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NARAYANA MATRIX SEQUENCE

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Abstract. In the present paper, we define Narayana matrix sequence associated with Narayana sequence. We obtain Binet's formula, sum of first n terms, sum of n terms having subscripts of the form $3k$, $3k+1$, and $3k+2$, Catalan identity, D'Ocagne's identity, sum of cubes of n terms, and other properties. Further, Binet's formula for Narayana sequence is derived with the help of Binet's formula for Narayana matrix sequence. Next, we define Narayana matrix sequence for negative subscripts and obtain some properties and find interesting relations between the general terms of Narayana matrix sequence for positive subscripts and negative subscripts.

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1. Introduction

Indian mathematician Narayana Pandit introduced Narayana sequence or numbers in the 14th century. The Narayana numbers are the solution to the problem of *cows and calves*. The problem is moduled as [1]: *"A cow produces one calf every year. Beginning in its fourth year, each calf produces one calf. How many calves are there together after 30 years?"*.
Narayana sequence is defined by the third order recurrence relation as

$$N_n = N_{n-1} + N_{n-3} \text{ for all } n \geq 3$$

with initial conditions $N_0 = 0,\ N_1 = 1,\ N_2 = 1$.
In [6] Ramirez and Sirvent generalised the Narayana sequence into one parameter, which is known as k-Narayana sequence. k-Narayana sequence is defined as $b_{k,n} = kb_{k,n-1} + b_{k,n-3}$ for all $n \geq 3$
with the initial conditions $b_{k,0} = 0,\ b_{k,1} = 1,\ b_{k,2} = k$.
The authors [6] gave the properties and relations of this sequence by a using matrix approach. They related these numbers to Hessenberg matrices.
In [7] Soykan gave another generalisation of Narayana sequence. In [4] Goy gave connections between Fibonacci numbers and Narayana numbers and also between Tribonacci numbers and Narayana numbers. He gave some Toeplitz-Hessenberg matrices, the entries of which are Fibonacci-Narayana numbers. In the literature various sequences are defined for negative subscripts [2, 3, 8].
Narayana sequence has applications in many areas including coding theory, cryptography, MIMO communication system, and graph theory [5].
There have been very few works in the field of Narayana sequence.

In the present paper, Narayana matrix sequence is defined and its properties are developed.

2. Narayana matrix sequence

Here we develop Narayana matrix sequence(NMS).

Definition 2.1. *Narayana matrix sequence is defined as*

$$\mathcal{N}_n = \mathcal{N}_{n-1} + \mathcal{N}_{n-3}, for\ all\ n \geq 3 \tag{2.1}$$

with the initial conditions

$$\mathcal{N}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathcal{N}_1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \mathcal{N}_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{2.2}$$

$$\{\mathcal{N}_i\}_{i=0}^{\infty} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \ldots \right\}$$

Infact

$$\mathcal{N}_n = \begin{pmatrix} N_{n+1} & N_{n-1} & N_n \\ N_n & N_{n-2} & N_{n-1} \\ N_{n-1} & N_{n-3} & N_{n-2} \end{pmatrix}. \tag{2.3}$$

Theorem 2.1. *Generating function of NMS is:* $\sum_{i=0}^{\infty} \mathcal{N}_i = \dfrac{z}{1-z-z^3}$.

Proof. By using recurrence relation (2.1)
$\mathcal{N}_n = \mathcal{N}_{n-1} + \mathcal{N}_{n-3}$
$\sum_{n=3}^{\infty} \mathcal{N}_n z^n - \sum_{n=3}^{\infty} \mathcal{N}_{n-1} z^n - \sum_{n=3}^{\infty} \mathcal{N}_{n-3} z^n = 0$
By using equation (2.2), we get
$\sum_{n=0}^{\infty} \mathcal{N}_n z^n = \dfrac{z}{1-z-z^3}$. □

