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A NOTE ON DEGENERATE BERNOULLI POLYNOMIALS ARISING FROM UMBRAL CALCULUS

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Abstract. In [18, 26], Kim-Kim-Kim defined the λ-analogue of Stirling number of the first and the second kind. In this paper, we find the relationships between various special functions by expressing degenerate Bernoulli polynomials as linear combinations of some special polynomials by using umbral calculus.

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1. Introduction

For nonzero integers n and k, the *Stirling numbers of the first kind* $S_1(n,k)$ and the *Stirling numbers of the second kind* $S_2(n,k)$, respectively, are given by

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k, \text{ (see [10, 13, 33])}, \tag{1.1}$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \geq 1)$ are the falling factorial sequences.

The *Bernoulli polynomials* are defined by the generating function to be

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \text{ (see [3, 6, 7, 15])}. \tag{1.2}$$

When $x = 0$, $B_n = B_n(0)$ are called the *Bernoulli numbers*.

There are many numbers that are important in combinatorics such as Fibonacci number, Bernoulli number, Euler number, Bell number, and Stirling number etc., and its applications and extensions are being actively studied by many researchers. In particular, in [30], authors defined the k-Fibonacci sequences and various factorizations of the k-Fibonacci and k-symmetric Fibonacci matrices are obtained. Some inequalities involving the eigenvalues of the k-symmetric Fibonacci matrices and some combinatorial identities are also obtained. In [1], authors gave some combinatorial properties of a new generalization of hyper-Lucas numbers, and investigated norms of some circulant and s-circulant matrices. Chakraborty-Komastu introduced generalized hypergeoemtric Bernoulli numbers with Dirichlet characters, and derived some expressions of these numbers in [7]. In [39], authors introduced a type 2 poly-Frobenius-Genocchi polynomials by using the polyexponential function, and derived some new relations and properties including the Stirling numbers of the first and second kinds. Kim-Kim-Dolgy-Park investigated the Poisson random variables related to the Lah-Bell polynomials and the degenerate binomial.

In addition, authors found some connections between degenerate Poisson random variables and the degenerate Lah-Bell polynomials in [23].

For any nonzero real number λ, the *degenerate exponential function* is defined to be

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}, \ e_\lambda(t) = (1+\lambda t)^{\frac{1}{\lambda}}, \ \text{(see [6, 19])}. \tag{1.3}$$

The study of degenerate version of a functions was first initiated by L. Carlitz, and since then, degenerate versions of various special functions have been defined and their properties have been actively studied by many researchers. In [21], Kim-Kim introduced the degenerate r-Whitney numbers of the first kind, of the second kind, and derived some properties, recurrence relations, orthogonality relations and several identities on those numbers. Khan-Younis-Nadeem introduced partially degenerate Laguerre–Bernoulli polynomials of the first kind and derived some implicit summation formulas for those polynomials (see [12]). In [4], Aydin-Acikgoz-Araci found some properties and identities for degenerate Hurwitz-zeta, modifined degenerate Hurwitz-zeta and degenerate digamma functions. Komatsu-Young found a general convolution formula involving the generalized Stirling numbers of Hsu and Shiue and the degenerate Bernoulli polynomials in [29].

In viewpoint of (1.2) and (1.3), the *degenerate Bernoulli polynomials* are defined by the generating function to be

$$\frac{t}{e_\lambda(t)-1}e_\lambda^x(t) = \sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!}, \ \text{(see [6, 22])}.$$

In the special case $x=0$, $\beta_{n,\lambda}=\beta_{n,\lambda}(0)$ are called the *degenerate Bernoulli numbers*. Note that $\lim_{\lambda\to 0}\beta_{n,\lambda}(x) = B_n(x)$.

As degenerate version of the Stirling numbers of the first and second kind, the *degenerate Stirling numbers of the first kind* $S_{1,\lambda}(n,k)$ and the *degenerate Stirling numbers of the second kind* $S_{2,\lambda}(n,k)$ are respectively introduced by Kim-Kim (see [13, 25]) as follows:

$$\frac{1}{k!}\left(\log_\lambda(1+t)\right)^k = \sum_{n=k}^{\infty}S_{1,\lambda}(n,k)\frac{t^n}{n!} \text{ and } \frac{1}{k!}\left(e_\lambda(t)-1\right)^k = \sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!}. \tag{1.4}$$

The λ-*analogue of the Stirling numbers of the first kind* and *the second kind* are defined by

$$(x)_{n,\lambda} = \sum_{k=0}^{n}S_\lambda^{(1)}(n,k)x^k, \text{ and } x^n = \sum_{k=0}^{n}S_\lambda^{(2)}(n,k)(x)_{k,\lambda}, \ \text{(see [18, 26])}. \tag{1.5}$$

By (1.5), we see that

$$\frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k = \sum_{n=k}^{\infty}S_\lambda^{(1)}(n,k)\frac{t^n}{n!}, \text{ and } \frac{1}{k!}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^k = \sum_{n=k}^{\infty}S_\lambda^{(2)}(n,k)\frac{t^n}{n!}, \tag{1.6}$$

(see [18, 26]).

