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PROPERTIES AND FURTHER GENERALIZATION ON THE EXTENSION OF τ-GAUSS HYPERGEOMETRIC FUNCTION

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Abstract. Recently, an extension of τ Gauss hypergeometric function was obtained in terms of the extended version of the pochhammer symbol[11]. We have established some properties on further generalization of the extended τ Gauss hypergeometric function containing extra parameters. We have also established some other properties and relationships involving the integral representations, derivative formulas and Mellin transforms.

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1. Introduction,Definitions and Preliminaries

Definition 1.1 *([11])Let a function $\Theta(\{k_l\}_{l\in N_0};z)$ be analytic within the disk $|z|<R(0<R<1)$ and let its Taylor-Maclaurin coefficients be explicitly denoted by the sequence $\{k_l\}_{l\in N_0}$. Suppose also that the function $\Theta(\{k_l\}_{l\in N_0};z)$ can be continued analytically in the right half-plane $\Re(z)>0$ with the asymptotic property given as follows:*

$$\Theta(\{k_l\}_{l\in N_0};z)=\begin{cases}\sum_{l=0}^{\infty}\{k_l\}\frac{z^l}{l!}; & (|z|<R;0<R<\infty;k_0=1);\\ M_0z^w\exp(z)[1+O(\frac{1}{z})]; & (\Re(z)\to\infty;M_0>0;w\in\mathbb{C})\end{cases}\tag{1}$$

for some suitable constants M_0 and w depending essentially on the sequence $\{k_l\}_{l\in N_0}$.They also defined extended Gamma function $\Gamma_p^{\{k_l\}}(z)$ and the extended Beta function.

$$\Gamma_p^{\{k_l\}}(z)=\int_0^{\infty}t^{z-1}\Theta(\{k_l\};-t-\frac{p}{t})dt,\qquad(\Re(p)\geqslant0,\Re(z)>0)\tag{2}$$

and

$$B_p^{\{k_l\}}(\alpha,\beta;p)=\int_0^1t^{\alpha-1}(1-t)^{\beta-1}\Theta\left(\{k_l\};\frac{-p}{t(1-t)}\right)dt,\tag{3}$$
$$(\Re(p)\geqslant0,min(\Re(\alpha),\Re(\beta))>0).$$

By introducing one additional parameter q with $\Re(q)\geqslant0$

$$B_{p,q}^{\{k_l\}}(\alpha,\beta)=\int_0^1t^{\alpha-1}(1-t)^{\beta-1}\Theta\left(\{k_l\};-\frac{p}{t}-\frac{q}{(1-t)}\right)dt,\tag{4}$$
$$min(\Re(p),\Re(q))>0;min(\Re(\alpha),\Re(\beta))>0.$$

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During the past few decades, various extensions and generalizations of well-known special functions have been studied by various researchers [3,5,7-9]. Chaudhry et.al. also defined a 2-parameter extension of Gamma function $\Gamma(\xi)$ with the parameter (p and v) in [5] as follows :

$$\Gamma_v(\xi;p) = \begin{cases} \sqrt{\frac{2p}{\pi}}\int_0^\infty t^{\xi-\frac{3}{2}}e^{-t}k_{v+\frac{1}{2}}(\frac{p}{t})dt, & (min(\Re(p),\Re(v))>0;\xi\in\mathbb{C}), \\ \Gamma_p(\xi) & (v=o;\Re(\xi)>0), \end{cases} \tag{5}$$

where $k_v(z)$is the modified Bessel function of order v and $\Gamma_p(\xi)$ was studied in [3,5]. Indeed if v=0 in (4) and make use of the following relationship

$$k_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}e^{-z}, \tag{6}$$

then the above-extended gamma function can be given as

$$\Gamma_p(\xi) = \int_0^\infty t^{\xi-1}e^{-t-\frac{p}{t}}dt, \qquad (\Re(p)\geqslant 0,\Re(\xi)>0). \tag{7}$$