Theorem 2.2. *Binet's formula of NMS is:* $A\alpha^n + B\beta^n + C\gamma^n$
where $\alpha,\ \beta,\ \gamma$ *are the roots of the equation* $x^3 - x^2 - 1 = 0$ *and* $A,\ B,\ C$ *are the matrices given by*

$$A = \frac{1}{(\alpha-\beta)(\alpha-\gamma)} \begin{pmatrix} (1-\beta)(1-\gamma) & 1 & 1-\beta-\gamma \\ 1-\beta-\gamma & \beta\gamma & 1 \\ 1 & -(\beta+\gamma) & \beta\gamma \end{pmatrix}$$

$$B = \frac{1}{(\beta-\gamma)(\beta-\alpha)} \begin{pmatrix} (1-\alpha)(1-\gamma) & 1 & 1-\alpha-\gamma \\ 1-\alpha-\gamma & \alpha\gamma & 1 \\ 1 & -(\alpha+\gamma) & \alpha\gamma \end{pmatrix}$$

$$C = \frac{1}{(\gamma-\alpha)(\gamma-\beta)} \begin{pmatrix} (1-\alpha)(1-\beta) & 1 & 1-\alpha-\beta \\ 1-\alpha-\beta & \alpha\beta & 1 \\ 1 & -(\alpha+\beta) & \alpha\beta \end{pmatrix}$$

or

$$A = \frac{(\alpha-1)\mathcal{N}_1 + \mathcal{N}_2 + \beta\gamma\mathcal{N}_0}{(\alpha-\beta)(\alpha-\gamma)},$$
$$B = \frac{(\beta-1)\mathcal{N}_1 + \mathcal{N}_2 + \alpha\gamma\mathcal{N}_0}{(\beta-\alpha)(\beta-\gamma)},$$
$$C = \frac{(\gamma-1)\mathcal{N}_1 + \mathcal{N}_2 + \alpha\beta\mathcal{N}_0}{(\gamma-\alpha)(\gamma-\alpha)}.$$

Proof. Take $\mathcal{N}_n = A\alpha^n + B\beta^n + C\gamma^n$ and $n = 0, n = 1, n = 2$.
We get three equations from which the values of A, B, and C can be obtained. □

Theorem 2.3. *Binet's formula for Narayana sequence from Binet's formula of Narayana matrix sequence is* $N_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}$ *where* α, β, *and* γ *are the roots of the equation* $x^3 - x^2 - 1 = 0$.

Proof. $\mathcal{N}_n = A\alpha^n + B\beta^n + C\gamma^n$

$$\begin{pmatrix} N_{n+1} & N_{n-1} & N_n \\ N_n & N_{n-2} & N_{n-1} \\ N_{n-1} & N_{n-3} & N_{n-2} \end{pmatrix} = \frac{(\alpha-1)\mathcal{N}_1 + \mathcal{N}_2 + \beta\gamma\mathcal{N}_0}{(\alpha-\beta)(\alpha-\gamma)}\alpha^n + \frac{(\beta-1)\mathcal{N}_1 + \mathcal{N}_2 + \alpha\gamma\mathcal{N}_0}{(\beta-\alpha)(\beta-\gamma)}\beta^n + \frac{(\gamma-1)\mathcal{N}_1 + \mathcal{N}_2 + \alpha\beta\mathcal{N}_0}{(\gamma-\alpha)(\gamma-\alpha)}\gamma^n$$

$$\begin{pmatrix} N_{n+1} & N_{n-1} & N_n \\ N_n & N_{n-2} & N_{n-1} \\ N_{n-1} & N_{n-3} & N_{n-2} \end{pmatrix}$$
$$= \frac{1}{(\alpha-\beta)(\alpha-\gamma)} \begin{pmatrix} (1-\beta)(1-\gamma) & 1 & 1-\beta-\gamma \\ 1-\beta-\gamma & \beta\gamma & 1 \\ 1 & -(\beta+\gamma) & \beta\gamma \end{pmatrix} \alpha^n$$
$$+ \frac{1}{(\beta-\gamma)(\beta-\alpha)} \begin{pmatrix} (1-\alpha)(1-\gamma) & 1 & 1-\alpha-\gamma \\ 1-\alpha-\gamma & \alpha\gamma & 1 \\ 1 & -(\alpha+\gamma) & \alpha\gamma \end{pmatrix} \beta^n$$
$$+ \frac{1}{(\gamma-\alpha)(\gamma-\beta)} \begin{pmatrix} (1-\alpha)(1-\beta) & 1 & 1-\alpha-\beta \\ 1-\alpha-\beta & \alpha\beta & 1 \\ 1 & -(\alpha+\beta) & \alpha\beta \end{pmatrix} \gamma^n$$

