Reciprocal Edge Transmission Based Topological Indices of Graphs

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Abstract

The reciprocal edge transmission of a edge e = uv in a connected graph G is defined as the sum of reciprocal of distances between the edge e and all other edges of a graph G. In this paper we introduce and study new topological indices based on the reciprocal edge transmission, such as reciprocal edge transmission sum-connectivity index, reciprocal edge transmission atom bond connectivity index, reciprocal edge transmission arithmetic-geometric index, reciprocal edge transmission geometric-arithmetic index, reciprocal edge transmission augmented Zagreb index and reciprocal edge transmission inverse sum indeg index. Further obtain bounds for reciprocal edge transmission based indices of any graph and also give explicite expression for some class of graphs.

Keywords: Degree of a vertex, degree of an edge, distance between edges, reciprocal edge transmission, reciprocal edge transmission indices.

Mathematics Subject Classification: 05C12, 05C09, 05C92, 92E10.

1 Introduction

A representation of an object giving information only about the number of elements composing it and their connectivity is named as topological representation of an object. A topological representation of a molecule is called molecular graph. A molecular graph is a collection of points representing the atoms in the molecule and set of lines representing the covalent bonds. These points are named vertices and the lines are named edges in graph theory language. The advantage of topological indices is in that they may be used directly as

simple numerical descriptors in a comparison with physical, chemical or biological parameters of molecules in Quantitative Structure Property Relationships (QSPR) and in Quantitative Structure Activity Relationships (QSAR).

The name line graph comes from a paper by Harary and Norman [14] although both Whitney [15] and Krausz [16] used the construction before this. Other terms used for the line graph include the covering graph, the derivative, the edge-to-vertex dual, the conjugate, the representative graph, as well as the edge graph, the interchange graph, the adjoint graph, and the derived graph.

The edge versions of augmented Zagreb index, hyper-Zagreb index, Harmonic index and sum-connectivity index are studied in [17], where degree of edge e = uv in L(G) is taken into account in the place of degree of vertex in G, that is $d_{L(G)}(e) = d_G(u) + d_G(v) - 2$.

In the recent years, mathematical techniques for the computation of the edge-Wiener index have been considered by a number of researchers, see [18, 19, 20, 21, 22, 23, 24, 25, 28] and a survey [27].

The distance between the edges $e, f \in E(G)$ is equal to the distance between the vertices e, f in the line graph of G, which is denoted by $d_G(e, f)$ and defined as [13],

$$d_G(e, f) = d_{L(G)}(e, f).$$
 (1)

Distance between edges $d_G(e, f)$ satisfies the definition of metric, therefore the concept of distance between edges in a graph is well defined.

In this paper we consider simple, connected graph G having n vertices and m edges. The vertex set and edge set of graph G denoted by V(G) and E(G) respectively. An edge joins vertices u and v are termed as uv. The number of edges connects to vertex v is known as degree of a vertex v and is denoted by $d_G(v)$. Further the degree of an edge e = uv is equal to $d_G(u) + d_G(v) - 2$.

The reciprocal edge transmission of a edge e = uv in a graph G is defined in [2] by

$$rs_G(e) = \sum_{f \in E(G)} \frac{1}{d_G(e, f)}.$$

The first reciprocal edge transmission index $RET_1(G)$ of a graph G is defined in [2] by

$$RET_1(G) = \sum_{e \sim f \in E(G)} \left[\sigma_G(e) + \sigma_G(f) \right].$$
(2)

The second reciprocal edge transmission index $RET_2(G)$ of a graph G is defined in [2] by

$$RET_2(G) = \sum_{e \sim f \in E(G)} \sigma_G(e) \sigma_G(f).$$
(3)

In the literature several degree based topological indices have been introduced and studied [6]. More studied topological indices based on the degree of vertices are Zagreb indices. [7, 9]. The first and second Zagreb indices of a graph G are defined in [8] by

$$M_1(G) = \sum_{uv \in E(G)} [d(u) + d(v)]$$
 and $M_2(G) = \sum_{uv \in E(G)} d(u)d(v).$

The sum-connectivity index of a graph G, denoted by SC(G), is defined in [12] by

$$SC(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d(u) + d(v)}}.$$

Estrada et al. [4] proposed a topological index called *atom-bond connectivity index*. It is defined as

$$ABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{d(u) + d(v) - 2}{d(u)d(v)}}.$$

The augmented Zagreb index of a graph G, proposed by Furtula et al. [5], is defined as

$$AZ(G) = \sum_{uv \in E(G)} \left[\frac{d(u)d(v)}{d(u) + d(v) - 2} \right]^3.$$

The arithmetic-geometric index of a graph G, proposed by Shigehalli and Kanabur [10], is defined as

$$AG(G) = \sum_{uv \in E(G)} \frac{d(u) + d(v)}{2\sqrt{d(u)d(v)}}.$$

The geometric-arithmetic index was invented by Vukicević and Furtula [11] and it is defined as

$$GA(G) = \sum_{uv \in E(G)} \frac{2\sqrt{d(u)d(v)}}{d(u) + d(v)}$$

The inverse sum indeg index of a graph G is defined in [28] by

$$ISI(G) = \sum_{uv \in E(G)} \left(\frac{d(u)d(v)}{d(u) + d(v)} \right).$$

Motivatd by the work on edge distance, reciprocal edge transmission and reciprocal edge transmission indices and we define the following indices.