In the year 2012, Srivastava et.al. [10] defined the following extensions and generalization of the pochhammer symbols as follows :

$$(\xi;p)_\mu = \begin{cases} \frac{\Gamma_p(\xi+\mu)}{\Gamma(\xi)} & (\Re(p)>0;\xi,\mu\in\mathbb{C}); \\ (\xi)_\mu & (p=0;\xi,\mu\in\mathbb{C}\setminus\{0\}); \end{cases} \tag{8}$$

$$(\xi;p,v)_\mu = \begin{cases} \frac{\Gamma_v(\xi+\mu;p)}{\Gamma(\xi)} & (min(\Re(p),\Re(v))>0;\xi,\mu\in\mathbb{C}); \\ (\xi;p)_\mu & (v=0;\xi,\mu\in\mathbb{C}\setminus\{0\}); \end{cases} \tag{9}$$

from (4) and (8), they get

$$(\xi;p,v)_\mu = \frac{1}{\Gamma(\xi)}\sqrt{\frac{2p}{\pi}}\int_0^\infty t^{\xi+\mu-\frac{3}{2}}e^{-t}k_{v+\frac{1}{2}}\left(\frac{p}{t}\right)dt, \tag{10}$$

$$(\xi;p)_\mu = \frac{1}{\Gamma(\xi)}\int_0^\infty t^{\xi+\mu-1}e^{-t-\frac{p}{t}}dt. \tag{11}$$

By using (8), an extension of generalized hypergeometric function ${}_pF_q$ was defined[11] as :

$${}_pF_q\left[\begin{cases}(\rho_1;p,v),\rho_2,\rho_3......,\rho_p \\ \sigma_1,\sigma_2,.....,\sigma_q;\end{cases} ;z\right] = \sum_{n=0}^\infty \frac{(\rho_1;p,v)_n(\rho_2)_n........(\rho_p)_n}{(\sigma_1)_n(\sigma_2)_n.......(\sigma_q)_n}\frac{z^n}{n!}, \tag{12}$$

$$where \qquad \rho_j\in\mathbb{C}(j=1,2,...p) \qquad \sigma_j\in\mathbb{C}\setminus\mathbb{Z}_0^-(j=1,2,...q).$$

Virchenko et al.[12] studied the following τ -Gauss hypergeometric function ${}_2R_1^\tau z$ defined as:

$${}_2R_1^\tau z = {}_2R_1\{\delta_1,\delta_2;\delta_3;\tau,z\} = \frac{\Gamma(\delta_3)}{\Gamma(\delta_2)}\sum_{n=0}^\infty\frac{(\delta_1)_n\Gamma(\delta_2+n\tau)}{\Gamma(\delta_3+n\tau)}\frac{z^n}{n!}. \tag{13}$$

$$(\tau>0;|z|<1;\Re(\delta_3)>\Re(\delta_2)>0 \;\; when\; |z|=1)$$

They also derive the following integral representation :

$$ {}_2R_1(\delta_1,\delta_2;\delta_3;\tau,z)=\frac{1}{B(\delta_2,\delta_3-\delta_2)}\int_0^\infty t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}(1-zt^\tau)^{-\delta_1}dt, \tag{14} $$

$$ (\tau>0;|arg(1-z)|<\pi;\Re(\delta_3)>\Re(\delta_2)>0) $$

,

in terms of classical Beta function $B(\alpha,\beta)$ defined as

$$ B(\alpha,\beta)=\begin{cases}\int_0^1 t^{\alpha-1}(1-t)^{\beta=\alpha}dt & (min(\Re(\alpha),\Re(\beta))>0),\\ \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha,\beta\in\mathbb{C}/\mathbb{Z}_0^-).\end{cases} \tag{15} $$

2. Properties on extension of τ-Gauss Hypergeometric Function

In this section, we first established the new extension of τ Gauss Hypergeometric Function using (1) for $\delta_1,\delta_2\in\mathbb{C}$ and $\delta_3\in\mathbb{C}\setminus\mathbb{Z}_0^-$ containing two parameters (μ_1 *and* μ_2)

$$ {}_2R_1^{\{k_l\}}(\delta_1,\delta_2;\delta_3;\tau;z;\mu_1,\mu_2)=(\delta_1)_n\sum_{n=0}^{\infty}\frac{B_{\mu_1,\mu_2}^{\{k_l\}}(\delta_2+n\tau,\delta_3-\delta_2)}{B(\delta_2,\delta_3-\delta_2)}\frac{z^n}{n!}, \tag{16} $$

$$ (\tau>0;|z|<1;\Re(\delta_3)>\Re(\delta_2)>0\ when\ |z|=1,min(\Re(\mu_1),\Re(\mu_2))>0). $$

