^2 - 4m + 7| + m\sum_{l=1}^{n-1}\left|1 + 6\cos\frac{2l\pi}{n}\right| + \sum_{p=1}^{m-1}\left|(7-2n) - 2n\cos\frac{2p\pi}{n}\right|$.
(iii) $E_{P_m}\overline{(G)_{m(i)}} = |4(n-2)| + m\sum_{l=1}^{n-1}\left|-2 - 6\cos\frac{2l\pi}{n}\right| + \sum_{p=1}^{m-1}\left|(2n-8) + 2n\cos\frac{2p\pi}{n}\right|$.

Proof. (i) The matrix $P_m(G)$ of the graph G is a block circulant matrix of the form

$$\begin{pmatrix} H_1 & H_2 & H_3 & \cdots & H_m \\ H_m & H_1 & H_2 & \cdots & H_{m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_2 & H_3 & H_4 & \cdots & H_1 \end{pmatrix}_{mn \times nm},$$

where $H_2 = H_m$ and $H_3 = H_4 = \cdots = H_{m-1}$, and

$$H_1 = \begin{pmatrix} 0 & 2 & -1 & \cdots & -1 & 2 \\ 2 & 0 & 2 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & -1 & -1 & \cdots & 2 & 0 \end{pmatrix}_{n\times n}, H_2 = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}_{n\times n} H_3 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{n\times n}.$$

The diagonal form of the matrix is

$$\begin{pmatrix} H_1 + 2H_2 & 0 & 0 & \cdots & 0 \\ 0 & H_1 + H_2\omega + H_2\omega^{m-1} & 0 & \cdots & 0 \\ 0 & 0 & H_1 + H_2\omega^2 + H_2\omega^{2(m-1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & H_1 + H_2\omega^{m-1} + H_2\omega^{(m-1)^2} \end{pmatrix}.$$

Let

$$A = H_1 + 2H_2 = \begin{pmatrix} 2 & 4 & 1 & \cdots & 1 & 4 \\ 4 & 2 & 4 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 4 & 1 & 1 & \cdots & 4 & 2 \end{pmatrix}_{n\times n}.$$

Then A is a circulant matrix of order n, and we get the eigenvalues as

$$\begin{cases} n+7 & \text{if } t = 0 \\ 1 + 6\cos\frac{2t\pi}{n} & \text{if } 1 \le t \le n-1. \end{cases}$$

$$\begin{aligned} \text{Let } B &= H_1 + H_2\omega + H_2\omega^{m-1} \\ &= \begin{pmatrix} \omega + 1/\omega & 2 + \omega + 1/\omega & -1 + \omega + 1/\omega & \cdots & -1 + \omega + 1/\omega & 2 + \omega + 1/\omega \\ 2 + \omega + 1/\omega & \omega + 1/\omega & 2 + \omega + 1/\omega & \cdots & -1 + \omega + 1/\omega & -1 + \omega + 1/\omega \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 + \omega + 1/\omega & -1 + \omega + 1/\omega & -1 + \omega + 1/\omega & \cdots & 2 + \omega + 1/\omega & \omega + 1/\omega \end{pmatrix} \end{aligned}$$

Then B is a circulant matrix of order n, with eigenvalues

$$\begin{cases} (7-n) + 2n\cos\frac{2\pi}{n} & \text{if } l = 0 \\ 1 + 6\cos\frac{2l\pi}{n} & \text{if } 1 \le l \le n-1. \end{cases}$$

Thus,

$$Spec(C_m[C_n]) = \begin{cases} n+7 & \text{once} \\ (7-n) + 2n\cos\frac{2p\pi}{n},\ 1 \leq p \leq m-1 & \text{once} \\ 1 + 6\cos\frac{2t\pi}{n},\ 1 \leq t \leq n-1 & \text{once} \\ 1 + 6\cos\frac{2l\pi}{n},\ 1 \leq l \leq n-1 & (m-1)\text{ times.} \end{cases}$$

Hence,

$$E_{P_m}(G) = |n+7| + \sum_{p=1}^{m-1} \left|(7-n) + 2n\cos\frac{2p\pi}{n}\right| + m\sum_{t=1}^{n-1}\left|1 + 6\cos\frac{2t\pi}{n}\right|.$$

□

With similar discussions as above one can prove (ii) and (iii).