The reciprocal edge transmission sum connectivity index of a graph G is defined by

$$RET_{SC}(G) = \sum_{e \sim f \in E(G)} \frac{1}{\sqrt{rs_G(e) + rs_G(f)}}.$$

The reciprocal edge transmission geometric airthmatic index of a graph G is defined by

$$RET_{GA}(G) = \sum_{e \sim f \in E(G)} \frac{2\sqrt{rs_G(e)rs_G(f)}}{rs_G(e) + rs_G(f)}.$$

The reciprocal edge transmission airthmatic geometric index of a graph G is defined by

$$RET_{AG}(G) = \sum_{e \sim f \in E(G)} \frac{rs_G(e) + rs_G(f)}{2\sqrt{rs_G(e)rs_G(f)}}.$$

The reciprocal edge transmission atom bond connectivity index of a graph G is defined by

$$RET_{ABC}(G) = \sum_{e \sim f \in E(G)} \sqrt{\frac{rs_G(e) + rs_G(f) - 2}{rs_G(e)rs_G(f)}}.$$

The reciprocal edge transmission augumented Zagreb index of a graph G is defined by

$$RET_{AZ}(G) = \sum_{e \sim f \in E(G)} \left(\frac{rs_G(e) rs_G(f)}{rs_G(e) + rs_G(f) - 2} \right)^3.$$

The reciprocal edge transmission inverse sum indeg index of a graph G is defined by

$$RET_{ISIS}(G) = \sum_{e \sim f \in E(G)} \left(\frac{rs_G(e) rs_G(f)}{rs_G(e) + rs_G(f)} \right).$$

The following results are used in the remaining paper:

Lemma 1.1. [3] Let \mathbb{P}_k be the set of all distinct paths of length $k \geq 1$ in a graph G. Then

$$|\mathbb{P}_1| = m$$
 and $|\mathbb{P}_2| = \frac{1}{2} [Z_1(G) - 2m].$

Note that $uvw \in \mathbb{P}_2$ means uvw is a path of length 2 with v as its middle vertex.

Theorem 1.2. [1] For a connected graph G, $diam(L(G)) \leq 2$ if and only if none of the three graphs F_1, F_2 and F_3 of Fig. 1.1 are an induced subgraph of G.



Figure 1.1 : Forbidden induced subgraphs.

2 Bounds for reciprocal edge transmission based topological indices of graphs

Theorem 2.1. Let G be a connected graph with n vertices, m edges and let diam(G) = D. Then

$$\sum_{uvw\in\mathbb{P}_2} \frac{1}{\sqrt{(m-1) + \frac{1}{2}(d(u) + 2d(v) + d(w) - 4)}} \le RET_{SC}(G)$$
$$\le \sum_{uvw\in\mathbb{P}_2} \frac{1}{\sqrt{\frac{2}{D}(m-1) + \left(1 - \frac{1}{D}\right)(d(u+2d(v) + d(w) - 4))}}.$$

Equality on both sides holds if and only if none of the three graphs F_1, F_2 and F_3 of Fig. 1.1 are an induced subgraph of G.

Proof. Lower bound: For any edge e of G, there are $d_G(e)$ number of edges which are at distance 1 from e and remaining $m - 1 - d_G(e)$ edges are at distance 2. Therefore

$$rs_G(e) \le \frac{1}{2}(m - 1 + d_G(e)).$$

We have

$$RET_{SC}(G) = \sum_{e \sim f \in E(G)} \frac{1}{\sqrt{rs_G(e) + rs_G(f)}}$$

$$\geq \sum_{e \sim f \in E(G)} \frac{1}{\sqrt{\frac{1}{2}(m - 1 + d_G(e)) + \frac{1}{2}(m - 1 + d_G(f))}}$$

$$\geq \sum_{e \sim f \in E(G)} \frac{1}{\sqrt{(m - 1) + \frac{1}{2}(d_G(e) + d_G(f))}}$$

$$\geq \sum_{uvw \in \mathbb{P}_2} \frac{1}{\sqrt{(m - 1) + \frac{1}{2}(d(u) + 2d(v) + d(w) - 4)}}.$$

Upper bound: For any edge e of G, there are $d_G(e)$ number of edges which are at distance 1 from e and remaining $m - 1 - d_G(e)$ edges are at distance D. Therefore

$$rs_G(e) \geq d_G(e) + \frac{1}{D}(m-1-d_G(e))$$

 $\geq \frac{1}{D}(m-1) + \left(1-\frac{1}{D}\right)d_G(e).$

$$RET_{SC}(G) = \sum_{e \sim f \in E(G)} \frac{1}{\sqrt{rs_G(e) + rs_G(f)}}$$

$$\leq \sum_{e \sim f \in E(G)} \frac{1}{\sqrt{\frac{1}{D}(m-1) + (1-\frac{1}{D}) d_G(e) + \frac{1}{D}(m-1) + (1-\frac{1}{D}) d_G(f)}}$$

$$\leq \sum_{e \sim f \in E(G)} \frac{1}{\sqrt{\frac{2}{D}(m-1) + (1-\frac{1}{D}) (d_G(e) + d_G(f))}}$$

$$\leq \sum_{uvw \in \mathbb{P}_2} \frac{1}{\sqrt{\frac{2}{D}(m-1) + (1-\frac{1}{D}) (d(u+2d(v) + d(w) - 4))}}.$$

