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SOME IDENTITIES OF TYPE 2 DEGENERATE POLY-GENOCCHI NUMBERS AND POLYNOMIALS

YUNJAE KIM, SANGBEOM PARK, SUNG-SOO PYO, AND JONGKYUM KWON

ABSTRACT. The purpose of this paper is to introduce a new version of the degenerate poly-Genocchi polynomials and numbers called the type 2 degenerate poly-Genocchi polynomials and numbers by the modification of degenerate polyexponential function in the generating function. To investigate the properties of the proposed polynomials and numbers, we derive several explicit expressions and identities induced from new generating function. Also, we present some relations between the type 2 degenerate poly-Genocchi polynomials and numbers and some other well-known special polynomials and numbers. In addition, we consider the higher-order type 2 degenerate Genocchi polynomials and show some interesting identities involving those polynomials and the type 2 higher-order Changhee polynomials.

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1. INTRODUCTION

Polynomials provides a fundamental tool in mathematics and its applications, especially in approximation theory. There are many different kinds of expressions for an approximation of a function in terms of difference polynomials. Even these expressions are analytically consistent, but when calculating numerically, there may be a big difference in terms of stability, efficiency and accuracy due to the difference in expression method. For example, the barycentric Lagrange interpolation polynomial is a variant of the Lagrange interpolation polynomial, but it is faster and more stable than the original one[1]. Therefore, it is important to develop various types of polynomials and find their properties for approximation theory as well as many different applications.

As a new branch of study for special polynomials and numbers, Carlitz in [3] introduced the degenerate types of special polynomials and numbers by modification of generating functions using new parameter λ, and found interesting relationships between Bernoulli polynomials and Eulerian polynomials and some important numbers in combinatorics. It turns out that these degenerate versions of special polynomials and numbers provide many useful properties and identities for the new type of functions as well as the original polynomials since the degenerate polynomials are approaching to the original ones as λ goes to 0. Especially, much attention has been paid to develop various degenerate types of Genocchi polynomials and numbers focusing on the relations and applications among them (see [7, 8, 23, 24, 26]). Genocchi numbers have applications in many different branches of mathematics such as elementary number theory, complex analytic number theory, p-adic analytic number theory, differential topology, theory of modular forms, quantum physics, and the combinatorial relations (see [2, 4, 5, 28, 30, 31] and references therein).

The main goal of this paper is to introduce a new type of degenerate version of Genocchi polynomials called type 2 degenerate poly-Genocchi polynomials and provide several identities and relations related to the well-known polynomials such as the degenerate Euler polynomials, the type 2 Changhee polynomials. To do this, we recall several preliminary definitions and properties. The polyexponential function firstly introduced by Hardy is defined by

$$e(x,a|s)=\sum_{n=0}^{\infty}\frac{x^n}{(n+a)^s n!},\ (\mathrm{Re}(a)>0),\quad (\text{see } [9,15]). \tag{1}$$

Recently, the modified polyexponential function was introduced by Kim-Kim and is given by

$$\mathrm{Ei}_k(x)=\sum_{n=1}^{\infty}\frac{x^n}{(n-1)!n^k},\ (k\in\mathbb{Z}),\quad (\text{see } [11,12,13,15]). \tag{2}$$

From (1) and (2), we note that

$$x\,e(x,1|k)=Ei_k(x).$$

The degenerate exponential functions are defined as

$$e_\lambda^x(t)=(1+\lambda t)^{\frac{x}{\lambda}},\quad e_\lambda(t)=e_\lambda^1(t)=(1+\lambda t)^{\frac{1}{\lambda}},\quad (\text{see } [6,7,8,11-14]). \tag{3}$$

Here we note that

$$e_\lambda^x(t)=\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!},\quad (\text{see } [12,13,14]), \tag{4}$$

where $(x)_{0,\lambda}=1$, $(x)_{n,\lambda}=x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda),(n\geq 1)$.

Let $\log_\lambda(t)$ be the compositional inverse function of $e_\lambda(t)$ such that

$$e_\lambda(\log_\lambda(t))=\log_\lambda(e_\lambda(t))=t.$$

The Genocchi polynomials $G_n(x)$ are defined by

$$\frac{2t}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!},\quad (\text{see } [2,4,5,6,23,25,26,28,29]).$$

When $x=0, G_n=G_n(0)$ are called the Genocchi numbers.

The degenerate Genocchi polynomial $g_{n,\lambda}$ is defined as

$$\frac{2t}{e_\lambda(t)+1}e_\lambda^x(t)=\sum_{n=0}^{\infty}g_{n,\lambda}(x)\frac{t^n}{n!}, \tag{5}$$

which is introduced by Lim at [24]. The degenerate Genocchi numbers correpondng to the degenerate Genocchi polynomials are defined as $g_{n,\lambda}=g_{n,\lambda}(0)$ when $x=0$.

Carlitz [3] introduced the degenerate Euler polynomials given by

$$\frac{2}{e_\lambda(t)+1}e_\lambda^x(t)=\sum_{n=0}^{\infty}E_{n,\lambda}(x)\frac{t^n}{n!}. \tag{6}$$

In particular, the numbers $E_{n,\lambda} = E_{n,\lambda}(0)$ when $x = 0$ are called the degenerate Euler numbers.

Dolgy-Jang in [4] introduced poly-Genocchi polynomials arising from polyexponential function as

$$\frac{2\mathrm{Ei}_k(\log(1+t))}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_n^{(k)}(x)\frac{t^n}{n!}, \tag{7}$$

and defined the poly-Genocchi numbers as $G_n^{(k)} = G_n^{(k)}(0)$.

Note that $G_n(x) = G_n^{(1)}(x)$, $(n \geq 0)$, are the Genocchi polynomials.

In [20], Kim-Kim introduced the degenerate Fubini polynomials given by

$$\frac{1}{1-x(e_\lambda(t)-1)} = \sum_{n=0}^{\infty} F_{n,\lambda}(x)\frac{t^n}{n!}. \tag{8}$$

In [10], the degenerate Bernoulli polynomials of the second kind are defined by the generating function to be

$$\frac{t}{\log_\lambda(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_{n,\lambda}(x)\frac{t^n}{n!}. \tag{9}$$

Note that $\lim_{\lambda\to 0} b_{n,\lambda}(x) = b_n(x), (n \geq 0)$.