Theorem 2.5. *Let $G = S_m[K_n]$, where S_m is a star graph with m vertices and K_n is the complete graph with n vertices. Let $P_m = \{V_1, V_2, \ldots, V_m\}$ where $V_i = \{v_{i1}, v_{i2}, \ldots, v_{in}\}$. Then*
$(i) E_{P_m}(G) = 4(mn - m - n + 1) + |(2n-2) + n\sqrt{m-1}| + |(2n-2) - n\sqrt{m-1}|$.
$(ii) E_{P_m}\overline{(G)_m} = 4m(n-1)$.
$(iii) E_{P_m}\overline{(G)_{m(i)}} = 2(mn - m - n + 1) + 2n\sqrt{m-1}$.

Proof. (i) The matrix of the graph $G = S_m[K_n]$ is

$$\begin{array}{c} - \\ v_{11} \\ v_{12} \\ v_{13} \\ \vdots \\ v_{1n} \\ v_{21} \\ v_{22} \\ \vdots \\ v_{2n} \\ \vdots \\ v_{m1} \\ v_{m2} \\ \vdots \\ v_{mn} \end{array} \begin{array}{c} \begin{array}{ccccccccccccccccc} v_{11} & v_{12} & v_{13} & \cdots & v_{1n} & v_{21} & v_{22} & \cdots & v_{2n} & v_{31} & v_{32} & \cdots & v_{3n} & \cdots\cdots & v_{m1} & v_{m2} & \cdots & v_{mn} \end{array} \\ \begin{pmatrix} 0 & 2 & 2 & \ldots & 2 & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \ldots\ldots & 1 & 1 & \ldots & 1 \\ 2 & 0 & 2 & \ldots & 2 & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \ldots\ldots & 1 & 1 & \ldots & 1 \\ 2 & 2 & 0 & \ldots & 2 & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \ldots\ldots & 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & 2 & \ldots & 0 & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \ldots\ldots & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & \ldots & 1 & 0 & 2 & \ldots & 2 & 0 & 0 & \ldots & 0 & \ldots\ldots & 0 & 0 & \ldots & 0 \\ 1 & 1 & 1 & \ldots & 1 & 2 & 0 & \ldots & 2 & 0 & 0 & \ldots & 0 & \ldots\ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 1 & 2 & 2 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots\ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots\ldots & 0 & 2 & \ldots & 2 \\ 1 & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots\ldots & 2 & 0 & \ldots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots\ldots & 2 & 2 & \ldots & 0 \end{pmatrix} \end{array}$$

Perform the following operation on $|\lambda I - P_m(G)| = 0$

$$\begin{aligned}
&Step\ 1: && C\,'_{1j} \to C_{1j} - C_{11},\\
& && C\,'_{2j} \to C_{2j} - C_{21},\\
& && \vdots\\
& && C\,'_{mj} \to C_{mj} - C_{m1}, for j = 2, 3, \ldots, n.\\
&Step\ 2: && Take\ (\lambda+2)^{m(n-1)} as\ common.\\
&Step\ 3: && R\,'_{i1} \to R_{i1} + R_{i2} + \cdots + R_{in}, i = 1, 2, \ldots, m\\
& && Determinant\ reduces\ to\ m \times m\ determinant\\
&Step\ 4: && C\,'_{i} \to C_i - C_2, i = 3, 4, \ldots, m\\
&Step\ 5: && Take\ (\lambda - 2(n-1))^{m-2}\ as\ common.
\end{aligned}$$

Determinant reduces to the form

$$\begin{vmatrix} \lambda - 2(n-1) & -n \\ -n(m-1) & \lambda - 2(n-1) \end{vmatrix}.$$

On expansion, we get

$$(\lambda+2)^{m(n-1)}[\lambda - 2(n-1)]^{m-2}\{\lambda^2 + \lambda(4-4n) + (5n^2 - 8n - n^2m + 4)\} = 0.$$

$$\text{Thus}, Spec(S_m[K_n]) = \begin{cases} -2 & m(n-1) \text{ times} \\ 2(n-1) & (m-2) \text{ times} \\ (2n-2) + n\sqrt{m-1} & \text{once} \\ (2n-2) - n\sqrt{m-1} & \text{once.} \end{cases}$$

Hence,

$$E_{P_m}(G) \quad = \quad 4(mn - m - n + 1) + |(2n-2) + n\sqrt{m-1}| + |(2n-2) - n\sqrt{m-1}|.$$