2. Review of umbral calculus

Let $\mathbb{C}$ be the complex numbers field,

$$\mathcal{F} = \left\{ f(t) = \sum_{n=0}^{\infty}a_k\frac{t^n}{n!} \,\middle|\, a_k\in\mathbb{C}\right\},$$

and let

$$\mathbb{P} = \mathbb{C}[x] = \left\{ \sum_{k=0}^{\infty} a_l x^k \,\middle|\, a_k \in \mathbb{C} \text{ with } a_k = 0 \text{ for all but finite number of } k \right\}.$$

Let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$.

Then linear functional $\langle f(t)|\cdot\rangle$ on $\mathbb{P}$ given by $f(t)$, is defined by

$$\langle f(t)|\, x^n\rangle = a_n, (n \geq 0), \text{ (see [11, 15, 35, 36]).} \tag{2.1}$$

From (2.1), we have

$$\left\langle t^k \middle|\, x^n \right\rangle = n!\delta_{n,k}, \ (n, k \geq 0), \tag{2.2}$$

where $\delta_{n,k}$ is Kronecker's symbol.

For each $\lambda \in \mathbb{R} - \{0\}$ and each $k \in \mathbb{N} \cup \{0\}$, the differential operator on $\mathbb{P}$ by

$$\left(t^k\right) x^n = \begin{cases} (n)_k x^{n-k}, & if\ k \leq n, \\ 0, & if\ k > n, \end{cases} \tag{2.3}$$

and for any $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathcal{F}$,

$$(f(t))\, x^n = \sum_{k=0}^{n} \binom{n}{k} a_k x^{n-k}, \text{ (see [11, 15, 35, 36]).} \tag{2.4}$$

In addition, they showed that for $f(t), g(t) \in \mathcal{F}$, and $p(x) \in \mathbb{P}$,

$$\langle f(t)g(t)|\, p(x)\rangle = \langle g(t)|\, (f(t))p(x)\rangle = \langle f(t)|\, (g(t))p(x)\rangle\,. \tag{2.5}$$

The *order* $o(f(t))$ of $f(t) \in \mathcal{F} - \{0\}$ is the smallest integer k for which the coefficient of t^k does not vanish. If $o(f(t)) = 0$, then $f(t)$ is called *invertible* and such series has a multiplicative inverse $\frac{1}{f(t)}$ of $f(t)$. If $o(f(t)) = 1$, then $f(t)$ is called *delta series* and it has a compositional inverse $\bar{f}(t)$ of $f(t)$ with $\bar{f}(f(t)) = f(\bar{f}(t)) = t$.

Let $f(t)$ be a delta series and let $g(t)$ be an invertible series. Then there exists a unique sequence $S_n(x)$, $(\deg S_n(x) = n)$ of polynomials satisfying the orthogonality conditions

$$\left\langle g(t)\big(f(t)\big)^k \,\middle|\, S_n(x)\right\rangle = n!\delta_{n,k}, \quad (n, k \geq 0), \text{ (see [11, 15, 35, 36]).} \tag{2.6}$$

Here $S_n(x)$ is called the *Sheffer sequence* for $(g(t), f(t))$, which is denoted by $S_n(x) \sim (g(t), f(t))$. The sequence $S_n(x)$ is the Sheffer sequence for $(g(t), f(t))$ if and only if

$$\frac{1}{g\left(\overline{f}(t)\right)} e^{y\overline{f}(t)} = \sum_{n=0}^{\infty} S_n(y) \frac{t^n}{n!}, \text{ (see [11, 15, 35, 36]),} \tag{2.7}$$

for all $y \in \mathbb{C}$, where $\overline{f}(t)$ is the compositional inverse of $f(t)$ such that $f(\overline{f}(t)) = \overline{f}(f(t)) = t$.

Lemma 2.1. *Let $S_n(x) \sim (g(t), f(t))_\lambda$ and let $h(x) = \sum_{l=0}^{n} a_l S_{l,\lambda}(x) \in \mathbb{P}$. Then*

$$a_k = \frac{1}{k!} \left\langle g(t)\,(f(t))^k \middle|\, h(x) \right\rangle.$$

Proof. Let $S_n(x) \sim (g(t), f(t))$ and let $h(x) = \sum_{l=0}^{n} a_l S_l(x)$.

$$\begin{aligned}\left\langle g(t)\,(f(t))^k \middle| h(x)\right\rangle =& \sum_{l=0}^{n} a_l \left\langle g(t)\,(f(t))^k \middle| S_l(x)\right\rangle \\ =& k!a_k,\end{aligned}$$

and thus our proof is completed. □

Theorem 2.2. *(*[35, 36, 15]*) Let $S_n \sim (g(t), f(t))$, $r_n \sim (h(t), l(t))$. Then we have*

$$S_n = \sum_{k=0}^{n} c_{n,k} r_k,$$

where

$$c_{n,k} = \frac{1}{k!}\left\langle \frac{h\left(\bar{f}(t)\right)}{g\left(\bar{f}(t)\right)}\left(l\left(\bar{f}(t)\right)\right)^k \middle| x^n \right\rangle.$$

In 1900's, umbral calculus consisted of symbolic techniques for sequence manipulation and its mathematical rigor was excluded. In the 1970s, G. C. Rota built a completely rigid foundation for theories based on modern ideas of linear functions, linear operators, and adjacency functions (see [6, 15, 32, 36]), and umbral calculus has been applied in many fields such as combinatorial counting, graph theory with chromatic polynomials, probability theory, statistics, topology, physics, etc (see [11, 14, 15, 24, 27, 28, 32, 35, 36]).

