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SKEW-ENERGY OF t-DUPLICATION GRAPH

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Abstract. Graph spectra provide much information about the structure of the graph. In this paper, we figure out the energy of a graph derived from simpler graphs by certain modifications. In the present work, we define t-duplication graph D_tG and disclosed that the energy $\mathcal{E}(D_tG) = 2\sqrt{t}\mathcal{E}(G)$ and $\mathcal{E}(D_tG^\sigma) = 2\sqrt{t}\mathcal{E}(G^\sigma)$.
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1. Introduction

Let $G = (V; E)$ be a finite undirected graph without loops or multi edges and $\mid V(G) \mid = m$. Let $A(G)$ be an adjacency matrix of order m. If $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the eigenvalues of G with multiplicity $k_1, k_2, \ldots, k_m$ respectively, then the spectrum of G denoted by $Spec(G)$ i.e

$$Spec(S) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_m \\ k_1 & k_2 & \cdots & k_m \end{pmatrix}$$

The notion graph energy was introduced by I. Gutman in 1978 [2], the energy of G is defined as $\mathcal{E}(G) = \sum_{i=1}^{m} \mid \lambda_i \mid$. In the present article, we construct t-duplication graph and find out energy for it. The notion skew energy was introduced by Adiga et al.in 2010 [1] and also we refer [4, 5, 10]. The definition of a digraph is utilized to compute the skew energy of some graphs.

Definition 1.1. *Let G^σ be a directed graph of order m with the vertex set $V(G^\sigma)$ and the arc set $\Gamma(G^\sigma) \subset V(G^\sigma) \times V(G^\sigma)$. The skew adjacent matrix of G^σ is the $n \times n$ matrix $S(G^\sigma) = [s_{ij}]$, where $s_{ij} = 1$ whenever $(v_i, v_j) \in \Gamma(G^\sigma)$, $s_{ij} = -1$ whenever $(v_j, v_i) \in \Gamma(G^\sigma)$ and $s_{ij} = 0$ otherwise.*
In [6] S. K. Vaidya et al. has estimated the energy of a t-splitting graph and a t-shadow graph. In this section, we calculate the energy of a t-duplication graph and its corresponding digraph.

2. energy of a Duplication graph

Let $A \in M_{m\times n}$ and $B \in M_{p\times q}$. Then the tensor product (or Kronecker product) of A and B is defined as the matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}$$

Proposition 2.1 ([3])**.** *Let $A \in M_{m\times n}$ and $B \in M_{p\times q}$ and let λ be the eigenvalue of matrix A with corresponding eigenvector x and μ be the eigenvalue of matrix B with corresponding eigenvector y. Then $\lambda\mu$ is an eigenvalue of $A \otimes B$ with corresponding eigenvector $x \otimes y$.*

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Definition 2.2 ([8]). *Let G be a graph with $V(G) = \{v_1, \ldots, v_m\}$. Take another set $U = \{u_1, \ldots, u_m\}$. Make u_j adjacent to all the vertices in $N(v_j)$ in G for each j and remove edges of G only. The resulting graph H is called the duplication graph of G denoted by DG.*

Motivated from above definition we develop following definition.

Definition 2.3. *The t-duplication graph $D_t(G)$ of a connected graph G with $V(G) = \{v_1, \ldots, v_m\}$ is constructed by taking t-sets say $U^j = \{u_1^j, \ldots, u_m^j\}$, $j = 1, 2, \ldots t$. Make each u_l^j adjacent to all the vertices in $N(v_l)$ in G for each l, for $j = 1, 2, \ldots m$ and remove edges of G only.*

Theorem 2.4. *Let $V(G) = \{v_1, \ldots, v_m\}$ be the vertex set of a graph G and $A(G)$ be its adjacency matrix. Then, $\mathcal{E}(D_tG) = 2\sqrt{t}\mathcal{E}(G)$.*

Proof. The adjacency matrix of G is given by

$$A(G) = \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & 0 & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & 0 & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & 0 \end{bmatrix}$$

Let $U^j = \{u_1^j, \ldots, u_m^j\}$, $j = 1, 2, \ldots t$. Make each u_l^j adjacent to all the vertices in $N(v_l)$ in G for each l and remove edges of G only. . We get t-duplication graph D_tG . The adjacency matrix of duplication graph D_tG of order $(t + 1)$ is given by

$$A(D_tG) = \begin{bmatrix} 0 & A(G) & \cdots & A(G) \\ A(G) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A(G) & 0 & \cdots & 0 \end{bmatrix}$$

$$A(D_tG) = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix} \otimes A(G).$$

Let

$$S = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}_{(t+1)\times(t+1)}$$

Since the rank of the matrix is two, we have two nonzero eigenvalues for the matrix S and $(t - 1)$ zero eigenvalues. By the matrix S we have,

$$\mu_1 + \mu_2 = 0 \tag{1}$$

Consider

$$S^2 = \begin{bmatrix} t & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}_{(t+1)\times(t+1)}$$

Here ,

(2) $$\mu_1^2 + \mu_2^2 = 2t$$

on solving equations (1) and (2) we get,

(3) $$\mu_1 = \sqrt{t} \quad and \quad \mu_2 = -\sqrt{t}$$

Thus,

$$Spec(S) = \begin{pmatrix} -\sqrt{t} & \sqrt{t} & 0 \\ 1 & 1 & t-1 \end{pmatrix}.$$