□

(ii) Perform the operation as above on $|\lambda I - P_m(G)| = 0$, we get

$$[(\lambda - (mn-2)][\lambda - 2(n-1)][(\lambda - (n-2)]^{m-2}[\lambda+2]^{m(n-1)} = 0.$$

$$\text{Thus}, Spec\overline{(S_m[K_n])_m} = \begin{cases} -2 & m(n-1) \text{ times} \\ (n-2) & (m-2) \text{ times} \\ 2(n-1) & \text{once} \\ (mn-2) & \text{once.} \end{cases}$$

Hence,

$$E_{P_m}\overline{(G)_m} = 4m(n-1).$$

(iii) Perform the operation as above on $|\lambda I - P_m(G)| = 0$, we get

$$(\lambda-1)^{m(n-1)}[\lambda+(n-1)]^{m-2}[\lambda^2+2\lambda(n-1)+(n^2(2-m)-2n+1)=0.$$

$$Spec\overline{(S_m[K_n])_{m(i)}}\begin{cases}1 & m(n-1) \text{ times}\\ -(n-1) & (m-2) \text{ times}\\ (1-n)+n\sqrt{m-1} & \text{once}\\ (1-n)-n\sqrt{m-1} & \text{once.}\end{cases}$$

Thus,

$$E_{P_m}\overline{(G)_{m(i)}} \quad = \quad 2(mn-m-n+1)+2n\sqrt{m-1}.$$

Theorem 2.6. *Let $G = K_{l\times 2}[K_n]$, where $K_{l\times 2}$ is cocktail party graph with $m = 2l$ vertices, whose vertex set is $\{u_1, u_2, \ldots, u_l, u'_1, u'_2, \ldots, u'_l\}$ and K_n is the complete graph with n vertices. Also $P_m = \{V_1, V_2, \ldots, V_l, V'_1, V'_2, \ldots, V'_l\}$ be the partition of vertex set of G, where $V_i = \{v_{i1}, v_{i2}, \ldots, v_{in}\}$ and $V'_i = \{v'_{i1}, v'_{i2}, \ldots, v'_{in}\}$, for $i = 1, 2, \ldots, l$. Then*
$(i) E_{P_m}(G) = 4[l(2n-1)-1]$.
$(ii) E_{P_m}\overline{(G)_m} = 8l(n-1)$.
$(iii) E_{P_m}\overline{(G)_{m(i)}} = (5ln-3l-3n+1)+|1-3n|(l-1)$.

Proof. (i) The matrix of the graph is of the form $\left(\begin{array}{c|c} A & B \\ \hline B & A\end{array}\right)$, whose eigenvalues constitute the eigenvalues of $A+B$ and $A-B$ matrix. The matrix

$$A = \begin{array}{c} - \\ v_{11} \\ v_{12} \\ \vdots \\ v_{1n} \\ v_{21} \\ v_{22} \\ \vdots \\ v_{2n} \\ \vdots \\ v_{l1} \\ v_{l2} \\ \vdots \\ v_{ln} \end{array} \begin{array}{c} \begin{array}{cccccccccccccc} v_{11} & v_{12} & \cdots & v_{1n} & v_{21} & v_{22} & \cdots & v_{2n} & \cdots\cdots & v_{l1} & v_{l2} & \cdots & v_{ln} \end{array} \\ \begin{pmatrix} 0 & 2 & \ldots & 2 & 1 & 1 & \ldots & 1 & \ldots\ldots & 1 & 1 & \ldots & 1\\ 2 & 0 & \ldots & 2 & 1 & 1 & \ldots & 1 & \ldots\ldots & 1 & 1 & \ldots & 1\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ldots\ldots & \vdots & \vdots & \ddots & \vdots\\ 2 & 2 & \ldots & 0 & 1 & 1 & \ldots & 1 & \ldots\ldots & 1 & 1 & \ldots & 1\\ 1 & 1 & \ldots & 1 & 0 & 2 & \ldots & 2 & \ldots\ldots & 1 & 1 & \ldots & 1\\ 1 & 1 & \ldots & 1 & 2 & 0 & \ldots & 2 & \ldots\ldots & 1 & 1 & \ldots & 1\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ldots\ldots & \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & \ldots & 1 & 2 & 2 & \ldots & 2 & \ldots\ldots & 1 & 1 & \ldots & 1\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\ 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \ldots\ldots & 0 & 2 & \ldots & 2\\ 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \ldots\ldots & 2 & 0 & \ldots & 2\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ldots\ldots & \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \ldots\ldots & 2 & 2 & \ldots & 0 \end{pmatrix} \end{array}$$