In this paper, we find some new and interesting identities related to the degenerate Bernoulli polynomials and some special polynomials by finding the coefficients when we express bernoulli polynomials as a linear combinations of other special polynomials with umbral calculus.

3. Main results

By the definition of the Bernoulli polynomials and degenerate Bernoulli polynomials, we see that

$$B_n(x) \sim \left(\frac{e^t - 1}{t}, t\right), \text{ and } \beta_{n,\lambda}(x) \sim \left(\frac{\lambda(e^t - 1)}{e^{\lambda t} - 1}, \frac{e^{\lambda t} - 1}{\lambda}\right). \tag{3.1}$$

Note that

$$\begin{aligned}\frac{t}{e_\lambda(t) - 1} e_\lambda^x(t) =& \left(\sum_{n=0}^{\infty} \beta_{n,\lambda} \frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}\right) \\ =& \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} \binom{n}{m} \beta_{n-m,\lambda}(x)_{n,\lambda}\right)\frac{t^n}{n!},\end{aligned}$$

and so we see that

$$\beta_{n,\lambda}(x) = \sum_{m=0}^{n} \binom{n}{m} \beta_{n-m,\lambda}(x)_{m,\lambda} = \sum_{m=0}^{n}\sum_{k=0}^{m} \binom{n}{m} S_\lambda^{(1)}(m,k) \beta_{n-m,\lambda} x^k. \tag{3.2}$$

Let $\beta_{n,\lambda}(x)=\sum_{l=0}^{n}a_{n,l}B_l(x)$. Then, by Theorem 2.2 and (3.1), we get

$$\begin{aligned}
a_{n,l}=&\frac{1}{l!}\left\langle \frac{\frac{e_\lambda(t)-1}{\frac{1}{\lambda}\log(1+\lambda t)}}{\frac{e_\lambda(t)-1}{t}}\left(\frac{1}{\lambda}\log(1+\lambda t)\right)^l \middle| x^n\right\rangle\\
=&\left\langle \frac{\lambda t}{\log(1+\lambda t)}\middle| \left(\frac{1}{l!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^l\right)x^n\right\rangle\\
=&\sum_{k=l}^{n}\binom{n}{k}S_\lambda^{(1)}(k,l)\left\langle \frac{\lambda t}{\log(1+\lambda t)}\middle| x^{n-k}\right\rangle\\
=&\sum_{k=l}^{n}\binom{n}{k}S_\lambda^{(1)}(k,l)\lambda^{n-k}b_{n-k},
\end{aligned}\tag{3.3}$$

where b_n are the Bernoulli numbers of the second kind which are defined by the generating function to be

$$\frac{t}{\log(1+t)}(1+t)^x=\sum_{n=0}^{\infty}b_n\frac{t^n}{n!},\ \text{(see [24])}.$$

In addition, by (1.5), Lemma 2.1 and (3.2), we get

$$\begin{aligned}
a_{n,l}=&\frac{1}{l!}\left\langle \frac{e^t-1}{t}t^l\middle| \beta_{n,\lambda}(x)\right\rangle\\
=&\sum_{k=0}^{n}\sum_{a=0}^{k}\binom{n}{k}\beta_{n-k,\lambda}S_\lambda^{(1)}(k,a)\frac{1}{l!}\left\langle \frac{e^t-1}{t}t^l\middle| x^a\right\rangle\\
=&\sum_{k=0}^{n}\sum_{a=0}^{k}\binom{n}{k}\binom{a}{l}\beta_{n-k,\lambda}S_\lambda^{(1)}(k,a)\left\langle \frac{e^t-1}{t}\middle| x^{a-l}\right\rangle\\
=&\sum_{k=0}^{n}\sum_{a=0}^{k}\binom{n}{k}\binom{a}{l}\frac{\beta_{n-k,\lambda}S_\lambda^{(1)}(k,a)}{a-l+1}\left\langle e^t-1\middle| x^{a-l+1}\right\rangle\\
=&\sum_{k=0}^{n}\sum_{a=l}^{k}\binom{n}{k}\binom{a}{l}\frac{\beta_{n-k,\lambda}S_\lambda^{(1)}(k,a)}{a-l+1}.
\end{aligned}\tag{3.4}$$

Conversely, assume that $B_n(x)=\sum_{l=0}^{n}b_{n,l}\beta_{l,\lambda}(x)$. Then, by (1.6), we get

$$\begin{aligned}
b_{n,l}=&\frac{1}{l!}\left\langle \frac{\frac{\lambda(e^t-1)}{e^{\lambda t}-1}}{\frac{e^t-1}{t}}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^l\middle| x^n\right\rangle\\
=&\left\langle \frac{\lambda t}{e^{\lambda t}-1}\middle| \left(\frac{1}{l!}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^l\right)x^n\right\rangle\\
=&\sum_{k=l}^{n}\binom{n}{k}S_\lambda^{(2)}(k,l)\left\langle \frac{\lambda t}{e^{\lambda t}-1}\middle| x^{n-k}\right\rangle\\
=&\sum_{k=l}^{n}\binom{n}{k}S_\lambda^{(2)}(k,l)\lambda^{n-k}B_{n-k}.
\end{aligned}\tag{3.5}$$

By (3.3), (3.4) and (3.5), we obtain the following theorem.