For $n \geq 0$, the Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l, \quad (\text{see } [4-26]). \tag{10}$$

where $(x)_0 = 1, (x)_n = x(x-1)\cdots(x-n+1)$ $(n \geq 1)$.

From (10), it is easy to see that

$$\frac{1}{k!}(\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}. \tag{11}$$

In the inverse expression to (10), for $n \geq 0$, the Stirling numbers of the second kind are defined by

$$x^n = \sum_{l=0}^{\infty} S_2(n,l)(x)_l, \quad (\text{see } [4-26]). \tag{12}$$

From (12), it is easy to see that

$$\frac{1}{k!}(e^t-1)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}. \tag{13}$$

The degenerate Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{l=0}^{n} S_{1,\lambda}(n,l)(x)_{l,\lambda}, \quad (n \geq 0), \quad (\text{see } [10]). \tag{14}$$

Note here that $\lim_{\lambda\to 0} S_{1,\lambda}(n,k) = S_1(n,k)$, where $S_1(n,k)$ are the Stirling numbers of the first kind.

From (14), we easily get

$$\frac{1}{k!}\left(\log_\lambda(1+t)\right)^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!}, \quad (k \geq 0), \quad (\text{see}[7,8,11-15,17,18,20,21,22]). \tag{15}$$

As an inversion formula of (14), the degenerate Stirling numbers of the second kind are given by

$$(x)_{n,\lambda} = \sum_{l=0}^{n} S_{2,\lambda}(n,l)(x)_l, \quad (n \geq 0), \quad (\text{see [10]}). \tag{16}$$

Observe here that $\lim_{\lambda\to 0} S_{2,\lambda}(n,l) = S_2(n,l)$, where $S_2(n,l)$ are the Stirling numbers of the second kind.

From (16), we note that

(17)

$$\frac{1}{k!}\left(e_\lambda(t)-1\right)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!}, \quad (k \geq 0), \quad (\text{see}[7,8,11-15,17,18,20,21,22]).$$

The type 2 degenerate Genocchi polynomials $G_{n,\lambda}(x)$ and the type 2 degenerate Euler polynomials $\mathscr{E}_{n,\lambda}(x)$ are defined

$$\frac{2t}{e_\lambda(t)+e_\lambda(-t)} e_\lambda^x(t) = \sum_{n=0}^{\infty} G_{n,\lambda}(x)\frac{t^n}{n!}, \tag{18}$$

and

$$\frac{2}{e_\lambda(t)+e_\lambda(-t)} e_\lambda^x(t) = \sum_{n=0}^{\infty} \mathscr{E}_{n,\lambda}(x)\frac{t^n}{n!}, \tag{19}$$

respectively.

When $x = 0$, $G_{n,\lambda} = G_{n,\lambda}(0)$ and $\mathscr{E}_{n,\lambda} = \mathscr{E}_{n,\lambda}(0)$ are called the type 2 degenerate Genocchi numbers and the type 2 degenerate Euler numbers, respectively.

To study the type 2 degenerate poly-Genocchi numbers and polynomials, we consider the modified degenerate polyexponential function which is given by

$$\mathrm{Ei}_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{x^n (1)_{n,\lambda}}{(n-1)!n^k}, (k \in \mathbb{Z}), \quad (\text{see } [7,11,12,13]). \tag{20}$$

Note that $\mathrm{Ei}_{1,\lambda}(x) = \sum_{n=1}^{\infty} (1)_{n,\lambda}\frac{x^n}{n!} = e_\lambda(x) - 1$.

The rest of the paper is organized as follows. In section 2, we introduce the type 2 degenerate poly-Genocchi numbers and polynomials arising from the modified degenerate polyexponential function and investigate some properties for those numbers and polynomials. Also, we derive some explicit identities and relations between the type 2 degenerate poly-Genocchi numbers and other special numbers in section 2. Section 3, we define the degenerate poly-Fubini polynomials and study their properties. Section 4 provides several figures of the type 2 degenerate poly-Genocchi polynomials and their scattering of roots. Lastly, we provide a summary in section 5.

2. Type 2 degenerate poly-Genocchi numbers and polynomials

In this section, we consider the type 2 degenerate poly-Genocchi polynomials given by

$$\frac{2\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{e_\lambda(t)+e_\lambda^{-1}(t)} e_\lambda^x(t) = \sum_{n=0}^{\infty} G_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}. \tag{21}$$

When $x=0$, $G_{n,\lambda}^{(k)}=G_{n,\lambda}^{(k)}(0)$ are called the type 2 degenerate poly-Genocchi numbers.

From (21),we get

$$\begin{aligned}
\sum_{n=0}^{\infty} G_{n,\lambda}^{(k)}(x)\frac{t^n}{n!} &= \frac{2\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^x(t) \\
&= \left(\sum_{l=0}^{\infty} G_{l,\lambda}^{(k)}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(x)_{m,\lambda}\frac{t^m}{m!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}(x)_{n-l,\lambda}G_{l,\lambda}^{(k)}\right)\frac{t^n}{n!}.
\end{aligned} \tag{22}$$

Therefore, by comparing the coefficients on the both sides of (22), we obtain the following theorem.

Theorem 1. *For $n\geq 0$, we have*

$$G_{n,\lambda}^{(k)}(x)=\sum_{l=0}^{n}\binom{n}{l}(x)_{n-l,\lambda}G_{l,\lambda}^{(k)}.$$

By (15) and (20), we note that

$$\begin{aligned}
\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t)) &= \sum_{n=1}^{\infty}\frac{(1)_{n,\lambda}(\log_\lambda(1+t))^n}{(n-1)!n^k} \\
&= \sum_{n=1}^{\infty}\frac{(1)_{n,\lambda}}{n^{k-1}}\frac{1}{n!}(\log_\lambda(1+t))^n \\
&= \sum_{n=1}^{\infty}\frac{(1)_{n,\lambda}}{n^{k-1}}\sum_{m=n}^{\infty}S_{1,\lambda}(m,n)\frac{t^m}{m!} \\
&= \sum_{m=1}^{\infty}\left(\sum_{n=1}^{m}\frac{(1)_{n,\lambda}}{n^{k-1}}S_{1,\lambda}(m,n)\right)\frac{t^m}{m!}.
\end{aligned} \tag{23}$$