and

$$B = \begin{array}{c} - \\ v_{11} \\ v_{12} \\ \vdots \\ v_{1n} \\ v_{21} \\ v_{22} \\ \vdots \\ v_{2n} \\ \vdots \\ v_{l1} \\ v_{l2} \\ \vdots \\ v_{ln} \end{array} \begin{array}{c} \begin{array}{cccccccccccccc} v'_{11} & v'_{12} & \cdots & v'_{1n} & v'_{21} & v'_{22} & \cdots & v'_{2n} & \cdots\cdots & v'_{l1} & v'_{l2} & \cdots & v'_{ln} \end{array} \\ \begin{pmatrix} 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 & \dots\dots & 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 & \dots\dots & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots\dots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 & \dots\dots & 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 & \dots\dots & 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 & \dots\dots & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots\dots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 & \dots\dots & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 & \dots\dots & 0 & 0 & \dots & 0 \\ 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 & \dots\dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots\dots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 & \dots\dots & 0 & 0 & \dots & 0 \end{pmatrix} \end{array}.$$

Let the matrix $C = A + B = \begin{pmatrix} 0 & 2 & 2 & 2 & \cdots & 2 \\ 2 & 0 & 2 & 2 & \cdots & 2 \\ 2 & 2 & 0 & 2 & \cdots & 2 \\ 2 & 2 & 2 & 0 & \cdots & 2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & 2 & 2 & \cdots & 0 \end{pmatrix}_{ln \times nl}$.

In $|\lambda I - C| = 0$ carry out the following operations

$$\begin{aligned} &Step\ 1: \ C\,'_i \to C_i - C_1, i = 2, \dots, n \\ &Step\ 2: \ Take\ (\lambda + 2)^{(ln-1)} as\ common. \\ &Step\ 3: \ R\,'_1 \to R_1 + R_2 + \cdots + R_n \end{aligned}$$

on further simplification,we get

$$Spec(K_{l\times 2}[K_n]) = \begin{cases} -2 & (ln-1) \text{ times} \\ 2(ln-1) & \text{once.} \end{cases}$$

The matrix $A - B$ is of the form

$$\begin{pmatrix}
0 & 2 & \cdots & 2 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
2 & 0 & \cdots & 2 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
2 & 2 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 2 & \cdots & 2 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 2 & 0 & \cdots & 2 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 2 & 2 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 2 & \cdots & 2 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 2 & 0 & \cdots & 2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 2 & 2 & \cdots & 0
\end{pmatrix}.$$

Using elementary row and column operations, we get,

$$Spec(K_{l\times 2}[K_n]) = \begin{cases} -2 & (ln - l) \text{ times} \\ 2(n-1) & l \text{ times.} \end{cases}$$

Hence,

$$E_{P_m}(G) \quad = \quad 4[l(2n-1)-1].$$

□

With similar discussions as above, one can prove (ii) and (iii).

Theorem 2.7. *The r-partition energy of the graph $G = K_{m,n}[K_s]$, where $K_{m,n}$ is a complete bipartite graph with vertex set $\{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n\}$ and K_s is a complete graph with vertex set $\{w_1, w_2, \ldots, w_s\}$. Let $P_r = \{U_1, U_2, \ldots, U_m, V_1, V_2, \ldots, V_n\}$ where $U_i = \{u_{i1}, u_{i2}, \ldots, u_{is}\}$, $V_l = \{v_{l1}, v_{l2}, \ldots, v_{ls}\}$ for $i = 1, 2, \ldots, m$, $l = 1, 2, \ldots, n$ and $r = m + n$. Then*
$(i) E_{P_r}(G) = 2(m+n)(s-1) + 2(s-1)(m+n-2) + 2s\sqrt{mn}$.
$(ii) E_{P_r}(G)_r = 4(m+n)(s-1)$.
$(iii) E_{P_r}(G)_{r(i)} = (m+n)(s-1) + |1-s|(m+n-2) + 2s\sqrt{mn}$.