Theorem 3.1. *For each $n \in \mathbb{N} \cup \{0\}$, we have*

$$\begin{aligned}\beta_{n,\lambda}(x) &= \sum_{l=0}^{n}\left(\sum_{k=l}^{n}\binom{n}{k}S_{\lambda}^{(1)}(k,l)\lambda^{n-k}b_{n-k}\right)B_l(x)\\ &= \sum_{l=0}^{n}\left(\sum_{k=0}^{n}\sum_{a=l}^{k}\binom{n}{k}\binom{a}{l}\frac{\beta_{n-k,\lambda}S_{\lambda}^{(1)}(k,a)}{a-l+1}\right)B_l(x),\end{aligned}$$

and

$$B_n(x) = \sum_{l=0}^{n}\left(\sum_{k=l}^{n}\binom{n}{k}S_{\lambda}^{(2)}(k,l)\lambda^{n-k}B_{n-k}\right)\beta_{l,\lambda}(x).$$

The *Euler polynomials* are defined by the generating function to be

$$\frac{2}{e^t+1}e^{xt} = \sum_{l=0}^{n}E_n(x)\frac{t^n}{n!}, \text{ (see [2, 15, 37]).}$$

In the special case $x = 0$, $E_n = E_n(0)$ are called the *Euler numbers*. By the definition of the Euler polynomials, the Sheffer sequences of the Euler polynomials are

$$E_n(x) \sim \left(\frac{e^t+1}{2}, t\right).$$

Let $\beta_{n,\lambda}(x) = \sum_{l=0}^{n} a_{n,l}E_l(x)$. Since

$$\frac{e_\lambda(t)+1}{2} = 1 + \frac{1}{2}\sum_{a=1}^{\infty}(1)_{a,\lambda}\frac{t^a}{a!}, \tag{3.6}$$

by Theorem 2.2 and (3.6), we get

$$\begin{aligned}a_{n,l} =& \frac{1}{l!}\left\langle \frac{\frac{e_\lambda(t)+1}{2}}{\frac{e_\lambda(t)-1}{t}}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^l \middle| x^n\right\rangle\\ =& \left\langle \frac{e_\lambda(t)+1}{2}\frac{t}{e_\lambda(t)-1}\middle| \left(\frac{1}{l!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^l\right)x^n\right\rangle\\ =& \sum_{k=l}^{n}\binom{n}{k}S_{\lambda}^{(1)}(k,l)\left\langle \frac{e_\lambda(t)+1}{2}\middle| \left(\frac{t}{e_\lambda(t)-1}\right)x^{n-k}\right\rangle\\ =& \sum_{k=l}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r}S_{\lambda}^{(1)}(k,l)\beta_{r,\lambda}\left\langle \frac{e_\lambda(t)+1}{2}\middle| x^{n-k-r}\right\rangle\\ =& \sum_{k=l}^{n}\binom{n}{k}S_{\lambda}^{(1)}(k,l)\beta_{n-k,\lambda} + \frac{1}{2}\sum_{k=l}^{n}\sum_{r=0}^{n-k-1}\binom{n}{k}\binom{n-k}{r}S_{\lambda}^{(1)}(k,l)\beta_{r,\lambda}(1)_{n-k-r,\lambda}.\end{aligned} \tag{3.7}$$

Conversely, assume that $E_n(x)=\sum_{l=0}^{n} b_{n,l}\beta_{l,\lambda}(x)$. Then

$$\begin{aligned}
b_{n,l} =&\frac{1}{l!}\left\langle \frac{\frac{\lambda(e^t-1)}{e^{\lambda t}-1}}{\frac{e^t+1}{2}}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^l \middle| x^n\right\rangle\\
=&\left\langle \frac{2}{e_\lambda(t)+1}\frac{\lambda t}{e^{\lambda t}-1}\frac{e^t-1}{t}\middle| \left(\frac{1}{l!}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^l\right)x^n\right\rangle\\
=&\sum_{k=l}^{n}\binom{n}{k}S_\lambda^{(2)}(k,l)\left\langle \frac{2}{e_\lambda(t)+1}\frac{\lambda t}{e^{\lambda t}-1}\frac{e^t-1}{t}\middle| x^{n-k}\right\rangle\\
=&\sum_{k=l}^{n}\sum_{a=0}^{n-k}\binom{n}{k}\binom{n-k}{a}S_\lambda^{(2)}(k,l)E_{a,\lambda}\left\langle \frac{\lambda t}{e^{\lambda t}-1}\frac{e^t-1}{t}\middle| x^{n-k-a}\right\rangle\\
=&\sum_{k=l}^{n}\sum_{a=0}^{n-k}\sum_{b=0}^{n-k-a}\binom{n}{k}\binom{n-k}{a}\binom{n-k-a}{b}S_\lambda^{(2)}(k,l)E_{a,\lambda}\lambda^b B_b\left\langle \frac{e^t-1}{t}\middle| x^{n-k-a-b}\right\rangle\\
=&\sum_{k=l}^{n}\sum_{a=0}^{n-k}\sum_{b=0}^{n-k-a}\binom{n}{k}\binom{n-k}{a}\binom{n-k-a}{b}\frac{S_\lambda^{(2)}(k,l)E_{a,\lambda}\lambda^b B_b}{n-k-a-b+1},
\end{aligned}\tag{3.8}$$

where $E_{n,\lambda}^{(r)}$ are the *degenerate Euler numbers of order* r which are defined by the generating function to be

$$\left(\frac{2}{e_\lambda(t)+1}\right)^r=\sum_{n=0}^{\infty}E_{n,\lambda}^{(r)}\frac{t^n}{n!},\ \text{(see [2, 5])}.$$

In the special case $r=1$, $E_{n,\lambda}=E_{n,\lambda}^{(1)}$ are called *degenerate Euler numbers.*

By (3.7) and (3.8), we obtain the following theorem.