Thus, by (23), we have

$$\begin{aligned}
&\frac{2\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^x(t) = \frac{2t}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^x(t)\frac{1}{t}\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t)) \\
&=\sum_{l=0}^{\infty}G_{l,\lambda}(x)\frac{t^l}{l!}\sum_{m=0}^{\infty}\frac{1}{m+1}\sum_{j=1}^{m+1}\frac{(1)_{j,\lambda}}{j^{k-1}}S_{1,\lambda}(m+1,j)\frac{t^m}{m!} \\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{j=1}^{m+1}\frac{\binom{n}{m}}{m+1}\frac{(1)_{j,\lambda}}{j^{k-1}}S_{1,\lambda}(m+1,j)G_{n-m,\lambda}(x)\right)\frac{t^n}{n!}.
\end{aligned} \tag{24}$$

Therefore, by (21) and (24), we obtain the following theorem.

Theorem 2. *For $n\geq 0$, we have*

$$G_{n,\lambda}^{(k)}(x)=\sum_{m=0}^{n}\sum_{j=1}^{m+1}\frac{\binom{n}{m}}{m+1}\frac{(1)_{j,\lambda}}{j^{k-1}}S_{1,\lambda}(m+1,j)G_{n-m,\lambda}(x). \tag{25}$$

From (21), we note that

$$\begin{aligned}
&\frac{2\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^x(t)=\frac{2}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^x(t)\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))\\
&=\sum_{l=0}^{\infty}\mathscr{E}_{l,\lambda}(x)\frac{t^l}{l!}\sum_{m=1}^{\infty}\sum_{j=1}^{m}\frac{(1)_{j,\lambda}}{j^{k-1}}S_{1,\lambda}(m,j)\frac{t^m}{m!}\\
&=\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\sum_{j=1}^{m}\binom{n}{m}\frac{(1)_{j,\lambda}}{j^{k-1}}S_{1,\lambda}(m,j)\mathscr{E}_{n-m,\lambda}(x)\right)\frac{t^n}{n!}.
\end{aligned}\tag{26}$$

Therefore, by (21) and (26), we obtain the following theorem.

Theorem 3. *For $n\in\mathbb{N}$, we have*

$$G_{n,\lambda}^{(k)}(x)=\sum_{m=1}^{n}\sum_{j=1}^{m}\binom{n}{m}\frac{(1)_{j,\lambda}}{j^{k-1}}S_{1,\lambda}(m,j)\mathscr{E}_{n-m,\lambda}(x).$$

From (20), we note that

$$\begin{aligned}
\frac{d}{dx}\mathrm{Ei}_{k,\lambda}(x)&=\frac{d}{dx}\sum_{n=1}^{\infty}\frac{(1)_{n,\lambda}x^n}{(n-1)!n^k}=\frac{1}{x}\sum_{n=1}^{\infty}\frac{(1)_{n,\lambda}x^n}{(n-1)!n^{k-1}}\\
&=\frac{1}{x}\mathrm{Ei}_{k-1,\lambda}(x).
\end{aligned}\tag{27}$$

Thus, by (27), we get

$$\begin{aligned}
\mathrm{Ei}_{k,\lambda}(x)&=\int_0^x\frac{1}{t}\mathrm{Ei}_{k-1,\lambda}(t)dt\\
&=\underbrace{\int_0^x\frac{1}{t}\int_0^t\cdots\frac{1}{t}\int_0^t}_{(k-2)-\text{times}}\frac{1}{t}\mathrm{Ei}_{1,\lambda}(t)dtdt\cdots dt\\
&=\underbrace{\int_0^x\frac{1}{t}\int_0^t\cdots\frac{1}{t}\int_0^t}_{(k-2)-\text{times}}\frac{e_\lambda(t)-1}{t}dtdt\cdots dt,
\end{aligned}\tag{28}$$

where k is a positive integer with $k\geq 2$.

From (21) and (28), we note that

(29)

$$\begin{aligned}
&\sum_{n=0}^{\infty}G_{n,\lambda}^{(k)}\frac{t^n}{n!}=\frac{2}{e_\lambda(t)+e_\lambda^{-1}(t)}\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+t)\right)\\
&=\frac{2}{e_\lambda(t)+e_\lambda^{-1}(t)}\underbrace{\int_0^t\frac{(1+t)^{\lambda-1}}{\log_\lambda(1+t)}\int_0^t\cdots\frac{(1+t)^{\lambda-1}}{\log_\lambda(1+t)}\int_0^t}_{(k-2)-\text{times}}\frac{t(1+t)^{\lambda-1}}{\log_\lambda(1+t)}dt\cdots dt\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}G_{n-l,\lambda}\sum_{l_1+\cdots+l_{k-1}=l}\binom{l}{l_1,\cdots,l_{k-1}}\frac{b_{l_1,\lambda}(\lambda-1)}{l_1+1}\frac{b_{l_2,\lambda}(\lambda-1)}{l_1+l_2+1}\cdots\frac{b_{l_{k-1},\lambda}(\lambda-1)}{l_1+\cdots+l_{k-1}+1}\right)\frac{t^n}{n!}.
\end{aligned}$$

where k is a positive integer with $k\geq 2$.

Therefore, by comparing the coefficients on both sides of (29), we obtain the following theorem.

Theorem 4. *For $n \geq 1$, we have*

$$G_{n,\lambda}^{(k)} = \sum_{l=0}^{n} \binom{n}{l} G_{n-l,\lambda} \sum_{l_1+\cdots+l_{k-1}=l} \binom{l}{l_1,\cdots,l_{k-1}} \frac{b_{l_1,\lambda}(\lambda-1)}{l_1+1} \frac{b_{l_2,\lambda}(\lambda-1)}{l_1+l_2+1} \cdots \frac{b_{l_{k-1},\lambda}(\lambda-1)}{l_1+\cdots+l_{k-1}+1}.$$

For $k=2$ in (29), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} G_{n,\lambda}^{(2)} \frac{t^n}{n!} &= \frac{2}{e_\lambda(t)+e_\lambda^{-1}(t)} \int_0^t \frac{t}{\log_\lambda(1+t)} (1+t)^{\lambda-1} dt \\
&= \frac{2t}{e_\lambda(t)+e_\lambda^{-1}(t)} \sum_{m=0}^{\infty} \frac{b_{m,\lambda}(\lambda-1)}{m+1} \frac{t^m}{m!} \\
&= \sum_{l=0}^{\infty} G_{l,\lambda} \frac{t^l}{l!} \sum_{m=0}^{\infty} \frac{b_{m,\lambda}(\lambda-1)}{m+1} \frac{t^m}{m!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} G_{l,\lambda} \frac{b_{n-l,\lambda}(\lambda-1)}{n-l+1} \right) \frac{t^n}{n!}.
\end{aligned} \tag{30}$$

Therefore, by comparing the coefficients on both sides of (30), we obtain the following theorem.