If $\Theta(k_l;z)=expz$, one can write

$$ {}_2R_1(\delta_1,\delta_2;\delta_3;\tau;z;\mu_1,\mu_2)=(\delta_1)_n\sum_{n=0}^{\infty}\frac{B_{\mu_1,\mu_2}(\delta_2+n\tau,\delta_3-\delta_2)}{B(\delta_2,\delta_3-\delta_2)}\frac{z^n}{n!}. $$

The following integral representation form holds true for (16) :

$$ {}_2R_1^{\{k_l\}}(\delta_1,\delta_2;\delta_3;\tau;z;\mu_1,\mu_2)= $$

$$ \frac{1}{B(\delta_2,\delta_3-\delta_2)}\int_0^1 t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}(1-zt^\tau)^{-\delta_1}\Theta(\{k_l\};-\frac{\mu_1}{t}-\frac{\mu_2}{1-t})dt. \tag{17} $$

$$ (\tau>0;|z|<1;\Re(\delta_3)>\Re(\delta_2)>0\ when\ |z|=1,min(\Re(\mu_1),\Re(\mu_2))>0). $$

If μ_1=μ_2=0, one can write

$$ {}_2R_1^{\{k_l\}}(\delta_1,\delta_2;\delta_3;\tau,z)=\frac{1}{B(\delta_2,\delta_3-\delta_2)}\int_0^1 t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}\ {}_1F_0{}^{\{k_l\}}\left[\delta_1;-;zt^\tau\right]dt, \tag{18} $$

$$ (\tau>0;|z|<1;\Re(\delta_3)>\Re(\delta_2)>0\ when\ |z|=1). $$

Corollary 2.1. *When $\tau=1$,(16) and (17) would immediately yield the standard form of Gauss Hypergeometric Function [7] and its integral representation*

$$ {}_2R_1^{\{k_l\}}(\delta_1,\delta_2;\delta_3;z;\mu_1,\mu_2)=(\delta_1)_n\sum_{n=0}^{\infty}\frac{B_{\mu_1,\mu_2}^{\{k_l\}}(\delta_2+n,\delta_3-\delta_2)}{B(\delta_2,\delta_3-\delta_2)}\frac{z^n}{n!}, \tag{19} $$

$$ {}_2R_1^{\{k_l\}}(\delta_1,\delta_2;\delta_3;z;\mu_1,\mu_2) \tag{20} $$

$$ =\frac{1}{B(\delta_2,\delta_3-\delta_2)}\int_0^1 t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}(1-zt)^{-\delta_1}\Theta(\{k_l\};-\frac{\mu_1}{t}-\frac{\mu_2}{1-t})dt. $$

$(\tau = 1; |z| < 1; \Re(\delta_3) > \Re(\delta_2) > 0 \; when \; |z| = 1, min(\Re(\mu_1), \Re(\mu_2)) > 0).$

3. Properties on the extension of τ-Gauss hypergeometric function in terms of pochhammer symbol

Now we discuss the extension of τ Gauss hypergeometric function ${}_2R_1^{\tau}(z)$in terms of pochhammer symbol $(\xi; p; v)_\mu$ defined in (8) for $\delta_1, \delta_2 \in \mathbb{C}$ and $\delta_3 \in \mathbb{C} \setminus \mathbb{Z}_0^-$ as:

$$(21) \qquad {}_2R_1^{\{k_l\}}[(\delta_1; p, v), \delta_2; \delta_3; \tau, z; \mu_1, \mu_2] = (\delta_1; p, v)_n \sum_{n=0}^{\infty} \frac{B_{\mu_1,\mu_2}^{\{k_l\}}(\delta_2 + n\tau, \delta_3 - \delta_2)}{B(\delta_2, \delta_3 - \delta_2)} \frac{z^n}{n!},$$

$(v > 0; \tau > 0; |z| < 1; \Re(\delta_3) > \Re(\delta_2) > 0 \; when \; |z| = 1 \; and \; p \geqslant 0, min(\Re(\mu_1), \Re(\mu_2)) > 0).$