Theorem 3.2. *For each $n\in\mathbb{N}\cup\{0\}$, we have*

$$\beta_{n,\lambda}(x)=\sum_{l=0}^{n}\left(\sum_{k=l}^{n}\binom{n}{k}S_\lambda^{(1)}(k,l)\beta_{n-k,\lambda}+\frac{1}{2}\sum_{k=l}^{n}\sum_{r=0}^{n-k-1}\binom{n}{k}\binom{n-k}{r}S_\lambda^{(1)}(k,l)\beta_{r,\lambda}(1)_{n-k-r,\lambda}\right)E_l(x),$$

and

$$E_n(x)=\sum_{l=0}^{n}\left(\sum_{k=l}^{n}\sum_{a=0}^{n-k}\sum_{b=0}^{n-k-a}\binom{n}{k}\binom{n-k}{a}\binom{n-k-a}{b}\frac{S_\lambda^{(2)}(k,l)E_{a,\lambda}\lambda^b B_b}{n-k-a-b+1}\right)\beta_{l,\lambda}(x).$$

The *Daehee polynomials* are defined by the generating function to be

$$\frac{\log(1+t)}{t}(1+t)^x=\sum_{n=0}^{\infty}D_n(x)\frac{t^n}{n!},\ \text{(see [8, 27, 31])}.$$

When $x=0$, $D_n=D_n(0)$ are called the *Daehee numbers.* By (1.1) and the definition of the Daehee polynomials, we see that

$$D_n(x)=\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}D_{n-m}S_1(m,k)x^k \tag{3.9}$$

and the Sheffer sequences of those polynomials are

$$D_n(x) \sim \left(\frac{e^t-1}{t}, e^t-1\right). \tag{3.10}$$

Let $\beta_{n,\lambda}(x) = \sum_{l=0}^{n} a_{n,l} D_l(x)$. Then, by Theorem 2.2 and (3.10), we get

$$\begin{aligned} a_{n,l} =& \frac{1}{l!}\left\langle \frac{\frac{e_\lambda(t)-1}{\frac{1}{\lambda}\log(1+\lambda t)}}{\frac{e_\lambda(t)-1}{t}} \left(e_\lambda(t)-1\right)^l \middle| x^n \right\rangle \\ =& \sum_{k=l}^{n}\binom{n}{k} S_{2,\lambda}(k,l) \left\langle \frac{\lambda t}{\log(1+\lambda t)} \middle| x^{n-k}\right\rangle \\ =& \sum_{k=l}^{n}\binom{n}{k} S_{2,\lambda}(k,l)\lambda^{n-k} b_{n-k}. \end{aligned} \tag{3.11}$$

Conversely, assume that $D_n(x) = \sum_{l=0}^{n} b_{n,l}\beta_{l,\lambda}(x)$. Then, by (1.6), Lemma 2.1 and (3.9), we get

$$\begin{aligned} b_{n,l} =& \frac{1}{l!}\left\langle \frac{\lambda(e^t-1)}{e^{\lambda t}-1}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^l \middle| D_n(x)\right\rangle \\ =& \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m} D_{n-m} S_1(m,k)\frac{1}{l!}\left\langle \frac{\lambda(e^t-1)}{e^{\lambda t}-1}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^l \middle| x^k\right\rangle \\ =& \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m} D_{n-m} S_1(m,k)\left\langle \frac{\lambda t}{e^{\lambda t}-1}\frac{e^t-1}{t} \middle| \left(\frac{1}{l!}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^l\right) x^k\right\rangle \\ =& \sum_{m=0}^{n}\sum_{k=l}^{m}\sum_{a=l}^{k}\binom{n}{m}\binom{k}{a} S_1(m,k) S_\lambda^{(2)}(a,l)\left\langle \frac{\lambda t}{e^{\lambda t}-1}\frac{e^t-1}{t} \middle| x^{k-a}\right\rangle \\ =& \sum_{m=0}^{n}\sum_{k=l}^{m}\sum_{a=l}^{k}\sum_{b=0}^{k-a}\binom{n}{m}\binom{k}{a}\binom{k-a}{b} S_1(m,k) S_\lambda^{(2)}(a,l)\lambda^{k-a}B_{k-a}\left\langle \frac{e^t-1}{t} \middle| x^{k-a-b}\right\rangle \\ =& \sum_{m=0}^{n}\sum_{k=l}^{m}\sum_{a=l}^{k}\sum_{b=0}^{k-a}\binom{n}{m}\binom{k}{a}\binom{k-a}{b} \frac{S_1(m,k) S_\lambda^{(2)}(a,l)\lambda^{k-a}B_{k-a}}{k-a-b+1}. \end{aligned} \tag{3.12}$$

By (3.11) and (3.12), we obtain the following theorem.