Theorem 2.2. Let G be a connected graph with n vertices, m edges and let diam(G) = D. Then

$$2 \sqrt{\frac{\frac{1}{4}[(m-1)^{2} + (m-1)(d(u) + 2d(v) + d(w) - 4)]}{(+(d(u)d(v) + d(v)d(w) + d(w)d(u))}}}_{uvw\in\mathbb{P}_{2}} \frac{2}{(m-1) + \frac{1}{2}(d(u) + 2d(v) + d(w)) + 4 + (d(v))^{2}]}}{(m-1) + \frac{1}{2}(d(u) + 2d(v) + d(w) - 4)} \leq \frac{(\frac{1}{D}(m-1)^{2}) + \frac{1}{D}(m-1)(1 - \frac{1}{D})(d(u) + 2d(v) + d(w) - 4)}{(+(1 - \frac{1}{D})^{2}(d(u)d(v) + d(v)d(w) + d(w)d(u))}}$$

$$RET_{GA}(G) \le \sum_{uvw \in \mathbb{P}_2} \frac{\sqrt{-2(d(u) + 2d(v) + d(w)) + 4 + d(v)^2}}{\frac{2}{D}(m-1) + \left(1 - \frac{1}{D}(d(u) + 2d(v) + d(w) - 4)\right)}.$$

Equality on both sides holds if and only if none of the three graphs F_1, F_2 and F_3 of Fig. 1.1 are an induced subgraph of G.

Proof. Lower bound: For any edge e of G, there are $d_G(e)$ edges which are at distance 1 from e and remaining $m - 1 - d_G(e)$ are at distance 2. Therefore

$$rs_G(e) \le \frac{1}{2}(m - 1 + d_G(e)).$$

We have

$$\begin{aligned} RET_{GA}(G) &= \sum_{e \sim f \in E(G)} \frac{rs_G(e) + rs_G(f)}{2\sqrt{rs_G(e)rs_G(f)}} \\ &\geq \sum_{e \sim f \in E(G)} \frac{2\sqrt{\frac{1}{4}(m-1+d_G(e))(m-1+d_G(f))}}{\frac{1}{2}(m-1+d_G(e)+m-1+d_G(f))} \\ &\geq \sum_{e \sim f \in E(G)} \frac{2\sqrt{\frac{1}{4}\left[(m-1)^2 + (m-1)(d_G(e) + d_G(f)) + d_G(e)d_G(f)\right]}}{(m-1) + \frac{1}{2}(d_G(e) + d_G(f))} \\ &\geq \sum_{uvw \in \mathbb{P}_2} \frac{2\sqrt{\frac{1}{4}\left[(m-1)^2 + (m-1)(d(u) + 2d(v) + d(w) - 4)\right]}}{(m-1) + \frac{1}{2}(d(u) + d(v) - 2)(d(v) + d(w) - 2)\right]}} \\ &\geq \sum_{uvw \in \mathbb{P}_2} \frac{2\sqrt{\frac{1}{4}\left[(m-1)^2 + (m-1)(d(u) + 2d(v) + d(w) - 4)\right]}}{(m-1) + \frac{1}{2}(d(u) + d(v) - 2 + d(v) + d(w) - 4)}} \\ &\geq \sum_{uvw \in \mathbb{P}_2} \frac{2\sqrt{\frac{1}{4}\left[(m-1)^2 + (m-1)(d(u) + 2d(v) + d(w) - 4)\right]}}{(m-1) + \frac{1}{2}(d(u) + 2d(v) + d(w) - 4)}} \\ &\geq \sum_{uvw \in \mathbb{P}_2} \frac{2\sqrt{\frac{1}{4}\left[(m-1)^2 + (m-1)(d(u) + 2d(v) + d(w) - 4)\right]}}{(m-1) + \frac{1}{2}(d(u) + 2d(v) + d(w) - 4)}}. \end{aligned}$$

Upper bound: For any edge e of G, there are $d_G(e)$ edges which are at distance 1 from e and remaining $m - 1 - d_G(e)$ are at distance D. Therefore