Theorem 5. *For $n \geq 1$, we have*

$$G_{n,\lambda}^{(2)} = \sum_{l=0}^{n} \binom{n}{l} G_{l,\lambda} \frac{b_{n-l,\lambda}(\lambda-1)}{n-l+1}.$$

From (21),we note that

$$\begin{aligned}
2\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t)) &= (e_\lambda(t)+e_\lambda^{-1}(t)) \sum_{l=0}^{\infty} G_{l,\lambda}^{(k)} \frac{t^l}{l!} \\
&= \left(\sum_{m=0}^{\infty} (1)_{m,\lambda} \frac{t^m}{m!} + \sum_{m=0}^{\infty} (1)_{m,\lambda} (-1)^m \frac{t^m}{m!} \right) \sum_{l=0}^{\infty} G_{l,\lambda}^{(k)} \frac{t^l}{l!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} (1)_{n-m,\lambda} G_{l,\lambda}^{(k)} + (1)_{n-m,\lambda} (-1)^m G_{l,\lambda}^{(k)} \right) \frac{t^n}{n!} \\
&= \sum_{n=1}^{\infty} \left(G_{l,\lambda}^{(k)}(1) + (-1)^m G_{l,\lambda}^{(k)}(1) \right) \frac{t^n}{n!}.
\end{aligned} \tag{31}$$

On the other hand, from (24), we have

$$\begin{aligned}
2\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t)) &= 2 \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda} (\log_\lambda(1+t))^n}{(n-1)! n^k} \\
&= 2 \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda}}{n^{k-1}} \frac{1}{n!} (\log_\lambda(1+t))^n \\
&= 2 \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda}}{n^{k-1}} \sum_{m=n}^{\infty} S_{1,\lambda}(m,n) \frac{t^m}{m!} \\
&= \sum_{m=1}^{\infty} \left(\sum_{n=1}^{m} \frac{2(1)_{n,\lambda}}{n^{k-1}} S_{1,\lambda}(m,n) \right) \frac{t^m}{m!}.
\end{aligned} \tag{32}$$

Therefore, by (31) and (32), we obtain the following theorem.

Theorem 6. *For $n \in \mathbb{N}$, we have*

$$G_{l,\lambda}^{(k)}(1)+(-1)^m G_{l,\lambda}^{(k)}(1)=\sum_{n=1}^{m}\frac{2(1)_{n,\lambda}}{n^{k-1}}S_{1,\lambda}(m,n).$$

It is known that the type 2 Changhee numbers are defined by

$$\frac{2}{(1+t)+(1+t)^{-1}}=\sum_{n=0}^{\infty}\widehat{Ch}_n\frac{t^n}{n!},\quad \text{(see [7,24,26])}. \tag{33}$$

By replacing t by $\log_\lambda(1+t)$ in (18), we get

$$\frac{2}{(1+t)+(1+t)^{-1}}\log_\lambda(1+t)=\sum_{m=1}^{\infty}G_{m,\lambda}\frac{1}{m!}\Big(\log_\lambda(1+t)\Big)^m$$

Thus, we have

$$\begin{aligned}\frac{2}{(1+t)+(1+t)^{-1}}&=\sum_{m=0}^{\infty}\frac{G_{m+1,\lambda}}{m+1}\frac{1}{m!}\Big(\log_\lambda(1+t)\Big)^m\\&=\sum_{m=0}^{\infty}\frac{G_{m+1,\lambda}}{m+1}\sum_{n=m}^{\infty}S_{1,\lambda}(n,m)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{G_{m+1,\lambda}}{m+1}S_{1,\lambda}(n,m)\right)\frac{t^n}{n!}.\end{aligned}\tag{34}$$

Therefore, by (33) and (34), we obtain the following theorem.

Theorem 7. *For $n \geq 0$, we have*

$$\widehat{Ch}_n=\sum_{m=0}^{n}\frac{G_{m+1,\lambda}}{m+1}S_{1,\lambda}(n,m).$$

By replacing t by $e_\lambda(t)-1$ in (33), we get

$$\begin{aligned}\frac{2}{e_\lambda(t)+e_\lambda^{-1}(t)}&=\sum_{m=0}^{\infty}\widehat{Ch}_m\frac{1}{m!}\Big(e_\lambda(t)-1\Big)^m\\&=\sum_{m=0}^{\infty}\widehat{Ch}_m\sum_{n=m}^{\infty}S_{2,\lambda}(n,m)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\widehat{Ch}_m S_{2,\lambda}(n,m)\right)\frac{t^n}{n!}\end{aligned}$$

Thus, we have

$$\begin{aligned}\sum_{n=1}^{\infty}G_{n,\lambda}\frac{t^n}{n!}&=\frac{2t}{e_\lambda(t)+e_\lambda^{-1}(t)}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\widehat{Ch}_m S_{2,\lambda}(n,m)\right)\frac{t^{n+1}}{n!}\\&=\sum_{n=1}^{\infty}\left(n\sum_{m=0}^{n-1}\widehat{Ch}_m S_{2,\lambda}(n-1,m)\right)\frac{t^n}{n!}.\end{aligned}\tag{35}$$

Therefore, by comparing the coefficients on both sides of (35), we obtain the following theorem.