Theorem 3.3. *For each $n \in \mathbb{N}\cup\{0\}$, we have*

$$\beta_{n,\lambda}(x) = \sum_{l=0}^{n}\left(\sum_{k=l}^{n}\binom{n}{k} S_{2,\lambda}(k,l)\lambda^{n-k} b_{n-k}\right) D_l(x),$$

and

$$D_n(x) = \sum_{l=0}^{n}\left(\sum_{m=0}^{n}\sum_{k=l}^{m}\sum_{a=l}^{k}\sum_{b=0}^{k-a}\binom{n}{m}\binom{k}{a}\binom{k-a}{b} \frac{S_1(m,k) S_\lambda^{(2)}(a,l)\lambda^{k-a}B_{k-a}}{k-a-b+1}\right)\beta_{l,\lambda}(x).$$

The *Changhee polynomials* are defined by the generating function to be

$$\frac{2}{2+t}(1+t)^x = \sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!}, \text{ (see [8, 34]).}$$

In the special case $x = 0$, $Ch_n = Ch_n(0)$ are called the *Changhee numbers*. By (1.1) and the definition of the Changhee polynomials, the Sheffer sequences of the Changhee polynomials are

$$Ch_n(x) \sim \left(\frac{e^t+1}{2}, e^t - 1\right),$$

and

$$Ch_n(x) = \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}Ch_{n-m}S_1(m,k)x^k. \tag{3.13}$$

Let $\beta_{n,\lambda}(x) = \sum_{l=0}^{n} a_{n,l}Ch_l(x)$. Since

$$\frac{e_\lambda(t)+1}{2} = 1 + \frac{1}{2}\sum_{n=1}^{\infty}(1)_{n,\lambda}\frac{t^n}{n!}, \tag{3.14}$$

by (1.1) , Theorem 2.2 and (3.14), we get

$$\begin{aligned}
a_{n,l} =& \frac{1}{l!}\left\langle \frac{\frac{e_\lambda(t)+1}{2}}{\frac{e_\lambda(t)-1}{t}}\left(e_\lambda(t)-1\right)^l \middle| x^n\right\rangle \\
=& \sum_{m=l}^{n}\binom{n}{m}S_2(m,l)\left\langle \frac{t}{e_\lambda(t)-1}\frac{e_\lambda(t)+1}{2}\middle| x^{n-m}\right\rangle \\
=& \sum_{m=l}^{n}\sum_{r=0}^{n-m}\binom{n}{m}\binom{n-m}{r}S_2(m,l)\beta_{r,\lambda}\left\langle \frac{e_\lambda(t)+1}{2}\middle| x^{n-m-r}\right\rangle \\
=& \sum_{m=l}^{n}\binom{n}{m}S_2(m,l)\beta_{n-m,\lambda} + \frac{1}{2}\sum_{m=l}^{n}\sum_{r=0}^{n-m-1}\binom{n}{m}\binom{n-m}{r}S_2(m,l)\beta_{r,\lambda}(1)_{n-m-r,\lambda}.
\end{aligned} \tag{3.15}$$

Conversely, assume that $Ch_n(x) = \sum_{l=0}^{n} b_{n,l}\beta_{l,\lambda}$. Then, by (1.6) and (3.13), we get

$$\begin{aligned}
b_{n,l} =& \frac{1}{l!}\left\langle \frac{\lambda(e^t-1)}{e^{\lambda t}-1}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^l \middle| Ch_n(x)\right\rangle \\
=& \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}Ch_{n-m}S_1(m,k)\left\langle \frac{\lambda t}{e^{\lambda t}-1}\frac{e^t-1}{t}\middle| \left(\frac{1}{l!}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^l\right)x^k\right\rangle \\
=& \sum_{m=0}^{n}\sum_{k=l}^{m}\sum_{a=l}^{k}\binom{n}{m}\binom{k}{a}Ch_{n-m}S_1(m,k)S_\lambda^{(2)}(a,l)\left\langle \frac{\lambda t}{e^{\lambda t}-1}\frac{e^t-1}{t}\middle| x^{k-a}\right\rangle \\
=& \sum_{m=0}^{n}\sum_{k=l}^{m}\sum_{a=l}^{k}\sum_{b=0}^{k-a}\binom{n}{m}\binom{k}{a}\binom{k-a}{b}Ch_{n-m}S_1(m,k)S_\lambda^{(2)}(a,l)\lambda^b B_b\left\langle \frac{e^t-1}{t}\middle| x^{k-a-b}\right\rangle \\
=& \sum_{m=0}^{n}\sum_{k=l}^{m}\sum_{a=l}^{k}\sum_{b=0}^{k-a}\binom{n}{m}\binom{k}{a}\binom{k-a}{b}\frac{Ch_{n-m}S_1(m,k)S_\lambda^{(2)}(a,l)\lambda^b B_b}{k-a-b+1}.
\end{aligned} \tag{3.16}$$

By (3.15) and (3.16), we obtain the following theorem.

