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# A REVISED FORMULA FOR A LOCALLY BOUNDED PSEUDOCHARACTER ON AN ALMOST CONNECTED LOCALLY COMPACT GROUP

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Abstract. As was proved in the paper Shtern A. I., Description of locally bounded pseudocharacters on almost connected locally compact groups, Russ. J. Math. Phys. **23** (2016), no. 4, 551–552, if $G$ is an almost connected locally compact group and $G_0$ is the connected component of the identity in $G$, then every locally bounded pseudocharacter of $G$ is a uniquely defined extension to $G$ of a locally bounded pseudocharacter on $G_0$. We prove here that every locally bounded pseudocharacter on $G_0$ admits an extension to a uniquely defined locally bounded pseudocharacter of $G$. Thus, all pseudocharacters on $G$ are in a one-to-one correspondence with the pseudocharacters on $G_0$ described in Theorem 1 of the aforementioned paper. We also correct the formula in the paper Shtern A. I., A formula for pseudocharacters on almost connected groups, Russ. J. Math. Phys. **25** (2018), no. 4, 531–533, connecting a locally bounded pseudocharacter of $G$ and its restriction to $G_0$.

## § 1. Introduction

For the definitions, notation, and generalities concerning pseudocharacters and quasicharacters, and also for the definition of the Guichardet–Wigner pseudocharacter on a locally compact group, see [1]–[3].

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As was proved in [4], if $G$ is an almost connected locally compact group and $G_0$ is the connected component of the identity in $G$, then every locally bounded pseudocharacter of $G$ is a uniquely defined extension to $G$ of a locally bounded pseudocharacter on $G_0$. We prove here that every locally bounded pseudocharacter on $G_0$ admits an extension to a uniquely defined locally bounded pseudocharacter of $G$. Thus, all pseudocharacters on $G$ are in a one-to-one correspondence with the pseudocharacters on $G_0$ described in Theorem 1 of [4], as was claimed in [4, 5] and proved there in one direction. We also correct a formula of [5] connecting a locally bounded pseudocharacter of $G$ and its restriction to $G_0$.

## § 2. Preliminaries

Let $G$ be an almost connected locally compact group and let $f$ be a locally bounded pseudocharacter on $G$. Let $N$ be a maximal compact normal subgroup of $G$ (see Lemma 4.2 of [7]). Then the quotient group $H = G/N$ is a Lie group. Since $f$ is locally bounded and $N$ is compact, it follows that the restriction of $f$ to $N$ is bounded, and hence zero (Corollary 4.1 of [8]). Therefore, there is a locally bounded pseudocharacter $\varphi$ on $H$ such that $f = \varphi \circ \pi$, where $\pi$ stands for the canonical epimorphism of $G$ onto $G/N$, and $\varphi$ is continuous (Theorem 4.1, (d) of [8]). Obviously, $H$ is an almost connected Lie group. We have thus proved the following assertion.

**Lemma 1.** *Every locally bounded pseudocharacter on an almost connected locally compact group is uniquely determined by a locally bounded pseudocharacter on the almost connected quotient Lie group $H$ of $G$ having no compact normal subgroups.*

Recall that a locally bounded pseudocharacter on a connected Lie group is automatically continuous (see [1–3, 8, 9]).

For this reason, we formulate or main theorem for continuous pseudocharacters on almost connected Lie groups.

Recall Dong Hoon Lee's supplement theorem (Theorem 2.13 of [6]): every almost connected locally compact group $G$ with the connected component $G_0$ (i.e., a locally compact group $G$ for which the quotient group $G/G_0$ is compact) admits a totally disconnected compact subgroup $D$ such that $G = G_0D$. Note that, if $H$ is a Lie group, then $H/H_0$ is a totally disconnected Lie group, and hence is discrete, which implies that $H/H_0$ is finite. Then, by Lemma 2.12 of [6], there exists a supplement for $H_0$, i.e., a finite subgroup

$D$ of $H$ such that $H = H_0D$, where $H_0$ stands for the connected component of $H$.

## § 3. Main theorems

**Theorem 1.** *Let $H$ be an almost connected Lie group, let $H_0$ be its connected component of the identity, and let $\varphi$ be a locally bounded (i.e., continuous) pseudocharacter on $H$. Let $D$ be a supplement for $H_0$ in $H$ (see Definition 1) and let $\psi$ be the restriction of $\varphi$ to $H_0$. Let us choose some representation of every element $h \in G$ in the form $h = h_0d$, where $h_0 \in H_0$ and $d \in D$. Then the formula*

$$F(h) = \psi(h_0), \qquad h = h_0d,$$

*well defines a quasicharacter on $H$. The pseudocharacter corresponding to $F$ by Theorem 4.1, (d) of [8] is $\varphi$.*