If λ_j, $j = 1, 2, ..., m$ are the eigenvalues of G, then by proposition 2.1

$$Spec(A(D_tG) = \begin{pmatrix} -\lambda_j\sqrt{t} & \lambda_j\sqrt{t} & 0 \\ 1 & 1 & t-1 \end{pmatrix}$$

$$\mathcal{E}(DG) = \sum_{j=1}^{m} \mid (\pm)\lambda_j\sqrt{t} \mid = 2\sqrt{t}\sum_{j=1}^{m} \mid \lambda_j \mid = 2\sqrt{t}\mathcal{E}(G).$$

□

Corollary 2.5. *Let $V(G^\sigma) = \{v_1, \ldots, v_m\}$ be the vertex set of a graph G^σ with arc set $\Gamma(G^\sigma)$ and $A(G^\sigma)$ be its adjacency matrix. Then, $\mathcal{E}(D_tG^\sigma) = 2\sqrt{t}\mathcal{E}(G^\sigma)$.*

Proof. The proof is similar to above theorem 2.4. □

The following illustrations gives better understanding of above theorems.

Illustration 1
Consider the cycle C_4 and its duplication graph.

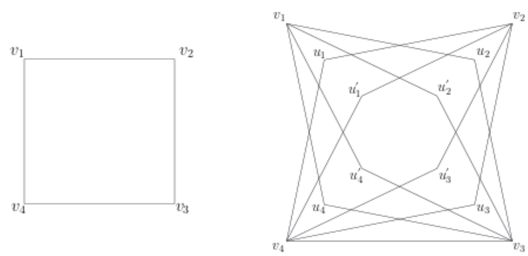

FIGURE 1. The duplication graph of C_4

Adjacency matrix of D_2C_4 is given by

$$A(D_2C_4) = \begin{bmatrix} 0 & C_4 & C_4 \\ C_4 & 0 & 0 \\ C_4 & 0 & 0 \end{bmatrix}$$

$$A(D_2C_4) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \otimes C_4$$

$$Spec(D_2C_4) = \begin{pmatrix} -2\sqrt{2} & 0 & 2\sqrt{2} \\ 2 & 8 & 2 \end{pmatrix} \text{ and } Spec(C_4) = \begin{pmatrix} -2 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

Therefore,

$$\mathcal{E}(D_2C_4) = 8\sqrt{2} \text{ and } \mathcal{E}(C_4) = 4$$

Hence,

$$\mathcal{E}(DC_4) = 2\sqrt{t}\mathcal{E}(C_4) \quad where \quad t = 2.$$

Illustration 2

Consider the digraph C_4^σ and its duplication graph $D_2C_4^\sigma$

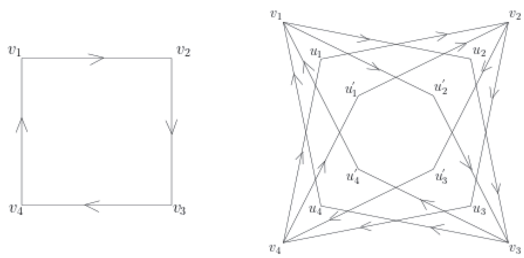

Figure 2. Duplication graph of digraph C_4^σ

Adjacency matrix of $D_2C_4^\sigma$ is given by

$$A(D_2C_4^\sigma) = \begin{bmatrix} 0 & C_4^\sigma & C_4^\sigma \\ C_4^\sigma & 0 & 0 \\ C_4^\sigma & 0 & 0 \end{bmatrix}$$

$$A(D_2C_4^\sigma) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \otimes C_4^\sigma$$

$$Spec(D_2C_4^\sigma) = \begin{pmatrix} -2\sqrt{2}i & 0 & 2\sqrt{2}i \\ 2 & 8 & 2 \end{pmatrix} \text{ and } Spec(C_4^\sigma) = \begin{pmatrix} -2i & 0 & 2i \\ 1 & 2 & 1 \end{pmatrix}$$

Therefore,

$$\mathcal{E}(D_2C_4^\sigma) = 8\sqrt{2} \text{ and } \mathcal{E}(C_4^\sigma) = 4$$

Hence,

$$\mathcal{E}(DC_4^\sigma) = 2\sqrt{t}\mathcal{E}(C_4^\sigma) \quad where \quad t = 2.$$

3. ON EQUIENERGETIC GRAPHS

The two non-isomorphic graphs G and H with same energy are equienergetic.

Definition 3.1. [8] *The t-shadow graph $S_t(G)$ of a connected graph G is constructed by taking t copies of G, say G_1, G_2,..., G_t, then join each vertex u in G_i to the neighbors of the corresponding vertex v in G_j, $1 \leq i, j \leq t$.*

Proposition 3.2. [8] $\mathcal{E}(S_t(G)) = t\mathcal{E}(G)$.

Theorem 3.3. *The t-duplication graph and t-shadow graph are equienergetic if and only if* $t = 2$.

Proof. Let $D_t(G)$ t-duplication graph and $S_t(G)$ be t-shadow graph and according to theorem 2.4 and proposition 3.2, $\mathcal{E}(D_tG) = 2\sqrt{t}\mathcal{E}(G)$ and $\mathcal{E}(S_t(G)) = t\mathcal{E}(G)$. So clearly, $t = 2$. □

4. Conclusion

We initiated investigation on energy of a derived graph obtained via some graph operations. The t-duplication graph D_tG is considered and it has been disclosed that $\mathcal{E}(D_tG) = 2\mathcal{E}(G)$ and $\mathcal{E}(D_tG^\sigma) = 2\mathcal{E}(G^\sigma)$ and also we characterized the equienergetic of these graphs.

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