Remark The following are some of the special-cases of τ Gauss hypergeometric function :

(1) When v=0, equation (21) will reduce to the following extended τ Gauss hypergeometric function

$$(22) \qquad {}_2R_1^{\{k_l\}}[(\delta_1; p), \delta_2; \delta_3; \tau, z; \mu_1, \mu_2] = (\delta_1; p)_n \sum_{n=0}^{\infty} \frac{B_{\mu_1,\mu_2}^{\{k_l\}}(\delta_2 + n\tau, \delta_3 - \delta_2)}{B(\delta_2, \delta_3 - \delta_2)} \frac{z^n}{n!}.$$

(2) When τ=1, equation (21) reduces the extended Gauss hypergeometric function

$$(23) \qquad {}_2R_1^{\{k_l\}}[(\delta_1; p, v), \delta_2; \delta_3; z; \mu_1, \mu_2] = (\delta_1; p, v)_n \sum_{n=0}^{\infty} \frac{B_{\mu_1,\mu_2}^{\{k_l\}}(\delta_2 + n, \delta_3 - \delta_2)}{B(\delta_2, \delta_3 - \delta_2)} \frac{z^n}{n!}.$$

(3) When v=0 and τ=1 equation (21) reduces to extended Gauss hypergeometric function

$$(24) \qquad {}_2R_1^{\{k_l\}}[(\delta_1; p), \delta_2; \delta_3; z; \mu_1, \mu_2] = (\delta_1; p)_n \sum_{n=0}^{\infty} \frac{B_{\mu_1,\mu_2}^{\{k_l\}}(\delta_2 + n, \delta_3 - \delta_2)}{B(\delta_2, \delta_3 - \delta_2)} \frac{z^n}{n!}.$$

4. Integral Representations and derivative formula

Theorem 4.1. *The following integral representation hold true for equation (21) is:*

$$(25) \qquad {}_2R_1^{\{k_l\}}[(\delta_1; p, v), \delta_2; \delta_3; \tau, z; \mu_1, \mu_2]$$

$$= \frac{1}{B(\delta_2, \delta_3 - \delta_2)} \int_0^1 t^{\delta_2 - 1}(1 - t)^{\delta_3 - \delta_2 - 1} \; {}_1F_0^{\{k_l\}} \left[(\delta_1, p, v); -; zt^\tau; \mu_1, \mu_2\right] dt,$$

$(\Re(p) > 0, v > 0, \tau > 0, |z| < 1, \Re(\delta_3) > \Re(\delta_2) > 0, min(\Re(\mu_1), \Re(\mu_2)) > 0)$

where $B(\alpha, \beta)$ denotes the classical Beta function defined in (14).

Special cases: When μ_1=μ_2=0

(26)

$${}_2R_1^{\{k_l\}}[(\delta_1; p, v), \delta_2; \delta_3; \tau, z] = \frac{1}{B(\delta_2, \delta_3 - \delta_2)} \int_0^1 t^{\delta_2 - 1}(1-t)^{\delta_3 - \delta_2 - 1} \; {}_1F_0^{\{k_l\}} \left[(\delta_1, p, v); -; zt^\tau\right] dt.$$

Corollary 4.2. *The following are some of the special-cases of integral representation of τ-gauss hypergeometric function:*

(1) *When v=0, equation(26) reduces to the following τ-Gauss hypergeometric function*

(27)
$$ {}_2R_1^{\{k_l\}}((\delta_1;p),\delta_2;\delta_3;\tau,z) = \frac{1}{B(\delta_2,\delta_3-\delta_2)}\int_0^1 t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}{}_1F_0^{\{k_l\}}[(\delta_1,p);-;zt^{\tau}]dt. $$

(2) *When $\tau=1$ equation (26) will yield the following extended Gauss hypergeometric function*

(28)
$$ {}_2F_1^{\{k_l\}}[(\delta_1;p,v),\delta_2;\delta_3;z] = \frac{1}{B(\delta_2,\delta_3-\delta_2)}\int_0^1 t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}{}_1F_0^{\{k_l\}}[(\delta_1,p,v);-;zt]dt. $$

(3) *When v=0 and $\tau=1$ equation (26) reduces to the extended gauss hypergeometric function*