*Proof.* Let $h = h_0d$ and $h' = h'_0d'$ be two elements of $H$ in the chosen forms. Then

$$hh' = h_0dh'0d' = h_0dh'h'_0d^{-1}dd' = (h_0dh'_0d^{-1})dd' = (h_0dh'_0d^{-1})dd'{d''}^{-1}d'',$$

where $(h_0dh'_0d^{-1})dd'{d''}^{-1}d''$ is the chosen representation of the corresponding element of $H$ with $(h_0dh'_0d^{-1})dd'{d''}^{-1} \in H_0$. Therefore, since $f$ vanishes on $D$, we have

$$\begin{aligned} F(hh') &= \psi(h_0dh'_0d^{-1})dd'{d''}^{-1}) = (\psi((h_0dh'_0d^{-1})dd'{d''}^{-1}) \\ &\quad - \psi(h_0dh'_0d^{-1}) - \psi(dd'{d''}^{-1}) + \psi(h_0dh'_0d^{-1}) \\ &= (\psi((h_0dh'_0d^{-1})dd'{d''}^{-1}) - \psi(h_0dh'_0d^{-1}) - \psi(dd'{d''}^{-1}) \\ &\quad + (\psi(h_0dh'_0d^{-1}) - \psi(h_0) - \psi(dh'_0d^{-1})) + \psi(h_0) + \psi(h'_0) \\ &= (\psi((h_0dh'_0d^{-1})dd'{d''}^{-1}) - \psi(h_0dh'_0d^{-1}) - \psi(dd'{d''}^{-1}) \\ &\quad + (\psi(h_0dh'_0d^{-1}) - \psi(h_0) - \psi(h'_0)) + \psi(h_0) + \psi(h'_0), \end{aligned}$$

due to the inner invariance of $\psi$. Thus, $|F(hh') - F(h) - F(h')|$ does not exceed two defects of the pseudocharacter $\varphi$ and thus is bounded on $H \times H$, which proves that $F$ is a quasicharacter on $H$ indeed.

For the chosen representation $h = h_0d \in G$, $h_0 \in H_0$, $d \in D$, we see that $|\varphi(h) - F(h)| = |\varphi(h_0d) - \varphi(h_0) - \varphi(d)|$ does not exceed the defect of $f$; since the difference between the pseudocharacter $\varphi$ and the quasicharacter

$F$ is bounded, it follows that $\varphi$ is the pseudocharacter defined by the quasicharacter $F$ on $H$ (see Theorem 4.1 of [8]). This completes the proof of the theorem.

Let us now correct the formula in the paper [5] connecting a locally bounded pseudocharacter of $H$ and its restriction to $H_0$.

**Theorem 2.** *Let $H$ be an almost connected Lie group and let $\varphi$ be a locally bounded (i.e., continuous) pseudocharacter on $H$. Let $D$ be a supplement for $H_0$ in $H$ (see Definition 1) and let $\psi$ be the restriction of $\varphi$ to $H_0$. Let $q$ be the order of $D$. Then*

$$\varphi(h_0 d) = q^{-1}\psi(\prod_{k=0}^{q-1} d^k h_0 d^{-k}), \qquad h_0 \in G_0, \quad d \in D. \tag{1}$$

*Proof.* Let us find the pseudocharacter on $H$ corresponding to the quasicharacter $F$ explicitly. Let $h = h_0 d \in H$ be the chosen representation of an element $h \in H$. Let $n \in \mathbb{N}$ be a positive integer. Then

$$F(h^n) - F(\prod_{k=0}^{n-1} d^k h_0 d^{-k}) = F(\prod_{k=0}^{n-1} d^k h_0 d^{-k} \cdot d^n) - F(\prod_{k=0}^{n-1} d^k h_0 d^{-k})$$

does not exceed the defect of $F$, and hence is bounded. Therefore,

$$\varphi(h) = \lim_{n\to\infty} n^{-1}\psi(\prod_{k=0}^{n-1} d^k h_0 d^{-k}). \tag{2}$$

Since $d^q = e_D = e_G$, it follows that, for $n = pq + r$, where $r = r(n) \in \{0.1.\dots, q-1\}$, we have

$$\psi(\prod_{k=0}^{n-1} d^k h_0 d^{-k}) = \psi(a^p \prod_{k=0}^{r-1} d^k h_0 d^{-k}), \tag{3}$$

where the product on the right-hand side is $e$ if $r = 0$ and

$$a = \prod_{k=0}^{q-1} d^k h_0 d^{-k}. \tag{4}$$

Combining (2) and (3), we see that the absolute value of

$$\psi(\prod_{k=0}^{n-1} d^k h_0 d^{-k}) - p\psi(a) - \psi(\prod_{k=0}^{r} d^k h_0 d^{-k}) \tag{4}$$

is bounded by the defect of $\psi$. Certainly, the last term in (4) is bounded. Therefore, the limit in (1) is equal to the limit

$$\lim_{n\to\infty} \frac{(n-r(n))/q}{n}\psi(a) = (1/q)\psi(a),$$

as was to be proved.

## § 4. Concluding remarks

Thus, the formula in [5] holds if and only if $\psi(\prod_{k=0}^{q-1} d^k h_0 d^{-k}) = q\psi(h_0)$ for all $h_0 \in H$ and $d \in D$.

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