$$rs_G(e) \geq d_G(e) + \frac{1}{D}(m - 1 - d_G(e))$$

 $rs_G(e) \geq \frac{1}{D}(m - 1) + \left(1 - \frac{1}{D}\right)d_G(e).$

We have

$$\begin{aligned} RET_{GA}(G) &= \sum_{e \sim f \in E(G)} \frac{rs_G(e) + rs_G(f)}{2\sqrt{rs_G(e)rs_G(f)}} \\ &\leq \sum_{e \sim f \in E(G)} \frac{2\sqrt{\left(\frac{1}{D}(m-1) + \left(1 - \frac{1}{D}\right)d_G(e)\right)\left(\frac{1}{D}(m-1) + \left(1 - \frac{1}{D}\right)d_G(f)\right)}}{\frac{1}{D}(m-1) + \left(1 - \frac{1}{D}\right)d_G(e) + \frac{1}{D}(m-1) + \left(1 - \frac{1}{D}\right)d_G(f)} \\ &\leq \sum_{e \sim f \in E(G)} \frac{2\sqrt{\left(\frac{1}{D}(m-1)\right)^2 + \frac{1}{D}(m-1)\left(1 - \frac{1}{D}\right)\left(d_G(e) + d_G(f)\right)}}{\frac{2}{D}(m-1) + \left(1 - \frac{1}{D}\right)^2d_G(e)d_G(f)} \\ &\leq \sum_{e \sim f \in E(G)} \frac{2\sqrt{\left(\frac{1}{D}(m-1)\right)^2 + \frac{1}{D}(m-1)\left(1 - \frac{1}{D}\right)\left(d(u) + 2d(v) + d(w) - 4\right)} \\ &+ \left(1 - \frac{1}{D}\right)^2(d(u) + d(v) - 2)(d(v) + d(w) - 4) \\ &\leq \sum_{e \sim f \in E(G)} \frac{2\sqrt{\left(\frac{1}{D}(m-1)^2\right) + \frac{1}{D}(m-1)\left(1 - \frac{1}{D}\right)\left(d(u) + 2d(v) + d(w) - 4\right)}}{\frac{2}{D}(m-1) + \left(1 - \frac{1}{D}\right)^2(d(u) + d(v) - 2)}{\frac{2}{D}(m-1) + \left(1 - \frac{1}{D}\right)(d(u) + 2d(v) + d(w) - 4)} \\ &+ \left(1 - \frac{1}{D}\right)^2(d(u)d(v) + d(v)d(w) + d(w)d(u)) \\ &\leq \sum_{uvw \in \mathbb{P}_2} \frac{\sqrt{\left(\frac{1}{D}(m-1)^2\right) + \frac{1}{D}(m-1)\left(1 - \frac{1}{D}\right)\left(d(u) + 2d(v) + d(w) - 4\right)}}{\frac{2}{D}(m-1) + \left(1 - \frac{1}{D}\right)(d(u) + 2d(v) + d(w) - 4)} \\ &= \sum_{uvw \in \mathbb{P}_2} \frac{\sqrt{\left(\frac{1}{D}(m-1)^2\right) + \frac{1}{D}(m-1)\left(1 - \frac{1}{D}\right)\left(d(u) + 2d(v) + d(w) - 4\right)}}{\frac{2}{D}(m-1) + \left(1 - \frac{1}{D}\right)(d(u) + 2d(v) + d(w) - 4)} \\ &= \sum_{uvw \in \mathbb{P}_2} \frac{\sqrt{\left(\frac{1}{D}(m-1)^2\right) + \frac{1}{D}(m-1)\left(1 - \frac{1}{D}\right)\left(d(u) + 2d(v) + d(w) - 4\right)}}{\frac{2}{D}(m-1) + \left(1 - \frac{1}{D}\right)(d(u) + 2d(v) + d(w) - 4)} \\ &= \sum_{uvw \in \mathbb{P}_2} \frac{\sqrt{\left(\frac{1}{D}(m-1)^2\right) + \frac{1}{D}(m-1)\left(1 - \frac{1}{D}\right)\left(\frac{1}{D}(w) + 2d(v) + d(w) - 4\right)}}{\frac{1}{D}(m-1) + \left(1 - \frac{1}{D}\right)\left(\frac{1}{D}(w) + 2d(v) + d(w) - 4\right)}} \\ &= \sum_{uvw \in \mathbb{P}_2} \frac{\sqrt{\left(\frac{1}{D}(w) + \frac{1}{D}\left(\frac{1}{D}(w) + \frac{1}{D}\left(\frac{1}{D}(w) + \frac{1}{D}\left(\frac{1}{D}(w) - \frac{1}{D}\left(\frac{1}{D}(w) + \frac{1}{D}\left(\frac{1}{D}(w) - \frac{1}{D}\right)}\right)}{\frac{1}{D}(w) + 2d(v) + d(w) - 4\right)}} \\ \\ &= \sum_{uvw \in \mathbb{P}_2} \frac{\sqrt{\left(\frac{1}{D}(w) + \frac{1}{D}\left(\frac{1}{D}(w) + \frac{1}{D}\left(\frac{1}{D}(w) - \frac{1}{D}\left(\frac{1}{D}(w) + \frac{1}{D}\left(\frac{1}{D}(w) - \frac$$

Theorem 2.3. Let G be a connected graph with n vertices, m edges and let diam(G) = D. Then

$$\sum_{uvw \in \mathbb{P}_2} \frac{(m-1) + \frac{1}{2}(d(u) + 2d(v) + d(w) - 4)}{\frac{1}{4}[(m-1)^2 + (m-1)(d(u) + 2d(v) + d(w) - 4)} \le \frac{1}{2} \sqrt{\frac{1}{4}[(m-1)^2 + (m-1)(d(u) + 2d(v) + d(w) - 4)}{(d(u)d(v) + d(v)d(w) + d(w)d(u)) - 2(d(u) + 2d(v) + d(w)) + 4 + (d(v))^2]}}{\frac{2}{D}(m-1) + \left(1 - \frac{1}{D}(d(u) + 2d(v) + d(w) - 4)\right)}{\left(\frac{1}{D}(m-1)^2\right) + \frac{1}{D}(m-1)\left(1 - \frac{1}{D}\right)(d(u) + 2d(v) + d(w) - 4)}{(d(u) + 2d(v) + d(w) - 4)} + \left(1 - \frac{1}{D}\right)^2(d(u)d(v) + d(v)d(w) + d(w)d(u)) - 2(d(u) + 2d(v) + d(w)) + 4 + (d(v))^2}$$

Equality on both sides holds if and only if none of the three graphs F_1, F_2 and F_3 of Fig. 1.1 are an induced subgraph of G.