Comparing (1,3)th element

$$N_n = \frac{1-\beta-\gamma}{(\alpha-\beta)(\alpha-\gamma)}\alpha^n + \frac{1-\alpha-\gamma}{(\beta-\gamma)(\beta-\alpha)}\beta^n + \frac{1-\alpha-\beta}{(\gamma-\alpha)(\gamma-\beta)}\gamma^n$$
$$N_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}.$$
□

Theorem 2.4. *Sum of first n terms of NMS*
$\mathcal{N}_1 + \mathcal{N}_2 + \mathcal{N}_3 + ... + \mathcal{N}_n = \mathcal{N}_{n+3} - \mathcal{N}_3$.

Proof. From Binet's formula, we get

$$\begin{aligned}\mathcal{N}_1+\mathcal{N}_2+...+\mathcal{N}_n &= A(\alpha^1+\alpha^2+...+\alpha^n)+B(\beta^1+\beta^2+...+\beta^n)\\ &+C(\gamma^1+\gamma^2+...+\gamma^n)\\ &= A\left(\frac{\alpha(\alpha^n-1)}{\alpha-1}\right)+B\left(\frac{\beta(\beta^n-1)}{\beta-1}\right)+C\left(\frac{\gamma(\gamma^n-1)}{\gamma-1}\right)\\ &= A(\alpha^3(\alpha^n-1))+B(\beta^3(\beta^n-1))+C(\gamma^3(\gamma^n-1))\\ &= A\alpha^{n+3}+B\beta^{n+3}+C\gamma^{n+3}-(A\alpha^3+B\beta^3+C\gamma^3)\\ &= \mathcal{N}_{n+3}-\mathcal{N}_3.\end{aligned}$$

□

Theorem 2.5. $\mathcal{N}_3+\mathcal{N}_6+\mathcal{N}_9+...+\mathcal{N}_{3n}=\mathcal{N}_{3n+1}-\mathcal{N}_1$.

Proof.

$$\begin{aligned}\mathcal{N}_3+\mathcal{N}_6+...+\mathcal{N}_{3n} &= A(\alpha^3+\alpha^6+...+\alpha^{3n})+B(\beta^3+\beta^6+...+\beta^{3n})\\ &+C(\gamma^3+\gamma^6+...+\gamma^{3n})\\ &= A\frac{\alpha^3(\alpha^{3n}-1)}{\alpha^3-1}+B\frac{\beta^3(\beta^{3n}-1)}{\beta^3-1}+C\frac{\gamma^3(\gamma^{3n}-1)}{\gamma^3-1}\\ &= A\alpha(\alpha^{3n}-1)+B\beta(\beta^{3n}-1)+C\gamma(\gamma^{3n}-1)\\ &= A\alpha^{3n+1}+B\beta^{3n+1}+C\gamma^{3n+1}-(A\alpha+B\beta+C\gamma)\\ &= \mathcal{N}_{3n+1}-\mathcal{N}_1.\end{aligned}$$

□

Theorem 2.6. $\mathcal{N}_4+\mathcal{N}_7+\mathcal{N}_{10}+...+\mathcal{N}_{3n+1}=\mathcal{N}_{3n+2}-\mathcal{N}_2$.

Proof. Proof is similar to Theorem 2.5. □

Theorem 2.7. $\mathcal{N}_5+\mathcal{N}_8+\mathcal{N}_{11}+...+\mathcal{N}_{3n+2}=\mathcal{N}_{3n+3}-\mathcal{N}_3$.

Proof. Proof is similar to Theorem 2.5. □

Theorem 2.8. $\mathcal{N}_n\mathcal{N}_m=\mathcal{N}_{n+m}$.