Theorem 8. *For $n \geq 1$, we have*

$$\frac{G_{n,\lambda}}{n} = \sum_{m=0}^{n-1} \widehat{Ch}_m S_{2,\lambda}(n-1,m).$$

For $r \in \mathbb{N}$, we consider the type 2 degenerate Genocchi polynomials of order r which are given by

$$\left(\frac{2t}{e_\lambda(t)+e_\lambda^{-1}(t)}\right)^r e_\lambda^x(t) = \sum_{n=0}^{\infty} \widehat{G}_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}. \tag{36}$$

Note here that $\widehat{G}_{0,\lambda}^{(r)}(x) = \widehat{G}_{1,\lambda}^{(r)}(x) = \cdots = \widehat{G}_{r-1,\lambda}^{(r)}(x) = 0$. When $x = 0$, $\widehat{G}_{n,\lambda}^{(r)} = \widehat{G}_{n,\lambda}^{(r)}(0)$ are called the type 2 degenerate Genocchi numbers of order r.

As is known, the type 2 Changhee polynomials of order r are defined by

$$\left(\frac{2}{(1+t)+(1+t)^{-1}}\right)^r (1+t)^x = \sum_{n=0}^{\infty} \widehat{Ch}_n^{(r)}(x)\frac{t^n}{n!}, \quad \text{(see [7,21,22])}. \tag{37}$$

When $x = 0$, $\widehat{Ch}_n^{(r)} = \widehat{Ch}_n^{(r)}(0)$ are called the type 2 Changhee numbers of order r.

By replacing t by $\log_\lambda(1+t)$ in (36), we get

(38)
$$\begin{aligned} r!\left(\frac{2}{(1+t)+(1+t)^{-1}}\right)^r (1+t)^x \frac{1}{r!}\Big(\log_\lambda(1+t)\Big)^r &= \sum_{m=0}^{\infty} \widehat{G}_{m,\lambda}^{(r)}(x)\frac{1}{m!}\Big(\log_\lambda(1+t)\Big)^m \\ &= \sum_{m=0}^{\infty} \widehat{G}_{m,\lambda}^{(r)}(x) \sum_{n=m}^{\infty} S_{1,\lambda}(n,m)\frac{t^n}{n!} \\ &= \sum_{n=r}^{\infty}\left(\sum_{m=r}^{n} \widehat{G}_{m,\lambda}^{(r)}(x) S_{1,\lambda}(n,m)\right)\frac{t^n}{n!}. \end{aligned}$$

On the other hand,

(39)
$$\begin{aligned} r!\left(\frac{2}{(1+t)+(1+t)^{-1}}\right)^r (1+t)^x \frac{1}{r!}\Big(\log_\lambda(1+t)\Big)^r &= r!\sum_{l=0}^{\infty} \widehat{Ch}_l^{(r)}(x)\frac{t^l}{l!}\sum_{m=r}^{\infty} S_{1,\lambda}(m,r)\frac{t^m}{m!} \\ &= r!\sum_{n=r}^{\infty}\sum_{m=r}^{n}\binom{n}{m}\widehat{Ch}_{n-m}^{(r)}(x) S_{1,\lambda}(m,r)\frac{t^n}{n!}. \end{aligned}$$

Therefore, by (38) and (39), we obtain the following theorem.

Theorem 9. *For $r \geq 1$, and $n \geq r$, we have*

$$\sum_{m=r}^{n} \widehat{G}_{m,\lambda}^{(r)}(x) S_{1,\lambda}(n,m) = r!\sum_{m=r}^{n}\binom{n}{m}\widehat{Ch}_{n-m}^{(r)}(x) S_{1,\lambda}(m,r).$$

From (36), we can derive the following equation.

$$\begin{aligned}\frac{1}{t^r}\left(\frac{2t}{e_\lambda(t)+e_\lambda^{-1}(t)}\right)^r e_\lambda^x(t) &= \sum_{m=0}^{\infty}\widehat{Ch}_m^{(r)}(x)\frac{1}{m!}\Big(e_\lambda(t)-1\Big)^m \\ &= \sum_{m=0}^{\infty}\widehat{Ch}_m^{(r)}(x)\sum_{n=m}^{\infty}S_{2,\lambda}(n,m)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\widehat{Ch}_m^{(r)}(x)S_{2,\lambda}(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{40}$$

On the other hand,

$$\begin{aligned}\frac{1}{t^r}\left(\frac{2t}{e_\lambda(t)+e_\lambda^{-1}(t)}\right)^r e_\lambda^x(t) &= \frac{1}{t^r}\sum_{n=r}^{\infty}\widehat{G}_{n,\lambda}^{(r)}(x)\frac{t^n}{n!} \\ &= \frac{1}{r!}\sum_{n=0}^{\infty}\frac{\widehat{G}_{n+r,\lambda}^{(r)}(x)}{\binom{n+r}{n}}\frac{t^n}{n!}.\end{aligned} \tag{41}$$

Therefore, by (40) and (41), we obtain the following theorem.

Theorem 10. *For $n \geq 0$, and $r \geq 1$, we have*

$$\widehat{G}_{n+r,\lambda}^{(r)}(x) = r!\binom{n+r}{n}\sum_{m=0}^{n}\widehat{Ch}_m^{(r)}(x)S_{2,\lambda}(n,m).$$

3. Degenerate poly-Fubini polynomials

In this section, we study a new version of the degenerate poly-Funini polynomials and investigate their properties.

We first define the degenerate poly-Fubini polynomials given by

$$\frac{Ei_{k,\lambda}(\log_\lambda(1+t))}{t(1-x(e_\lambda(t)-1))} = \sum_{n=0}^{\infty}F_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}. \tag{42}$$

Now, we consider

$$\frac{Ei_{k,\lambda}(\log_\lambda(1+t))}{1-x(e_\lambda(t)-1)} = \sum_{n=0}^{\infty}nF_{n-1,\lambda}^{(k)}(x)\frac{t^n}{n!}. \tag{43}$$