Theorem 2.4. Let G be a connected graph with n vertices, m edges and let diam(G) = D. Then

$$\sum_{uvw \in \mathbb{P}_2} \sqrt{\frac{(m-1) + \frac{1}{2}(d(u) + 2d(v) + d(w)) - 4}{\frac{1}{4}\left[(m-1)^2 + (m-1)(d(u) + 2d(v) + d(w) - 4\right)}}{+(d(u)d(v) + d(v)d(w) + d(w)d(u)) - 2(d(u) + 2d(v) + d(w)) + 4 + (d(v))^2\right]}} \leq RET_{ABC}(G) \leq \sum_{uvw \in \mathbb{P}_2} \sqrt{\frac{\frac{2}{D}(m-1) + \left(1 - \frac{1}{D}\right)(d(u) + 2d(v) + d(w) - 4\right) - 2}{\left(\frac{1}{D}(m-1)^2\right) + \frac{1}{D}(m-1)\left(1 - \frac{1}{D}\right)(d(u) + 2d(v) + d(w) - 4)}} + \left(1 - \frac{1}{D}\right)^2 (d(u)d(v) + d(v)d(w) + d(w)d(u))}{-2(d(u) + 2d(v) + d(w)) + 4 + (d(v))^2}}$$

Equality on both sides holds if and only if none of the three graphs F_1, F_2 and F_3 of Fig. 1.1 are an induced subgraph of G.

Theorem 2.5. Let G be a connected graph with n vertices, m edges and let diam(G) = D. Then

$$\sum_{uvw\in\mathbb{P}_{2}} \left(\frac{\frac{1}{4} [(m-1)^{2} + (m-1)(d(u) + 2d(v) + d(w) - 4) \\ + (d(u)d(v) + d(v)d(w) + d(w)d(u)) - 2(d(u) + 2d(v) + d(w)) + 4 + (d(v))^{2}]}{(m-1) + \frac{1}{2}(d(u) + 2d(v) + d(w)) - 4} \right)^{3} \leq RET_{AZ}(G) \leq \sum_{uvw\in\mathbb{P}_{2}} \left(\frac{\left(\frac{1}{D}(m-1)^{2}\right) + \frac{1}{D}(m-1)\left(1 - \frac{1}{D}\right)(d(u) + 2d(v) + d(w) - 4)}{+\left(1 - \frac{1}{D}\right)^{2}(d(u)d(v) + d(v)d(w) + d(w)d(u))} \\ - 2(d(u) + 2d(v) + d(w)) + 4 + (d(v))^{2} \\ \frac{2}{D}(m-1) + \left(1 - \frac{1}{D}\right)(d(u) + 2d(v) + d(w) - 4) - 2 \end{array} \right)^{3}$$

Equality on both sides holds if and only if none of the three graphs F_1, F_2 and F_3 of Fig. 1.1 are an induced subgraph of G.

Theorem 2.6. Let G be a connected graph with n vertices, m edges and let diam(G) = D.

Then

$$\sum_{uvw\in\mathbb{P}_{2}} \left(\frac{\frac{1}{4} [(m-1)^{2} + (m-1)(d(u) + 2d(v) + d(w) - 4) \\ + (d(u)d(v) + d(v)d(w) + d(w)d(u)) - 2(d(u) + 2d(v) + d(w)) + 4 + (d(v))^{2}] \\ (m-1) + \frac{1}{2}(d(u) + 2d(v) + d(w) - 4) \\ (m-1) + \frac{1}{2}(d(u) + 2d(v) + d(w) - 4) \\ + (1 - \frac{1}{D})^{2} (d(u)d(v) + d(v)d(w) + d(w)d(u)) \\ - 2(d(u) + 2d(v) + d(w)) + 4 + (d(v))^{2} \\ \frac{2}{D}(m-1) + (1 - \frac{1}{D}) (d(u) + 2d(v) + d(w) - 4) \\ \end{array} \right).$$

Equality on both sides holds if and only if none of the three graphs F_1, F_2 and F_3 of Fig. 1.1 are an induced subgraph of G.

3 Reciprocal edge transmission based topological indices of some graphs

Proposition 3.1. For a complete graph K_n on n vertices. Then

$$RET_{SC}(K_n) = \frac{1}{2}n(n-1)(n-2)\frac{1}{\sqrt{\frac{1}{2}(n^2+3n-10)}}$$

Proof. From Theorem 2.1, we have

$$RET_{SC}(K_n) = \sum_{uvw \in \mathbb{P}_2} \frac{1}{\sqrt{(m-1) + \frac{1}{2} (d(u) + 2d(v) + d(w) - 4)}}$$

= $\frac{1}{2} (M_1 - 2m) \frac{1}{\sqrt{\frac{n(n-1)}{2} - 1 + \frac{1}{2} (4(n-1) - 4)}}$
= $\frac{1}{2} \left(n(n-1)^2 - 2\frac{n(n-1)}{2} \right) \frac{1}{\sqrt{\frac{n(n-1)-2}{2} + 2(n-2)}}$
= $\frac{1}{2} n(n-1)(n-2) \frac{1}{\sqrt{\frac{1}{2} (n^2 + 3n - 10)}}.$

Analogous to Proposition 3.1 by using Theorem 2.2 to 2.6 we get the following results.