Proof. We will prove the theorem by induction on m.
For $m=1$
By using equation (2.3) and initial conditions of the sequence

$$\begin{aligned}\mathcal{N}_n\mathcal{N}_1 &= \begin{pmatrix}N_{n+1} & N_{n-1} & N_n\\ N_n & N_{n-2} & N_{n-1}\\ N_{n-1} & N_{n-3} & N_{n-2}\end{pmatrix}\begin{pmatrix}1 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\\ &= \begin{pmatrix}N_{n+2} & N_n & N_{n+1}\\ N_{n+1} & N_{n-1} & N_n\\ N_n & N_{n-2} & N_{n-1}\end{pmatrix}=\mathcal{N}_{n+1}\end{aligned}$$

Now suppose result is true for $m = k$ i.e. $\mathcal{N}_{n+k} = \mathcal{N}_n \mathcal{N}_k$
We will prove result for $m = k + 1$.

$$\mathcal{N}_n \mathcal{N}_{k+1} = \mathcal{N}_{n+k} \mathcal{N}_1 = \begin{pmatrix} N_{n+k+1} & N_{n+k-1} & N_{n+k} \\ N_{n+k} & N_{n+k-2} & N_{n+k-1} \\ N_{n+k-1} & N_{n+k-3} & N_{n+k-2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} N_{n+k+2} & N_{n+k} & N_{n+k+1} \\ N_{n+k+1} & N_{n+k-1} & N_{n+k} \\ N_{n+k} & N_{n+k-2} & N_{n+k-1} \end{pmatrix} = \mathcal{N}_{n+k+1}$$

So theorem holds for all m. □

Corollary 2.9. $\mathcal{N}_n \mathcal{N}_m = \mathcal{N}_m \mathcal{N}_n$.

Proof. From Theorem 2.8, we get
$\mathcal{N}_n \mathcal{N}_m = \mathcal{N}_{n+m} = \mathcal{N}_{m+n} = \mathcal{N}_m \mathcal{N}_n$. □

Corollary 2.10. *(Catalan's identity:)* $\mathcal{N}_{n-r} \mathcal{N}_{n+r} = \mathcal{N}_n^2$.

Proof. From Theorem 2.8, we get
$\mathcal{N}_{n-r} \mathcal{N}_{n+r} = \mathcal{N}_{n-r+n+r} = \mathcal{N}_{2n} = \mathcal{N}_n \mathcal{N}_n = \mathcal{N}_n^2$. □

Corollary 2.11. *(D'Ocagne's identity:)* $\mathcal{N}_m \mathcal{N}_{n+1} - \mathcal{N}_{m+1} \mathcal{N}_n = 0$.

Proof. From Theorem 2.8, we get
$\mathcal{N}_m \mathcal{N}_{n+1} - \mathcal{N}_{m+1} \mathcal{N}_n = \mathcal{N}_{m+n+1} - \mathcal{M}_{m+1+n} = 0$. □

Corollary 2.12. *(Cassini's identity:)* $\mathcal{N}_n \mathcal{N}_{n+1} - \mathcal{N}_{n+1} \mathcal{N}_n = 0$.

Proof. $\mathcal{N}_n \mathcal{N}_{n+1} - \mathcal{N}_{n+1} \mathcal{N}_n = \mathcal{N}_{2n+1} - \mathcal{N}_{2n+1} = 0$. □

Corollary 2.13. $\mathcal{N}_n^m = \mathcal{N}_{mn}$.

Proof. $\mathcal{N}_n^m = \mathcal{N}_n \mathcal{N}_n \mathcal{N}_n ... \mathcal{N}_n$
$= \mathcal{N}_{n+n+n+n+...+n} = \mathcal{N}_{mn}$. □

Corollary 2.14. $\mathcal{N}_1^3 + \mathcal{N}_2^3 + \mathcal{N}_3^3 + ... + \mathcal{N}_n^3 = \mathcal{N}_{3n+1} - \mathcal{N}_1$.