Theorem 3.4. *For each $n \in \mathbb{N}\{0\}$, we have*

$$\beta_{n,\lambda}(x) = \sum_{l=0}^{n} \left(\sum_{m=l}^{n} \binom{n}{m} S_2(m,l)\beta_{n-m,\lambda} + \frac{1}{2}\sum_{m=l}^{n}\sum_{r=0}^{n-m-1}\binom{n}{m}\binom{n-m}{r} S_2(m,l)\beta_{r,\lambda}(1)_{n-m-r,\lambda} \right) Ch_l(x),$$

and

$$Ch_n(x) = \sum_{l=0}^{n} \left(\sum_{m=0}^{n}\sum_{k=l}^{m}\sum_{a=l}^{k}\sum_{b=0}^{k-a} \binom{n}{m}\binom{k}{a}\binom{k-a}{b} \frac{Ch_{n-m}S_1(m,k)S_\lambda^{(2)}(a,l)\lambda^b B_b}{k-a-b+1} \right) \beta_{l,\lambda}(x).$$

The *Bell polynomials* are defined by the generating function to be

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} Bel_n(x)\frac{t^n}{n!}, \text{ (see [14, 19]).}$$

When $x = 1$, $Bel_n = Bel_n(1)$ are called the *Bell numbers*. Note that

$$\begin{aligned} \frac{1}{l!}\left(\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)\right)^l &= \sum_{m=l}^{\infty} S_1(m,l)\frac{1}{m!}\left(\frac{1}{\lambda}\log(1+\lambda t)\right)^m \\ &= \sum_{n=l}^{\infty}\sum_{m=l}^{n} S_1(m,l)S_\lambda^{(1)}(n,m)\frac{t^n}{n!}, \end{aligned} \tag{3.17}$$

and by the definition of the Sheffer sequences of the Bell polynomials are

$$Bel_n(x) \sim (1, \log(1+t)). \tag{3.18}$$

Let $\beta_{n,\lambda}(x) = \sum_{l=0}^{n} a_{n,l} Bel_l(x)$. Then by Theorem 2.2, (3.17) and (3.18), we get

$$\begin{aligned} a_{n,l} &= \frac{1}{l!}\left\langle \frac{1}{\frac{e_\lambda(t)-1}{t}}\left(\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)\right)^l \middle| x^n \right\rangle \\ &= \left\langle \frac{t}{e_\lambda(t)-1} \middle| \left(\frac{1}{l!}\left(\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)\right)^l\right) x^n \right\rangle \\ &= \sum_{r=l}^{n}\sum_{m=l}^{r}\binom{n}{r} S_1(m,l)S_\lambda^{(1)}(r,m)\left\langle \frac{t}{e_\lambda(t)-1} \middle| x^{n-r}\right\rangle \\ &= \sum_{r=l}^{n}\sum_{m=l}^{r}\binom{n}{r} S_1(m,l)S_\lambda^{(1)}(r,m)\beta_{n-r}. \end{aligned} \tag{3.19}$$

Conversely, assume that $Bel_n(x) = \sum_{l=0}^{n} a_{n,l}\beta_{l,\lambda}(x)$. Since

$$\begin{aligned} \sum_{n=0}^{\infty} Bel_n(x)\frac{t^n}{n!} &= e^{x(e^t-1)} \\ &= \sum_{m=0}^{\infty} x^m \frac{1}{m!}(e^t-1)^m \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} S_2(n,m)x^m\frac{t^n}{n!}, \end{aligned}$$

we see that

$$Bel_n(x) = \sum_{m=0}^{n} S_2(n,m)x^m. \tag{3.20}$$

By (3.20), we get

$$\begin{aligned} b_{n,l} =& \frac{1}{l!}\left\langle \frac{\lambda(e^t-1)}{e^{\lambda t}-1}\left(\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right)^l \middle| Bel_n(x)\right\rangle \\ =& \sum_{m=0}^{n} S_2(n,m)\left\langle \frac{\lambda t}{e^{\lambda t}-1}\frac{e^t-1}{t}\middle| \left(\frac{1}{l!}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^l\right)x^m\right\rangle \\ =& \sum_{m=0}^{n}\sum_{a=l}^{m}\binom{m}{a}S_2(n,m)S_\lambda^{(2)}(a,l)\left\langle \frac{\lambda t}{e^{\lambda t}-1}\frac{e^t-1}{t}\middle| x^{m-a}\right\rangle \\ =& \sum_{m=0}^{n}\sum_{a=l}^{m}\sum_{b=0}^{m-a}\binom{m}{a}\binom{m-a}{b}S_2(n,m)S_\lambda^{(2)}(a,l)\lambda^b B_b\left\langle \frac{e^t-1}{t}\middle| x^{m-a-b}\right\rangle \\ =& \sum_{m=0}^{n}\sum_{a=l}^{m}\sum_{b=0}^{m-a}\binom{m}{a}\binom{m-a}{b}\frac{S_2(n,m)S_\lambda^{(2)}(a,l)\lambda^b B_b}{m-a-b+1}. \end{aligned} \tag{3.21}$$

By (3.19) and (3.21), we obtain the following theorem.