Theorem 3.2. For a complete graph K_n on n vertices. Then

(i)
$$RET_{GA}(K_n) = 2n(n-1)(n-2) \frac{\sqrt{\frac{1}{4} \left[\left(\frac{n(n-1)-2}{2} \right)^2 + 2(n-2)(n(n-1)-2) + 4(n^2-4n+4) \right]}{n^2+3n-10}}{n^2+3n-10}$$

(*ii*)
$$RET_{AG}(K_n) = \frac{1}{8}n(n-1)(n-2) \frac{n^2 + 3n - 10}{\sqrt{\frac{1}{4} \left[\left(\frac{n(n-1)-2}{2} \right)^2 + 2(n-2)(n(n-1)-2) + 4(n^2 - 4n + 4) \right]}}$$

(*iii*)
$$RET_{ABC}(K_n) = \frac{1}{2}n(n-1)(n-2)\sqrt{\frac{2(n^2+3n-14)}{\left(\frac{n(n-1)-2}{2}\right)^2+2n^3-2n^2-16n+24}}$$

$$(iv) RET_{AZ}(K_n) = \frac{1}{2}n(n-1)(n-2) \left(\frac{\frac{1}{2} \left[\left(\frac{n(n-1)-2}{2} \right)^2 + 2(n-2)(n(n-1)-2) + 4(n^2-4n+4) \right]}{n^2+3n-14} \right)^3.$$

$$(v) \ RET_{ISIS}(K_n) = \frac{1}{2}n(n-1)(n-2) \left(\frac{\frac{1}{2} \left[\left(\frac{n(n-1)-2}{2} \right)^2 + 2(n-2)(n(n-1)-2) + 4(n^2-4n+4) \right] \right]}{n^2 + 3n - 14} \right)$$

Proposition 3.3. For a complete bipartite graph $K_{p,q}$,

$$RET_{SC}(K_{p,q}) = \frac{1}{2}pq(p+q-2)\frac{1}{\sqrt{pq+p+q-3}}.$$

Proof. The vertex set $V(K_{p,q})$ can be partitioned into two independent sets V_1 and V_2 such that the vector $u \in V_1$ and $v \in V_2$ for every edge $e = uv \in E(K_{p,q})$. Therefore d(u) = q and d(v) = p. And the graph $K_{p,q}$ has n = p + q vertices and m = pq edges.

$$M_1(K_{p,q}) = pq(p+q).$$

Also $diam(G) \leq 2$. Therefore by the equality part of Theorem 2.1 we get,

$$RET_{K_{p,q}} = \sum_{uvw \in \mathbb{P}_2} \frac{1}{\sqrt{(m-1) + \frac{1}{2}(d(u) + 2d(v) + d(w) - 4)}}$$

= $\frac{1}{2}(M_1 - 2m) \frac{1}{\sqrt{(m-1) + \frac{1}{2}(d(u) + 2d(v) + d(w) - 4)}}$
= $\frac{1}{2}(pq(p+q) - 2pq) \frac{1}{\sqrt{(pq-1) + \frac{1}{2}(2p+2q-4)}}$
= $\frac{1}{2}pq(p+q-2) \frac{1}{\sqrt{pq+p+q-3}}.$

Analogous to Proposition 3.3 by using Theorem 2.2 to 2.6 we get the following results **Theorem 3.4.** For a complete bipartite graph $K_{p,q}$,

(i)
$$RET_{GA}(K_{p,q}) = pq(p+q-2) \frac{\sqrt{\frac{\frac{1}{4}[(pq-1)^2 + 2(pq-1)(p+q-2) + 3pq]}{-4(p+q) + 4 + q^2]}}}{(pq+p+q-3)}.$$

(*ii*)
$$RET_{AG}(K_{p,q}) = pq(p+q-2) \frac{pq+p+q-3}{\sqrt{\frac{1}{4}[(pq-1)^2+2(pq-1)(p+q-2)]}} \cdot \sqrt{\frac{1}{4}[(pq-1)^2+2(pq-1)(p+q-2)]} \cdot \frac{1}{\sqrt{\frac{1}{4}[(pq-1)^2+2(pq-1)(p+q-2)]}} \cdot \frac{1}{\sqrt{\frac{1}{4}[(pq-1)^2+2($$

(*iii*)
$$RET_{ABC}(K_{p,q}) = \frac{1}{2}pq(p+q-2) \sqrt{\frac{pq+p+q-5}{\frac{1}{4}[(pq-1)^2+2(pq-1)(p+q-2)]}} + 3pq-4(p+q)+4+q^2]}$$

$$(iv) RET_{AZ}(K_{p,q}) = \frac{1}{2}pq(p+q-2) \left(\frac{\frac{1}{4} \left[(pq-1)^2 + 2(pq-1)(p+q-2) + 3pq - 4(p+q) + 4 + q^2 \right]}{pq+p+q-5} \right)^3$$