Proof. The matrix of the graph $P_r(G)$ is of the form $\left(\begin{array}{c|c} A & B \\ \hline B^T & C \end{array}\right)$, where the matrix

$$\begin{array}{c|ccccccccccccc}
- & u_{11} & u_{12} & \cdots & u_{1s} & u_{21} & u_{22} & \cdots & u_{2s} & \cdots & u_{m1} & u_{m2} & \cdots & u_{ms} \\
\hline
u_{11} & 0 & 2 & \ldots & 2 & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 \\
u_{12} & 2 & 0 & \ldots & 2 & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ldots & \vdots & \vdots & \ddots & \vdots \\
u_{1s} & 2 & 2 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 \\
u_{21} & 0 & 0 & \ldots & 0 & 0 & 2 & \ldots & 2 & \ldots & 0 & 0 & \ldots & 0 \\
u_{22} & 0 & 0 & \ldots & 0 & 2 & 0 & \ldots & 2 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ldots & \vdots & \vdots & \ddots & \vdots \\
u_{2s} & 0 & 0 & \ldots & 0 & 2 & 2 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ldots & \vdots & \vdots & \ddots & \vdots \\
u_{m1} & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 2 & \ldots & 2 \\
u_{m2} & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 2 & 0 & \ldots & 2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ldots & \vdots & \vdots & \ddots & \vdots \\
u_{ms} & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 2 & \ldots & 2
\end{array}_{ms\times s}$$

$$B = \begin{array}{c|ccccccccccccc}
- & v_{11} & v_{12} & \cdots & v_{1s} & v_{21} & v_{22} & \cdots & v_{2s} & \cdots & v_{n1} & v_{n2} & \cdots & v_{ns} \\
\hline
u_{11} & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 \\
u_{12} & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ldots & \vdots & \vdots & \ddots & \vdots \\
u_{1s} & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 \\
u_{21} & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 \\
u_{22} & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ldots & \vdots & \vdots & \ddots & \vdots \\
u_{2s} & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ldots & \vdots & \vdots & \ddots & \vdots \\
u_{m1} & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 \\
u_{m2} & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ldots & \vdots & \vdots & \ddots & \vdots \\
u_{ms} & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1
\end{array}_{ms\times ns}$$

$$= \begin{array}{c|ccccccccccccc}
- & v_{11} & v_{12} & \cdots & v_{1s} & v_{21} & v_{22} & \cdots & v_{2s} & \cdots & v_{n1} & v_{n2} & \cdots & v_{ns} \\
\hline
v_{11} & 0 & 2 & \ldots & 2 & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 \\
v_{12} & 2 & 0 & \ldots & 2 & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ldots & \vdots & \vdots & \ddots & \vdots \\
v_{1s} & 2 & 2 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 \\
v_{21} & 0 & 0 & \ldots & 0 & 0 & 2 & \ldots & 2 & \ldots & 0 & 0 & \ldots & 0 \\
v_{22} & 0 & 0 & \ldots & 0 & 2 & 0 & \ldots & 2 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ldots & \vdots & \vdots & \ddots & \vdots \\
v_{2s} & 0 & 0 & \ldots & 0 & 2 & 2 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ldots & \vdots & \vdots & \ddots & \vdots \\
v_{n1} & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 2 & \ldots & 2 \\
v_{n2} & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 2 & 0 & \ldots & 2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ldots & \vdots & \vdots & \ddots & \vdots \\
v_{ns} & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 2 & \ldots & 2
\end{array}_{ns\times ns}$$

In $|P_r(G)-\lambda I|$, carryout the following operations.