(29)
$$ {}_2F_1^{\{k_l\}}[(\delta_1;p),\delta_2;\delta_3;z] = \frac{1}{B(\delta_2,\delta_3-\delta_2)}\int_0^1 t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}{}_1F_0^{\{k_l\}}[(\delta_1,p);-;zt]dt. $$

Theorem 4.3. *The following Laplace type representation holds true for (21) :*

(30)
$$ {}_2R_1^{\{k_l\}}[(\delta_1;p,v),\delta_2;\delta_3;\tau,z;\mu_1,\mu_2] = \frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(\delta_1)}\int_0^\infty t^{\delta_1-\frac{3}{2}}e^{-t}k_{v+\frac{1}{2}}(\frac{p}{t}){}_1\Phi_1^{\{k_l\}}[\delta_2,\delta_3;\tau;zt,\mu_1,\mu_2]dt, $$
$$ (\Re(p)>0;v>0;\tau>0;\Re(z)<1;\Re(\delta_1)>0,min(\Re(\mu_1),\Re(\mu_2))>0) $$

where ${}_1\Phi_1^{\{k_l\}}[\delta_2,\delta_3;\tau;zt,\mu_1,\mu_2]$ is extended τ-kummer hypergeometric function defined as :

$$ {}_1\Phi_1^{\{k_l\}}[\delta_2,\delta_3;\tau;zt,\mu_1,\mu_2] = \sum_{n=0}^{\infty}\frac{B_{\mu_1,\mu_2}^{\{k_l\}}(\delta_2+n\tau,\delta_3-\delta_2)}{B(\delta_2,\delta_3-\delta_2)}\frac{z^n}{n!}, $$
$$ (\tau>0;\delta_2\in\mathbb{C};\delta_3\in\mathbb{C}\setminus\mathbb{Z}_0^-,min(\Re(\mu_1),\Re(\mu_2))>0). $$

Proof : By first utilizing (9) in (21) and then applying ${}_1\Phi_1^{\{k_l\}}[\delta_2,\delta_3;\tau;zt,\mu_1,\mu_2]$; We obtain the assertion (30) of theorem 4.3.

Corollary 4.4. *Let μ_1=μ_2=0 in assertion (30), we get*

$$ {}_2R_1^{\{k_l\}}[(\delta_1;p,v),\delta_2;\delta_3;\tau,z] = \frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(\delta_1)}\int_0^\infty t^{\delta_1-\frac{3}{2}}e^{-t}k_{v+\frac{1}{2}}(\frac{p}{t}){}_1\Phi_1^{\{k_l\}}[\delta_2,\delta_3;\tau;zt]dt. \quad (31) $$

Special cases

(1) When τ=1, (31) will yield the following special-cases :

$$ {}_2F_1^{\{k_l\}}[(\delta_1;p,v),\delta_2;\delta_3;z] = \frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(\delta_1)}\int_0^\infty t^{\delta_1-\frac{3}{2}}e^{-t}k_{v+\frac{1}{2}}(\frac{p}{t}){}_1F_1^{\{k_l\}}[\delta_2,\delta_3;zt]dt. $$

(2) When v=0, (31) will yield the following extended τ-Gauss hypergeometric function

$$ {}_2R_1^{\{k_l\}}[(\delta_1;p),\delta_2;\delta_3;\tau,z] = \frac{1}{\Gamma(\delta_1)}\int_0^\infty t^{\delta_1-1}e^{-t-\frac{p}{t}}\,{}_1\Phi_1{}^{\{k_l\}}[\delta_2,\delta_3;\tau;zt]dt. $$

(3) When τ=1 and v=0, (31) will reduce to the following extended Gauss hypergeometric function

$$ {}_2F_1^{\{k_l\}}[(\delta_1;p),\delta_2;\delta_3;z] = \frac{1}{\Gamma(\delta_1)}\int_0^\infty t^{\delta_1-1}e^{-t-\frac{p}{t}}\,{}_1F_1{}^{\{k_l\}}[\delta_2,\delta_3;zt]dt. $$