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$$(v) \ RET_{ISI}(K_{p,q}) = \frac{1}{2}pq(p+q-2) \left(\frac{\frac{1}{4}[(pq-1)^2 + 2(pq-1)(p+q-2) + 3pq - 4(p+q) + 4 + q^2]}{pq+p+q-7} \right)$$

Proposition 3.5. For a wheel W_{n+1} , $n \ge 3$. Then

$$RET_{SC}(W_{n+1}) = \sum_{e \sim f \in E_1} \frac{1}{\sqrt{3n}} + \sum_{e \sim f \in E_2} \frac{1}{\sqrt{2n+3}}$$

Proof. The wheel W_{n+1} has n + 1 vertices and 2n edges. If $diam(W_{n+1}) \leq 2$ then $d_{W_{n+1}}(e)$ number of edges are at distance 1 from e and remaining $2m - 1 - d_{W_{n+1}}(e)$ edges are at distance 2. Therefore for each edge e in W_{n+1} .

$$rs_{W_{n+1}}(e) = d_{W_{n+1}}(e) + \frac{1}{2}(2n - 1 - d_{W_{n+1}}(e))$$
$$= \frac{1}{2}(2n - 1 + d_{W_{n+1}}(e)).$$

By the definition of reciprocal edge transmission sum connectivity index of graph we have

$$RET_{SC}(W_{n+1}) = \sum_{e \sim f \in E(W_{n+1})} \frac{1}{\sqrt{rs_{W_{n+1}}(e) + rs_{W_{n+1}}(f)}}$$
$$= \sum_{e \sim f \in E(W_{n+1})} \frac{1}{\sqrt{(2n-1) + \frac{1}{2}(d_{W_{n+1}}(e) + d_{W_{n+1}}(f))}}$$

The edge set $E(W_{n+1})$ can be partitioned into two sets E_1 and E_2 such that

$$E_1 = \{uv / d(u) = n \text{ and } d(v) = 3\},\$$

$$E_2 = \{uv / d(u) = 3 \text{ and } d(v) = 3\}$$

$$\begin{array}{rcl} d_{W_{n+1}}(e) &=& d(u) + d(v) - 2 \ for \ each \ edge \ e = uv \ \in E_1 \\ &=& n+3-2 = n+1 \ for \ each \ edge \ e = uv \ \in E_1, \\ d_{W_{n+1}}(e) &=& d(u) + d(v) - 2 \ for \ each \ edge \ e = uv \ \in E_2 \\ &=& 3+3-2 = 4 \ for \ each \ edge \ e = uv \ \in E_2. \end{array}$$

$$RET_{SC}(W_{n+1}) = \sum_{e \sim f \in E_1} \frac{1}{\sqrt{(2n-1) + \frac{1}{2}(d_{W_{n+1}}(e) + d_{W_{n+1}}(f))}}} \\ + \sum_{e \sim f \in E_2} \frac{1}{\sqrt{(2n-1) + \frac{1}{2}(d_{W_{n+1}}(e) + d_{W_{n+1}}(f))}}} \\ = \sum_{e \sim f \in E_1} \frac{1}{\sqrt{(2n-1) + \frac{1}{2}(2(n+1))}} + \sum_{e \sim f \in E_2} \frac{1}{\sqrt{(2n-1) + \frac{1}{2}(8)}} \\ = \sum_{e \sim f \in E_1} \frac{1}{\sqrt{(2n-1) + (n+1)}} + \sum_{e \sim f \in E_2} \frac{1}{\sqrt{(2n-1) + 4}} \\ = \sum_{e \sim f \in E_1} \frac{1}{\sqrt{3n}} + \sum_{e \sim f \in E_2} \frac{1}{\sqrt{2n+3}}.$$

Analogous to Theorem 3.5 we have the following results.

Theorem 3.6. For a wheel W_{n+1} , $n \geq 3$. Then

$$(i) RET_{GA}(W_{n+1}) = \sum_{e \sim f \in E_1} \frac{2\sqrt{\frac{1}{4}} \left[(2n-1)^2 + 2(2n-1)(n+1) + (n+1)^2\right]}{3n} \\ + \sum_{e \sim f \in E_2} \frac{2\sqrt{\frac{1}{4}} \left[4n^2 + 12n + 9\right]}{2n+3}.$$

$$(ii) RET_{AG}(W_{n+1}) = \sum_{e \sim f \in E_1} \frac{3n}{2\sqrt{\frac{1}{4}} \left[(2n-1)^2 + 2(2n-1)(n+1) + (n+1)^2\right]} \\ + \sum_{e \sim f \in E_2} \frac{2n+3}{2\sqrt{\frac{1}{4}} \left[4n^2 + 12n + 9\right]}.$$

$$(iii) RET_{ABC}(W_{n+1}) = \sum_{e \sim f \in E_1} \sqrt{\frac{3n-2}{\frac{1}{4} \left[(2n-1)^2 + 2(2n-1)(n+1) + (n+1)^2\right]}} \\ + \sum_{e \sim f \in E_2} \sqrt{\frac{2n+1}{\frac{1}{4} \left[4n^2 + 12n + 9\right]}}.$$

$$(iv) RET_{AZ}(W_{n+1}) = \sum_{e \sim f \in E_1} \left(\frac{\frac{1}{4} \left[(2n-1)^2 + 2(2n-1)(n+1) + (n+1)^2\right]}{3n-2}\right)^3 \\ + \sum_{e \sim f \in E_2} \left(\frac{\frac{1}{4} \left[4n^2 + 12n + 9\right]}{2n+1}\right)^3.$$

