Proceedings of the Jangjeon Mathematical Society
25 (2022), No. 4, pp. 365 – 368
www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2022.25.4.365

A ONE-DIMENSIONAL PSEUDOREPRESENTATION OF AN AMENABLE GROUP WITH A DEFECT LESS THAN 1/4 IS AN ORDINARY CHARACTER OF THE GROUP

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Abstract. We prove that a one-dimensional pseudorepresentation of an amenable group with a defect less than 1/4 is an ordinary character of the group.

§ 1. Introduction

For the definitions, notation, and generalities concerning pseudocharacters, quasicharacters, pseudorepresentations, and quasirepresentations, see [1]–[3]. In particular, recall that a mapping π of a given group G into the family of invertible operators in the algebra $\mathcal{L}(E)$ of bounded linear operators on a Banach space E is said to be a *quasirepresentation* of G on E if $\pi(e_G) = 1_E$, where e_G stands for the identity element of G and 1_E for the identity operator on E, and if

$$\|\pi(g_1g_2) - \pi(g_1)\pi(g_2)\|_{\mathcal{L}(E)} \le \varepsilon, \qquad g_1, g_2 \in G,$$

for some ε, which is usually assumed to be sufficiently small, and the greatest lower bound of $|\pi(g_1g_2) - \pi(g_1)\pi(g_2)|$ for a one-dimensional quasirepresentation π is referred to as the *defect* of π; a one-dimensional quasirepresentation π of G is said to be a one-dimensional *pseudorepresentation* of G if $\pi(g^n) = \pi(g)^n$ for any $n \in \mathbb{Z}$ and $g \in G$.

2020 *Mathematics Subject Classification.* Primary 20C99.
Submitted August 5, 2022.
Key words and phrases. Pseudorepresentation, one-dimensional pseudorepresentation, defect of pseudorepresentation.

We prove here that a one-dimensional pseudorepresentation of an amenable group with a defect less than 1/4 is an ordinary character of the group.

§ 2. Preliminaries

We need a well-known lemma. For the convenience of the reader, we present it with a proof.

Lemma 1. *Let T be a bounded linear operator on a dual Banach space E such that*

$$\|T^n - 1_E\| \le q < 1, \qquad n \in \mathbb{N},$$

with respect to the operator norm. Then $T = 1_E$.

Proof. Let us apply an invariant mean I on $\mathbb{N}$ extended to the bounded linear operators on E as in [5]. Let $S = I_n(T^n)$. Then $\|S - I_E\| \le q < 1$, and hence S is invertible. By the invariance of I, we have

$$TS = TI(T^n) = I(T^{n+1}) = S,$$

which implies that $T = 1_E$.

This immediately implies the following corollary.

Corollary 1. *Let φ and ψ be unitary characters of the group $\mathbb{Z}$ of integers. If*

$$|\varphi(n) - \psi(n)| \le q < 1 \tag{1}$$

for all $n \in \mathbb{Z}$, then $\varphi = \psi$.

Proof. It follows from (1) that $|\varphi \circ \psi^{-1}(n) - 1| \le q < 1$ for all $n \in \mathbb{N}$. Applying an invariant mean I on $\mathbb{N}$, we see that $|S - 1| < q$ for $S = I_n(\varphi \circ \psi^{-1}(n))$. It follows from the invariance of I that $S\varphi \circ \psi^{-1}(1) = S$, where $S \ne 0$. Thus, $\varphi \circ \psi^{-1}(1) = 1$, $\varphi(1) = \psi(1)$, and $\varphi = \psi$, as was to be proved.

§ 3. Main result

Recall a known result (Lemma 3 of [4]).

Lemma 2. *Let G be an amenable group (for example, a commutative group), let f be a one-dimensional bounded ε-quasirepresentation of G ($\varepsilon > 0$) satisfying the condition $f(e) = 1$. If $\varepsilon < 1/3$, then there is an ordinary unitary character ψ of G for which*

$$|f(g) - \psi(g)| < \varepsilon/(1 - 3\varepsilon) \quad \textit{for any} \quad g \in G.$$

If $\varepsilon < \sqrt{3}/(2+3\sqrt{3})$ (*e.g., if* $\varepsilon < 0.24$), *then there is an ordinary unitary character* ψ *of* G *such that* $|f(g) - \psi(g)| < \sqrt{3}/2$ *for any* $g \in G$.

This result gives an immediate possibility to prove the desired fact.

Theorem. *Every one-dimensional pseudorepresentation of an amenable group with a defect less than* $1/4$ *is an ordinary character of the group.*

Proof. Certainly, if $\varepsilon < 1/4$, then there is an ordinary unitary character ψ of G for which $|f(g) - \psi(g)| < \varepsilon/(1-3\varepsilon) < 1$. The restriction of f to every cyclic subgroup of G is an ordinary character of the subgroup. Therefore, for any cyclic subgroup of G, the restrictions of f and ψ to this subgroup coincide by Corollary 1. Hence, $f(g) = \psi(g)$ for every $g \in G$, as was to be proved.

Corollary 2. *Let* G *be a group, and let* π *and* ρ *be two one-dimensional bounded* (*and thus unitary*) *pseudorepresentations of* G. *If* $|\pi(g) - \psi(g)| \le r < 1/4$ *for all* $g \in G$, *then* $\pi = \rho$.

Proof. Applying Corollary 1 to the restrictions of π and ψ to every cyclic subgroup of G, we see that these restrictions coincide, and thus $\pi(g) = \psi(g)$ for all $g \in G$.

§ 4. Comments

Thus, the "topology of uniform convergence on the group" is discrete for one-dimensional pseudorepresentations.

Acknowledgments

I thank Professor Taekyun Kim for the invitation to publish this paper in the Proceedings of the Jangjeon Mathematical Society.

Funding

The research was partially supported by the Moscow Center for Fundamental and Applied Mathematics.

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