Theorem 4.5. *The following Derivative formulas hold true for the extended τ Gauss hypergeometric function defined in (18)*

$$ \frac{\partial^n}{\partial z^n}\,{}_2R_1^{\{k_l\}}[(\delta_1;p,v),\delta_2;\delta_3;\tau,z] \tag{32} $$

$$ = \frac{(\delta_1)_n\Gamma(\delta_2+n\tau)\Gamma(\delta_3)}{\Gamma(\delta_3+n\tau)\Gamma(\delta_2)}\,{}_2R_1^{\{k_l\}}[(\delta_1+n;p,v),\delta_2+n\tau;\delta_3+n\tau;\tau,z]. $$

Proof:

$$ {}_2R_1^{\{k_l\}}[(\delta_1;p,v),\delta_2;\delta_3;\tau,z] = (\delta_1;p,v)_n\sum_{n=0}^{\infty}\frac{B_{\mu_1,\mu_2}^{\{k_l\}}(\delta_2+n\tau,\delta_3-\delta_2)}{B(\delta_2,\delta_3-\delta_2)}\frac{z^n}{n!} $$

$$ \frac{\partial}{\partial z}{}_2R_1^{\{k_l\}}[(\delta_1;p,v),\delta_2;\delta_3;\tau,z] = (\delta_1;p,v)_n\sum_{n=0}^{\infty}\frac{B_{\mu_1,\mu_2}^{\{k_l\}}(\delta_2+n\tau,\delta_3-\delta_2)}{B(\delta_2,\delta_3-\delta_2)}\frac{z^{n-1}}{(n-1)!} $$

Replacing n$\to$ n+1 in the above equation and after simplification, we get

$$ \frac{\partial}{\partial z}{}_2R_1^{\{k_l\}}[(\delta_1;p,v),\delta_2;\delta_3;\tau,z] = (\delta_1;p,v)_{n+1}\sum_{n=0}^{\infty}\frac{B_{\mu_1,\mu_2}^{\{k_l\}}(\delta_2+n\tau+\tau,\delta_3-\delta_2)}{B(\delta_2,\delta_3-\delta_2)}\frac{z^n}{n!} $$

$$ = \frac{\Gamma(\delta_3)\Gamma(\delta_2+\tau)(\delta_1)}{\Gamma(\delta_2)\Gamma(\delta_3+\tau)}\sum_{n=0}^{\infty}\frac{(\delta_1+1;p,v)_n\Gamma(\delta_2+n\tau+\tau)(\delta_3+\tau)}{\Gamma(\delta_3+n\tau+\tau)(\delta_2+\tau)}\frac{z^n}{n!} $$

$$ = \frac{\Gamma(\delta_3)\Gamma(\delta_2+\tau)(\delta_1)}{\Gamma(\delta_2)\Gamma(\delta_3+\tau)}{}_2R_1^{\{k_l\}}[(\delta_1+1;p,v),\delta_2+\tau;\delta_3+\tau;\tau,z] $$

So iterating this differential equation n-times, we get

$$ \frac{\partial^n}{\partial z^n}{}_2R_1^{\{k_l\}}[(\delta_1;p,v),\delta_2:\delta_3:\tau,z] $$

$$ = \frac{(\delta_1)_n\Gamma(\delta_2+n\tau)\Gamma(\delta_3)}{\Gamma(\delta_3+n\tau)\Gamma(\delta_2)}{}_2R_1^{\{k_l\}}[(\delta_1+n;p,v);\delta_2+n\tau;\delta_3+n\tau;\tau,z] $$

$$ = \frac{(\delta_1)_n\Gamma(\delta_2+n\tau)\Gamma(\delta_3)}{\Gamma(\delta_3+n\tau)\Gamma(\delta_2)}{}_2R_1^{\{k_l\}}[(\delta_1+n;p,v);\delta_2+n\tau;\delta_3+n\tau;\tau,z]. $$

Hence, the proof is completed.

5. Application of the Mellin Transform

The well known Mellin transform of a given integrable function f(t) is

$$\mathcal{M}\{f(t):t\to s\}=\int_0^\infty t^{s-1}f(t)dt \tag{33}$$

provided that the improper integral in (33) exists.