On the other hand,

(44)

$$\begin{aligned}\frac{Ei_{k,\lambda}(\log_\lambda(1+t))}{1-x(e_\lambda(t)-1)} &= \sum_{\ell=1}^{\infty}\frac{(1)_{\ell,\lambda}(\log_\lambda(1+t))^\ell}{(\ell-1)!\ell^k}\times\sum_{j=0}^{\infty}x^j j!\frac{1}{j!}(e_\lambda(t)-1)^j \\ &= \sum_{\ell=1}^{\infty}\frac{(1)_{\ell,\lambda}}{\ell^{k-1}}\frac{1}{\ell!}(\log_\lambda(1+t))^\ell\times\sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}x^jS_2(i,j)j!\right)\frac{t^i}{i!} \\ &= \left(\sum_{m=1}^{\infty}\sum_{\ell=1}^{m}\frac{(1)_{\ell,\lambda}}{\ell^{k-1}}S_{1,\lambda}(m,\ell)\frac{t^m}{m!}\right)\times\left(\sum_{i=0}^{\infty}\sum_{j=0}^{i}x^jS_2(i,j)j!\frac{t^i}{i!}\right) \\ &= \sum_{n=1}^{\infty}\left(\sum_{i=0}^{n}\sum_{\ell=1}^{n-i}\sum_{j=0}^{i}\binom{n}{i}\frac{(1)_{\ell,\lambda}}{\ell^{k-1}}S_{1,\lambda}(n-i,\ell)x^jS_2(i,j)j!\right)\frac{t^n}{n!}.\end{aligned}$$

Therefore, by comparing the coefficients on the both sides of (43) and (44), we obtain the following result.

Theorem 11. *For $n \geq 1$, we have*

$$nF^{(k)}_{n-1,\lambda}(x) = \sum_{i=0}^{n}\sum_{\ell=1}^{n-i}\sum_{j=0}^{i}\binom{n}{i}\frac{(1)_{\ell,\lambda}}{\ell^{k-1}}S_{1,\lambda}(n-i,\ell)x^j j!S_2(i,j).$$

In (8), we consider

$$\sum_{n=0}^{\infty}F_{n,\lambda}(-\frac{1}{2})\frac{t^n}{n!} = \frac{2}{2e_\lambda(t)+1}.$$

Then,

$$\begin{aligned}\frac{2}{2e_\lambda(t)+1} &= t\sum_{n=0}^{\infty}F_{n,\lambda}(-\frac{1}{2})\frac{t^n}{n!}\\ &= \sum_{n=1}^{\infty}nF_{n-1,\lambda}(-\frac{1}{2})\frac{t^n}{n!}.\end{aligned} \tag{45}$$

On the other hand,

$$\begin{aligned}\frac{2t}{e_\lambda(t)+1} &= \frac{2t}{e_\lambda^{\frac{1}{2}}(t)(e_\lambda^{\frac{1}{2}}(t)+e_\lambda^{-\frac{1}{2}}(t))}\\ &= 2\frac{2(\frac{t}{2})}{e_{2\lambda}(\frac{t}{2})+e_{2\lambda}^{-1}(\frac{t}{2})}e_{2\lambda}^{-1}(\frac{t}{2})\\ &= 2\sum_{n=0}^{\infty}G_{n,2\lambda}(-1)\frac{(\frac{t}{2})^n}{n!}\\ &= \sum_{n=0}^{\infty}\frac{G_{n,2\lambda}(-1)}{2^{n-1}}\frac{t^n}{n!}.\end{aligned} \tag{46}$$

Thus, comparing the coefficients on the both sides of (45) and (46) provides the following result.

Theorem 12. *For $n \geq 1$, we have*

$$nF^{(k)}_{n-1,\lambda}(x) = \frac{G_{n,2\lambda}(-1)}{2^{n-1}}.$$

In (42), we consider

$$\begin{aligned}\sum_{n=0}^{\infty}F^{(k)}_{n,\lambda}(-\frac{1}{2})\frac{t^n}{n!} &= \frac{Ei_{k\lambda}(\log_\lambda(1+t))}{t(1+\frac{1}{2}(e_\lambda(t)-1))}\\ &= \frac{2}{e_\lambda(t)+1}\frac{Ei_{k\lambda}(\log_\lambda(1+t))}{t}\\ &= \left(\sum_{j=1}^{\infty}jF_{j-1,\lambda}(-\frac{1}{2})\frac{t^j}{j!}\right)\times\left(\sum_{m=1}^{\infty}\sum_{\ell=1}^{m}\frac{(1)_{\ell,\lambda}}{\ell^{k-1}}S_{1,\lambda}(m,\ell)\frac{t^{m-1}}{m!}\right)\\ &= \left(\sum_{j=1}^{\infty}jF_{j-1,\lambda}(-\frac{1}{2})\frac{t^j}{j!}\right)\times\left(\sum_{m=0}^{\infty}\sum_{\ell=1}^{m+1}\frac{(1)_{\ell,\lambda}}{\ell^{k-1}}\frac{S_{1,\lambda}(m+1,\ell)}{m+1}\frac{t^m}{m!}\right)\\ &= \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\sum_{\ell=1}^{m+1}\binom{n}{m}(n-m)F_{n-m-1,\lambda}(-\frac{1}{2})\frac{(1)_{\ell,\lambda}}{\ell^{k-1}}\frac{S_{1,\lambda}(m+1,\ell)}{m+1}\right)\frac{t^n}{n!}\end{aligned}$$

Therefore, the following result is established.

Theorem 13. *For $n \geq 1$, we have*

$$F_{n,\lambda}^{(k)}(-\frac{1}{2}) = \sum_{m=1}^{\infty} \sum_{\ell=1}^{m+1} \binom{n}{m} (n-m) F_{n-m-1,\lambda}(-\frac{1}{2}) \frac{(1)_{\ell,\lambda}}{\ell^{k-1}} \frac{S_{1,\lambda}(m+1,\ell)}{m+1}.$$

4. Illustration of $G_{n,\lambda}^{(k)}(x)$

We finally present graphs and scattering of zeros of the proposed polynomials. In order to observe the relationship among the parameters in the polynomial $G_{n,\lambda}^{(k)}(x)$, we compute the polynomial $G_{n,\lambda}^{(k)}(x)$ by varying the parameters, n and k for different λ, and the results are plotted in Figures 1–3 when $\lambda = 0.5$ and 1 for $k = 0, 1$, and 2, respectively. For further observation of the polynomials, we dis-

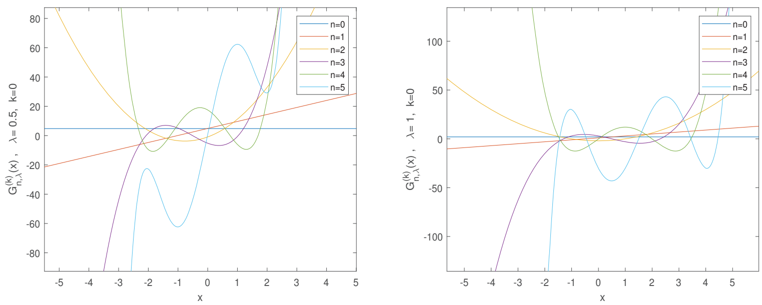

FIGURE 1. The graphs of $G_{n,\lambda}^{(k)}(x)$, where $\lambda = 0.5$(left) and $\lambda = 1$(right) for $k = 0$.