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$$(v) RET_{ISIS}(W_{n+1}) = \sum_{e \sim f \in E_1} \left(\frac{\frac{1}{4} \left[(2n-1)^2 + 2(2n-1)(n+1) + (n+1)^2 \right]}{3n} \right) \\ + \sum_{e \sim f \in E_2} \left(\frac{\frac{1}{4} \left[4n^2 + 12n + 9 \right]}{2n+3} \right).$$

Proposition 3.7. For friendship graph F_n , $n \ge 2$. Then

$$RET_{SC}(F_n) = \sum_{e \sim f \in E_1} \frac{1}{\sqrt{5n-1}} + \sum_{e \sim f \in E_2} \frac{1}{\sqrt{3n+1}}.$$

Proof. The friendship graph F_n has 2n + 1 vertices and 3n edges. If $diam(F_n) \leq 2$, then $d_{F_n}(e)$ number of edges are at distance 1 from e and remaining $3n - 1 - d_{F_n}(e)$ edges are at distance 2. Therefore for each edge e in F_n .

$$rs_{F_n}(e) = d_{F_n}(e) + \frac{1}{2}(3n - 1 - d_{F_n}(e))$$

= $\frac{1}{2}(3n - 1 + d_{F_n}(e)).$

By the definition of reciprocal edge transmission sum connectivity index of graph we have

$$RET_{SC}(F_n) = \sum_{e \sim f \in E(F_n)} \frac{1}{\sqrt{rs_{F_n}(e) + rs_{F_n}(f))}}$$
$$= \sum_{e \sim f \in E(F_n)} \frac{1}{\sqrt{(3n-1) + \frac{1}{2}(d_{F_n}(e) + d_{F_n}(f))}}$$

The edge set $E(F_n)$ can be partitioned into two sets E_1 and E_2 , such that

$$E_1 = \{uv \mid d(u) = 2n \text{ and } d(v) = 2\},\$$

$$E_2 = \{uv \mid d(u) = 2 \text{ and } d(v) = 2\}.$$

$$d_{F_n}(e) = d(u) + d(v) - 2 \text{ for each edge } e = uv \in E_1$$

= $2n + 2 - 2 = 2n$ for each edge $e = uv \in E_1$,
$$d_{F_n}(e) = d(u) + d(v) - 2 \text{ for each edge } e = uv \in E_2$$

= $2 + 2 - 2 = 2$ for each edge $e = uv \in E_2$.

$$RET_{SC}(F_n) = \sum_{e \sim f \in E_1} \frac{1}{\sqrt{(3n-1) + \frac{1}{2}(2(2n))}} + \sum_{e \sim f \in E_2} \frac{1}{\sqrt{(3n-1) + \frac{1}{2}4}}$$
$$= \sum_{e \sim f \in E_1} \frac{1}{\sqrt{5n-1}} + \sum_{e \sim f \in E_2} \frac{1}{\sqrt{3n+1}}.$$

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Analogous to Theorem 3.7 we have the following results.

Theorem 3.8. For friendship graph F_n , $n \ge 2$. Then

$$\begin{aligned} (i) \ RET_{GA}(F_n) \ &= \ \sum_{e \sim f \in E_1} \frac{2\sqrt{\frac{1}{4}[25n^2 - 10n + 1]}}{5n - 1} + \sum_{e \sim f \in E_2} \frac{2\sqrt{\frac{1}{4}[9n^2 + 6n + 1]}}{3n + 1}. \\ (ii) \ RET_{AG}(F_n) \ &= \ \sum_{e \sim f \in E_1} \frac{5n - 1}{2\sqrt{\frac{1}{4}[25n^2 - 10n + 1]}} + \sum_{e \sim f \in E_2} \frac{3n + 1}{2\sqrt{\frac{1}{4}[9n^2 + 6n + 1]}}. \\ (iii) \ RET_{ABC}(F_n) \ &= \ \sum_{e \sim f \in E_1} \sqrt{\frac{5n - 3}{\frac{1}{4}[25n^2 - 10n + 1]}} + \sum_{e \sim f \in E_2} \sqrt{\frac{3n - 1}{\frac{1}{4}[9n^2 + 6n + 1]}}. \\ (iv) \ RET_{ABC}(F_n) \ &= \ \sum_{e \sim f \in E_1} \left(\frac{\frac{1}{4}[25n^2 - 10n + 1]}{5n - 3}\right)^3 + \sum_{e \sim f \in E_2} \left(\frac{\frac{1}{4}[9n^2 + 6n + 1]}{3n - 1}\right)^3. \\ (v) \ RET_{ISI}(F_n) \ &= \ \sum_{e \sim f \in E_1} \left(\frac{\frac{1}{4}[25n^2 - 10n + 1]}{5n - 1}\right) + \sum_{e \sim f \in E_2} \left(\frac{\frac{1}{4}[9n^2 + 6n + 1]}{3n + 1}\right). \end{aligned}$$

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