$$\begin{aligned}
&Step\ 1: && C'_{u_{1j}} \to C_{u_{1j}} - C_{u_{11}}\\
& && C'_{u_{2j}} \to C_{u_{2j}} - C_{u_{21}}\\
& && \vdots\\
& && C'_{u_{mj}} \to C_{u_{mj}} - C_{u_{m1}} \ for\ j=2,3,\dots,s\\
& && C'_{v_{1j}} \to C_{v_{1j}} - C_{v_{11}}\\
& && C'_{v_{2j}} \to C_{v_{2j}} - C_{v_{21}}\\
& && C'_{v_{nj}} \to C_{v_{nj}} - C_{v_{n1}} \ for\ j=2,3,\dots,s\\
&Step\ 2: && Take\ (\lambda+2)^{(m+n)(s-1)} as\ common.\\
&Step\ 3: && R'_{u_{11}} \to R_{u_{11}} + R_{u_{12}} + \dots + R_{u_{1s}}\\
& && R'_{u_{21}} \to R_{u_{21}} + R_{u_{22}} + \dots + R_{u_{2s}}\\
& && \vdots\\
& && R'_{u_{m1}} \to R_{u_{m1}} + R_{u_{m2}} + \dots + R_{u_{ms}}\\
& && R'_{v_{11}} \to R_{v_{11}} + R_{v_{12}} + \dots + R_{v_{1s}}\\
& && R'_{v_{21}} \to R_{v_{21}} + R_{v_{22}} + \dots + R_{v_{2s}}\\
& && \vdots\\
& && R'_{v_{n1}} \to R_{v_{n1}} + R_{v_{n2}} + \dots + R_{v_{ns}}
\end{aligned}$$

Thus the determinant reduces to the form

$$\begin{array}{c|cccccccc|}
 & u_1 & u_2 & \cdots & u_s & v_1 & v_2 & \cdots & v_s\\
u_1 & -\lambda+2(s-1) & 0 & \cdots & 0 & s & s & \cdots & s\\
u_2 & 0 & -\lambda+2(s-1) & \cdots & 0 & s & s & \cdots & s\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
u_s & 0 & 0 & \cdots & -\lambda+2(s-1) & s & s & \cdots & s\\
v_1 & s & s & \cdots & s & -\lambda+2(s-1) & 0 & \cdots & 0\\
v_2 & s & s & \cdots & s & 0 & -\lambda+2(s-1) & \cdots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
v_s & s & s & \cdots & s & 0 & 0 & \cdots & -\lambda+2(s-1)
\end{array}$$

$$\begin{aligned}
&Step\ 4: && C'_{u_i} \to C_{u_i} - C_{u_1} \ for\ i=2,3,\dots,m\\
& && C'_{v_j} \to C_{v_j} - C_{v_1} \ for\ j=2,3,\dots,n\\
&Step\ 5: && Take\ [\lambda-2(s-1)]^{m+n-2}\ as\ common\\
&Step\ 6: && R'_{u_1} \to R_{u_1} + R_{u_2} + \dots + R_{u_m}\\
& && R'_{v_1} \to R_{v_1} + R_{v_2} + \dots + R_{v_n}
\end{aligned}$$

Thus, we have

$$\begin{vmatrix} -\lambda+2(s-1) & ms\\ ns & -\lambda+2(s-1)\end{vmatrix},$$

which on expansion gives

$$\lambda^2 + (4-4s)\lambda + [s^2(4-mn) - 8s + 4] = 0.$$

Therefore, $\lambda = 2(s-1) \pm s\sqrt{mn}$.

$$Spec(K_{m,n}[K_s]) = \begin{cases} -2 & (m+n)(s-1) \text{ times} \\ 2(s-1) & (m+n-2) \text{ times} \\ 2(s-1)+s\sqrt{mn} & \text{once} \\ 2(s-1)-s\sqrt{mn} & \text{once.} \end{cases}$$

Thus,

$$E_{P_r}(G) = 2(m+n)(s-1) + 2(s-1)(m+n-2) + 2s\sqrt{mn}.$$

□

Proof. (*ii*) With similar discussions as above,

$$Spec(\overline{K_{m,n}[K_s]})_r = \begin{cases} -2 & (m+n)(s-1) \text{ times} \\ (s-2) & (m+n-2) \text{ times} \\ s(m+1)-2 & \text{once} \\ s(n+1)-2 & \text{once.} \end{cases}$$

$$E_{P_r}\overline{(G)_r} = 4(m+n)(s-1).$$

□

Proof. (*iii*) With similar discussions, we get

$$Spec(\overline{K_{m,n}[K_s]})_{r(i)} = \begin{cases} 1 & (m+n)(s-1) \text{ times} \\ (1-s) & (m+n-2) \text{ times} \\ (1-s)+s\sqrt{mn} & \text{once} \\ (1-s)-s\sqrt{mn} & \text{once.} \end{cases}$$

Thus,

$$E_{P_r}\overline{(G)_{r(i)}} = (m+n)(s-1) + |1-s|(m+n-2) + 2s\sqrt{mn}.$$

□

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