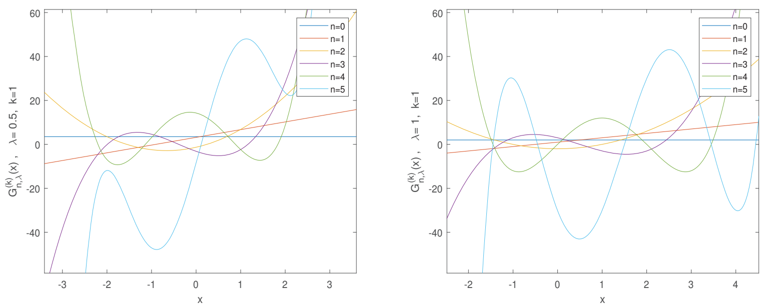

FIGURE 2. The graphs of $G_{n,\lambda}^{(k)}(x)$, where $\lambda = 0.5$(left) and $\lambda = 1$(right) for $k = 1$.

play the roots of polynomials $G_{n,\lambda}^{(k)}(x) = 0$, where $n = 1, 2, \cdots, 15$ when $\lambda = 0.1$ and 1 for $k = 0$ and 2, which are presented in Figures 4–5.

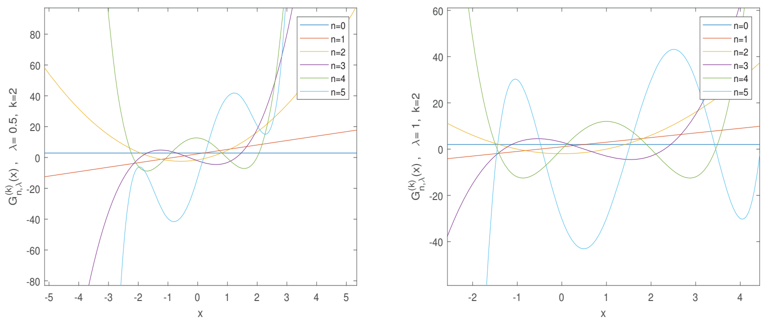

FIGURE 3. The graphs of $G_{n,\lambda}^{(k)}(x)$, where $\lambda = 0.5$(left) and $\lambda = 1$(right) for $k = 2$.

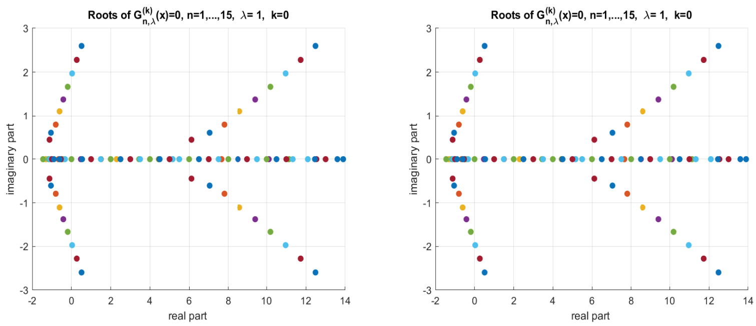

FIGURE 4. Scattering of roots of $G_{n,\lambda}^{(k)}(x) = 0,\ n = 1, 2, \cdots, 15$ when $\lambda = 0.1$(left) and $\lambda = 1$(right) for $k = 0$

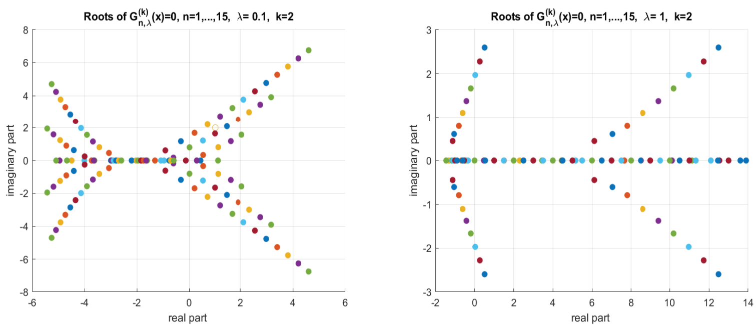

FIGURE 5. Scattering of roots of $G_{n,\lambda}^{(k)}(x) = 0,\ n = 1, 2, \cdots, 15$ when $\lambda = 0.1$(left) and $\lambda = 1$(right) for $k = 2$

5. CONCLUSION

This paper introduced a new version of the degenerate poly-Genocchi polynomials and numbers. We derived explicit expressions for those polynomials and corresponding numbers. The role of the parameters n,k and λ included in the poly-Genocchi polynomial $G_{n,\lambda}^{(k)}(x)$ was investigated. Moreover, some identities involving those polynomials and numbers and some other special numbers and polynomials are investigated. Also, we introduced the degenerate poly-Fubini polynomials and their properties, which are a new version of degenerate Fubini polynomials. In addition, to better understand the degenerate poly-Genocchi polynomials, we provided several graphs and scattering of roots for the proposed polynomials. In the next study, we will focus on how to effectively apply the newly developed polynomials to solve several partial differential equations for applications.

References

[1] Berrut, J.P; Trefethen, L.N. *Barycentric Lagrange Interpolation,* SIAM Review **46**(3) (2004) no. 501–517.

[2] Cangul, I. N.; Kurt, V.; Ozden, H.; Simsek, Y. *On the higher-order w-q-Genocchi numbers,* Adv. Stud. Contemp. Math. (Kyungshang) **19** (2009), no. 1, 39–57.