Theorem 5.1. *The Mellin Transform of the extended τ-Gauss hypergeometric function*

$$_2R_1^{\{k_l\}}[(\delta_1;p,v),\delta_2;\delta_3;\tau,z;\mu_1,\mu_2]$$

is given as

$$\mathcal{M}\{_2R_1^{\{k_l\}}[(\delta_1;p,v),\delta_2;\delta_3;\tau,z;\mu_1,\mu_2]:t\to s\}$$

(34)

$$=\frac{2^{s-1}}{\sqrt{\pi}}\Gamma(\frac{s-v}{2})\Gamma(\frac{s+v+1}{2})\frac{\Gamma(\delta_3)}{\Gamma(\delta_2)}(\delta_1)_s\sum_{n=0}^{\infty}\frac{(\delta_1+s)_n\Gamma(\delta_2+n\tau)}{\Gamma(\delta_3+n\tau)}\Theta(\{k_l\};-\frac{\mu_1}{t}-\frac{\mu_2}{1-t});$$

$$(\Re(s-v)>0;\Re(\delta_1+s)>-1,min(\Re(\mu_1),\Re(\mu_2))>0).$$

Proof Using (33) of the Mellin transform on both the sides of (21), we get

$$\mathcal{M}\{_2R_1^{\{k_l\}}[(\delta_1;p,v),\delta_2;\delta_3;\tau,z;\mu_1,\mu_2]:t\to s\}$$

$$=\int_0^\infty t^{s-1}{}_2R_1^{\{k_l\}}[(\delta_1;p,v),\delta_2;\delta_3;\tau,z;\mu_1,\mu_2]dt$$

$$=\frac{\Gamma(\delta_3)}{\Gamma(\delta_2)}\sum_{n=0}^{\infty}\frac{\Gamma(\delta_2+n\tau)}{\Gamma(\delta_3+n\tau)}\int_0^\infty t^{s-1}(\delta_1;p,v)_n\Theta(\{k_l\},-\frac{\mu_1}{t}-\frac{\mu_2}{1-t})dt$$

Using the result (9) in the above equation, we get

$$(\delta_1;p,v)_n=\frac{\Gamma_v(\delta_1+n;p)}{\Gamma(\delta_1)};$$

$$=\frac{\Gamma(\delta_3)}{\Gamma(\delta_2)}\sum_{n=0}^{\infty}\frac{\Gamma(\delta_2+n\tau)}{\Gamma(\delta_3+n\tau)}\int_0^\infty t^{s-1}\frac{\Gamma_v(\delta_1+n;p)}{\Gamma(\delta_1)}\Theta(\{k_l\},-\frac{\mu_1}{t}-\frac{\mu_2}{1-t})dt.$$

Now using the result given by Chaudhry et.al. ([5]), we get

$$\int_0^\infty t^{s-1}\Gamma_v(\delta_1+n;p)dt=\frac{2^{s-1}}{\sqrt{\pi}}\Gamma(\frac{s-v}{2})\Gamma(\frac{s+v+1}{2})\Gamma(\delta_1+n+s)$$

$$=\frac{1}{\Gamma(\delta_1)}\frac{\Gamma(\delta_3)}{\Gamma(\delta_2)}\sum_{n=0}^{\infty}\frac{\Gamma(\delta_2+n\tau)}{\Gamma(\delta_3+n\tau)}\frac{2^{s-1}}{\sqrt{\pi}}\Gamma(\frac{s-v}{2})\Gamma(\frac{s+v+1}{2})\Gamma(\delta_1+n+s)\Theta(\{k_l\},-\frac{\mu_1}{t}-\frac{\mu_2}{1-t});$$

$$=\frac{2^{s-1}}{\sqrt{\pi}}\Gamma(\frac{s-v}{2})\Gamma(\frac{s+v+1}{2})\frac{\Gamma(\delta_3)}{\Gamma(\delta_2)}(\delta_1)_s\sum_{n=0}^{\infty}\frac{(\delta_1+s)_n\Gamma(\delta_2+n\tau)}{\Gamma(\delta_3+n\tau)}\Theta(\{k_l\};-\frac{\mu_1}{t}-\frac{\mu_2}{1-t}).$$

which yields the Mellin transform formula (34).

6. Conclusion

In this paper, We introduce a new generalization of the extended τ Gauss hypergeometric function. In the light of techniques used by Gupta and Kim ([1] and [6]), this study can be further extended in the field of q-Calculus and Degenerate hypergeometric functions.

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