Proof.

$$\begin{aligned} \mathcal{N}_1^3 + \mathcal{N}_2^3 + \mathcal{N}_3^3 + ... + \mathcal{N}_n^3 &= \mathcal{N}_3 + \mathcal{N}_6 + \mathcal{N}_9 + ... + \mathcal{N}_{3n} \\ &= \mathcal{N}_{3n+1} - \mathcal{N}_1. \end{aligned}$$

□

3. Narayana matrix sequence with negative indices

We define Narayana matrix sequence with negative indices as

$$\mathcal{N}_{-n} = -\mathcal{N}_{-n+2} + \mathcal{N}_{-n+3} \; for \; all \; n \geq 3 \tag{3.4}$$

with the initial conditions

$\mathcal{N}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathcal{N}_{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$, $\mathcal{N}_{-2} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$

$\{\mathcal{N}_{-i}\}_{i=0}^{\infty}$
$= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}, ... \right\}.$

Theorem 3.1. $\mathcal{N}_{-n} = \begin{pmatrix} N_{-n+1} & N_{-n-1} & N_{-n} \\ N_{-n} & N_{-n-2} & N_{-n-1} \\ N_{-n-1} & N_{-n-3} & N_{-n-2} \end{pmatrix}$, *for all* $n \geq 1$.

Proof. The theorem is proved by induction on n.
For $n = 1$

$$\mathcal{N}_{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} N_0 & N_{-2} & N_{-1} \\ N_{-1} & N_{-3} & N_{-2} \\ N_{-2} & N_{-4} & N_{-3} \end{pmatrix}$$

Assume result is true for $n = k$, so that

$$\mathcal{N}_{-k} = \begin{pmatrix} N_{-k+1} & N_{-k-1} & N_{-k} \\ N_{-k} & N_{-k-2} & N_{-k-1} \\ N_{-k-1} & N_{-k-3} & N_{-k-2} \end{pmatrix}$$

Now we will prove the result for $n = k + 1$.

$$\begin{aligned} \mathcal{N}_{-(k+1)} &= N_{-(k-2)} - N_{-(k-1)} \\ &= \begin{pmatrix} N_{-k+3} & N_{-k+1} & N_{-k+2} \\ N_{-k+2} & N_{-k} & N_{-k+1} \\ N_{-k+1} & N_{-k-1} & N_{-k} \end{pmatrix} - \begin{pmatrix} N_{-k+2} & N_{-k} & N_{-k+1} \\ N_{-k+1} & N_{-k-1} & N_{-k} \\ N_{-k} & N_{-k-2} & N_{-k-1} \end{pmatrix} \\ &= \begin{pmatrix} N_{-k} & N_{-k-2} & N_{-k-1} \\ N_{-k-1} & N_{-k-3} & N_{-k-2} \\ N_{-k-2} & N_{-k-4} & N_{-k-3} \end{pmatrix} \end{aligned}$$

This completes the proof. □

Theorem 3.2. *Binet's formula for negative subscripted Narayana matrix sequence is* $\mathcal{N}_{-n} = A\alpha^{-n} + B\beta^{-n} + C\gamma^{-n}$
where $\alpha,\ \beta,\ \gamma$ *are the roots of recurrence relation given by equation (3.4), and*

$$\begin{aligned} A &= \frac{\alpha^2\mathcal{N}_0 + \alpha\mathcal{N}_{-2} + \alpha^2(\alpha-1)\mathcal{N}_{-1}}{(\alpha-\beta)(\alpha-\gamma)} \\ B &= \frac{\beta^2\mathcal{N}_0 + \beta\mathcal{N}_{-2} + \beta^2(\beta-1)\mathcal{N}_{-1}}{(\beta-\gamma)(\beta-\alpha)}, \\ C &= \frac{\gamma^2\mathcal{N}_0 + \gamma\mathcal{N}_{-2} + \gamma^2(\gamma-1)\mathcal{N}_{-1}}{(\gamma-\alpha)(\gamma-\beta)}. \end{aligned}$$

Proof. Proof is similar to Theorem 2.2. □

Theorem 3.3. *For* $n \geq 1$, *we have* $\mathcal{N}_{-n} = \mathcal{N}_n^{-1}$.