[3] Carlitz, L. *Degenerate Stirling, Bernoulli and Eulerian numbers,* Utilitas Math. **15** (1979), 51–88.

[4] Dolgy, D. V.; Jang, L.-C. *A note on the polyexponential Genocchi polynomials and numbers,* Symmetry 2020 (in press).

[5] Duran, U.; Acikgoz, M.; Araci, S. *Symmetric identities involving weighted q-Genocchi polynomials under S_4,* Proc. Jangjeon Math. Soc. **18** (2015), no. 4, 455–465.

[6] Jang, L.-C. *A study on the distribution of twisted q-Genocchi polynomials,* Adv. Stud. Contemp. Math. (Kyungshang) **18** (2009), no. 2, 181–189.

[7] Jang, L.-C.; Kim, H.Y. *A note on degenerate poly-Genocchi numbers and polynomaisl,* Adv. Difference Equ. 2020, (2020:392).

[8] Khan, W. A.; Ali, R.; Alzobydi K. A. H.; Ahmed, N. *A New Family of Degenerate Poly-Genocchi Polynomials with Its Certain Properties,* J. Funct. Spaces. **2020** (20221).

[9] Kim, D. S.; Kim, T. *A note on polyexponential and unipoly functions,* Russ. J. Math. Phys. **26** 2019, no. 1, 40–49.

[10] Kim, D. S.; Kim, T. *A note on a new type of degenerate Bernoulli numbers,* Russ. J. Math. Phys. **27** 2020, no. 2, 227–235.

[11] Kim, T.; Kim, H. K. *Degenerate poly-Bell polynomials and numbers,* Adv. Difference Equ. 2021, (2021:361).

[12] Kim, T.; Kim, D. S.; Kim, H. Y.; Jang, L-C. *Degenerate poly-Bernoulli polynomials arising from degenerate polygarithm,* Adv. Difference Equ. 2020, (2020:444).

[13] Kim, T.; Kim, D. S.; Kwon, J.; Lee, H. *Degenerate polyexponential functions and type 2 degenerate poly-Bernoulli numbers and polynomials,* Adv. Difference Equ. 2020, (2020:168).

[14] Kim, T.; Jang. L. -C.; Kim, D. S.; Kim, H. -Y. *Some identities of type 2 degenerate Bernoulli polynomials of the second kind,* Symmetry **2020** (2020).

[15] Kim, T.; Kim, D. S. *Degenerate polyexponential functions and degenerate Bell polynomials,* J. Math. Anal. appl. **487** (2020), no. 2, 124017.

[16] Kim, T.; Kim, D. S. *Some identities of extended degenerate r-central Bell polynomials arising from umbral calculus,* Rev. R. Acad. Cienc. Exactas Fı́s. Nat. Ser. A Mat. RACSAM **114** (2020), no.1, Art., 19pp.

[17] Kim, T.; Kim, D. S. *A note on central Bell numbers and polynomials,* Russ. J. Math. Phys. **27** (2020), no. 1, 76–81.

[18] Kim, T.; Kim, D. S.;Kim, H. Y.,; Kwon, J. *Some results on degenerate Daehee and Bernoulli numbers and polynomials.* Adv. Difference Equ. 2020, (2020:311).

[19] Kim, T.; Kim, D. S.; Lee, H.; Kwon, J. *Degenerate binomial coefficients and degenerate hypergeometric functions.* Adv. Difference Equ. 2020, (2020:115).

[20] Kim, T.; Kim, D. S.; Jang, G.-W. *A note on degenerate Fubini polynomials,* Proc. Jangjeon Math. Soc. **20** (2017), no. 4, 521–531.

[21] Kurt, B.; Simsek. Y. *On the Hermite based Genocchi polynomials,* Adv. Stud. Contemp. Math. (Kyungshang) **23** (2013), no. 1, 13–17.

[22] Kwon, J.; Kim, T.; Kim, D. S.; Kim, H.-Y. *Some identities for degenerate complete and incomplete r-Bell polynomials,* J. Inequal. Appl. 2020, 2020:23, 9pp.
[23] Kwon, J.; Park. J.-W.; Yun. S. J. *A note on degenerate generalized q-Genocchi polynomials,* J. Comput. Anal. Appl. **22** (2017), no. 5, 923–928.
[24] Lim, D. *Some identities of degenerate Genocchi polynomials,* Bull. Korean Math. Soc. **53** (2016), no. 2, 569–579.
[25] Liu, C.; Wuyungaowa. *Application of probabilistic method on Daehee sequences,* Eur. J. Pure Appl. Math. **11** (2018), no. 1, 69–78.
[26] Lee, D. S.; Kin ,H.K. *On the new type of degenerate poly-Genocchi numbers and polynomials,* Adv. Differ. Equ. **2020** (2020), no 431.
[27] Pak. H.-k.; Jeong, J.; Kang, D.-J.; Rim, S. -H. *Changhee-Genocchi numbers and their applications,* Ars Combin. **136** (2018), 153–159.
[28] Rim. S. -H.; Lee, S.J. *On the q-Genocchi numbers and polynomials associated with q-zeta function,* Proc. Jangjeon Math. Soc. **12** (2009), no. 3, 261–267.
[29] Roman, S. *The umbral calculus,* Pure and Applied Mathematics 111, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1984.
[30] Simsek, Y.; Cangul. I. N.I.; Kurt, V.; Kim, D. S. *q-Genocchi numbers and polynomials associated with q-Genocchi type functions,* Adv. Difference Equ. **2008**, Art. ID 815750, 12pp.
[31] Wuyungaowa. *Identities on products of Genocchi numbers,* J. Inequal. Appl. **2013**, 2013:422, 10 pp.

Department of Mathematics, Kyungpook National University, Daegu, 41566, Republic of Korea
Email address: kimholzi@gmail.com

Department of Mathematics, Kyungpook National University, Daegu, 41566, Republic of Korea
Email address: piaoxf76@gmail.com

Department of Mathematics Education, Silla University, Busan, 46958, Republic of Korea
Email address: ssoopyo@gmail.com

Department of Mathematics Education, Gyeongsang National University, Jinju, 52828, Republic of Korea,
Email address: mathkjk26@gnu.ac.kr