Proof. For $n = 1$

$$\mathcal{N}_{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} = \mathcal{N}_1^{-1}$$

Suppose theorem holds for $n = k$.
For $n = k + 1$

$$
\begin{aligned}
\mathcal{N}_{k+1}^{-1} &= (\mathcal{N}_k\mathcal{N}_1)^{-1} = \mathcal{N}_1^{-1}\mathcal{N}_k^{-1} = \mathcal{N}_{-1}\mathcal{N}_{-k} \\
&= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} N_{-k+1} & N_{-k-1} & N_{-k} \\ N_{-k} & N_{-k-2} & N_{-k-1} \\ N_{-k-1} & N_{-k-3} & N_{-k-2} \end{pmatrix} \\
&= \begin{pmatrix} N_{-k} & N_{-k-2} & N_{-k-1} \\ N_{-k-1} & N_{-k-3} & N_{-k-2} \\ N_{-k-2} & N_{-k-4} & N_{-k-3} \end{pmatrix} \\
&= \mathcal{N}_{-(k+1)}.
\end{aligned}
$$

□

Corollary 3.4. $\mathcal{N}_{-m}\mathcal{N}_{-n} = \mathcal{N}_{-m-n}$.

Proof.

$$
\begin{aligned}
\mathcal{N}_{-m}\mathcal{N}_{-n} &= \mathcal{N}_m^{-1}\mathcal{N}_n^{-1} = (\mathcal{N}_n\mathcal{N}_m)^{-1} \\
&= (\mathcal{N}_{m+n})^{-1} = \mathcal{N}_{-m-n}.
\end{aligned}
$$

□

Corollary 3.5. $\mathcal{N}_{-n}^m = \mathcal{N}_{-mn}$.

Proof.

$$
\begin{aligned}
\mathcal{N}_{-n}^m &= (\mathcal{N}_n^{-1})^m = (\mathcal{N}_n^m)^{-1} \\
&= (\mathcal{N}_{mn})^{-1} = \mathcal{N}_{-mn}.
\end{aligned}
$$

□

Corollary 3.6. $(\mathcal{N}_{-n-1})^m = (\mathcal{N}_{-1})^m\mathcal{N}_{-mn}$.

Proof.

$$
\begin{aligned}
(\mathcal{N}_{-n-1})^m &= \mathcal{N}_{m(-n-1)} = \mathcal{N}_{-1}\mathcal{N}_{-m+1}\mathcal{N}_{-mn} \\
&= (\mathcal{N}_{-1})^2\mathcal{N}_{-m+2}\mathcal{N}_{-mn} \\
&\cdots \\
&= (\mathcal{N}_{-1})^m\mathcal{N}_{-m+m}\mathcal{N}_{-mn} \\
&= (\mathcal{N}_{-1})^m\mathcal{N}_{-mn}.
\end{aligned}
$$

□

Corollary 3.7. $\mathcal{N}_{-n-r}\mathcal{N}_{-n+r} = (\mathcal{N}_{-n})^2 = (\mathcal{N}_{-2})^n$.

Proof.

$$
\begin{aligned}
\mathcal{N}_{-n-r}\mathcal{N}_{-n+r} &= \mathcal{N}_{-n}\mathcal{N}_{-r}\mathcal{N}_{-n}\mathcal{N}_r \\
&= \mathcal{N}_{-n}(\mathcal{N}_r)^{-1}\mathcal{N}_r\mathcal{N}_{-n} \\
&= (\mathcal{N}_{-n})^2 = (\mathcal{N}_{-2})^n.
\end{aligned}
$$

□

Corollary 3.8. $\mathcal{N}_{-m}\mathcal{N}_{-n+1} - \mathcal{N}_{-m+1}\mathcal{N}_{-n} = 0$.

Proof. $\mathcal{N}_{-m}\mathcal{N}_{-n+1} - \mathcal{N}_{-m+1}\mathcal{N}_{-n} = \mathcal{N}_{-m-n+1} - \mathcal{N}_{-m+1-n} = 0.$ □

Corollary 3.9. $\mathcal{N}_{-n}\mathcal{N}_{-n+1} - \mathcal{N}_{-n+1}\mathcal{N}_{-n} = 0.$

Proof. $\mathcal{N}_{-n}\mathcal{N}_{-n+1} - \mathcal{N}_{-n+1}\mathcal{N}_{-n} = \mathcal{N}_{-n-n+1} - \mathcal{N}_{-n+1-n} = 0.$ □

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