Theorem 3.5. *For each $n \in \mathbb{N}\cup\{0\}$, we have*

$$\beta_{n,\lambda}(x) = \sum_{l=0}^{n}\left(\sum_{r=l}^{n}\sum_{m=l}^{r}\binom{n}{r}S_1(m,l)S_\lambda^{(1)}(r,m)\beta_{n-r}\right)Bel_l(x),$$

and

$$Bel_n(x) = \sum_{l=0}^{n}\left(\sum_{m=0}^{n}\sum_{a=l}^{m}\sum_{b=0}^{m-a}\binom{m}{a}\binom{m-a}{b}\frac{S_2(n,m)S_\lambda^{(2)}(a,l)\lambda^b B_b}{m-a-b+1}\right)\beta_{l,\lambda}(x).$$

The *unsigned Lah number* $L(n,k)$ has the explicit formula

$$L(n,k) = \binom{n-1}{k-1}\frac{n!}{k!} \text{ and } \frac{1}{k!}\left(\frac{t}{1-t}\right)^k = \sum_{n=k}^{\infty} L(n,k)\frac{t^n}{n!}, \text{ (see [14, 17, 38])}. \tag{3.22}$$

The *Lah-Bell polynomials* are defined by the generating function to be

$$e^{\frac{xt}{1-t}} = \sum_{n=0}^{\infty} B_n^L(x)\frac{t^n}{n!} \text{ (see [14, 17, 23])}.$$

In the special case $x = 1$, $B_n^L = B_n^L(1)$ are called the *Lah-Bell numbers*. Note that

$$\begin{aligned} e^{x\frac{t}{1-t}} =& \sum_{n=0}^{\infty} x^n\frac{1}{n!}\left(\frac{t}{1-t}\right)^n \\ =& \sum_{m=0}^{\infty}\sum_{n=0}^{m} x^n L(m,n)\frac{t^m}{m!}, \end{aligned}$$

and so

$$B_n^L(x) = \sum_{m=0}^{n} L(n,m)x^m. \tag{3.23}$$

Let $\beta_{n,\lambda}(x)=\sum_{l=0}^{n}a_{n,l}B_l^L(x)$. Since

$$\begin{aligned}\left(\frac{t}{1+t}\right)^l =&t^l(1+t)^{-l}\\ =&\sum_{r=0}^{\infty}(-1)^r<l>_r\frac{t^{r+l}}{r!},\end{aligned}\tag{3.24}$$

and

$$B_n^L(x)\sim\left(1,\frac{t}{1+t}\right),$$

by Theorem 2.2, (3.2), (3.23) and (3.24), we get

$$\begin{aligned}a_{n,l}=&\frac{1}{l!}\left\langle\left(\frac{t}{1+t}\right)^l\middle|\beta_{n,\lambda}(x)\right\rangle\\ =&\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}S_\lambda^{(1)}(m,k)\beta_{n-m,\lambda}\frac{1}{l!}\left\langle\left(\frac{t}{1+t}\right)^l\middle|x^k\right\rangle\\ =&\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}\binom{k}{l}(-1)^{k-l}<1>_{k-l}S_\lambda^{(1)}(m,k)\beta_{n-m,\lambda}.\end{aligned}\tag{3.25}$$

Conversely, assume that $B_n^L(x)=\sum_{l=0}^{n}b_{n,l}\beta_{l,\lambda}(x)$. By (1.6) and (3.23), we get

$$\begin{aligned}b_{n,l}=&\frac{1}{l!}\left\langle\frac{\lambda(e^t-1)}{e^{\lambda t}-1}\left(\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right)^l\middle|B_n^L(x)\right\rangle\\ =&\sum_{m=0}^{n}L(n,m)\left\langle\frac{\lambda t}{e^{\lambda t}-1}\frac{e^t-1}{t}\middle|\left(\frac{1}{l!}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^l\right)x^m\right\rangle\\ =&\sum_{m=0}^{n}\sum_{a=l}^{m}\binom{m}{a}L(n,m)S_\lambda^{(2)}(a,l)\left\langle\frac{\lambda t}{e^{\lambda t}-1}\frac{e^t-1}{t}\middle|x^{m-a}\right\rangle\\ =&\sum_{m=0}^{n}\sum_{a=l}^{m}\sum_{b=0}^{m-a}\binom{m}{a}\binom{m-a}{b}L(n,m)S_\lambda^{(2)}(a,l)\lambda^bB_b\left\langle\frac{e^t-1}{t}\middle|x^{m-a-b}\right\rangle\\ =&\sum_{m=0}^{n}\sum_{a=l}^{m}\sum_{b=0}^{m-a}\binom{m}{a}\binom{m-a}{b}\frac{L(n,m)S_\lambda^{(2)}(a,l)\lambda^bB_b}{m-a-b+1}.\end{aligned}\tag{3.26}$$

By the (3.25) and (3.26), we obtain the following theorem.

Theorem 3.6. *For each $n\in\mathbb{N}\cup\{0\}$, we have*

$$\beta_{n,\lambda}(x)=\sum_{l=0}^{n}\left(\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}\binom{k}{l}(-1)^{k-l}<1>_{k-l}S_\lambda^{(1)}(m,k)\beta_{n-m,\lambda}\right)B_l^L(x),$$

and

$$B_n^L(x)=\sum_{l=0}^{n}\left(\sum_{m=0}^{n}\sum_{a=l}^{m}\sum_{b=0}^{m-a}\binom{m}{a}\binom{m-a}{b}\frac{L(n,m)S_\lambda^{(2)}(a,l)\lambda^bB_b}{m-a-b+1}\right)\beta_{l,\lambda}(x).$$

The *Frobenius-Euler polynomials of order* r are defined by the generating function to be

$$\left(\frac{1-u}{e^t-u}\right)^r e^{xt}=\sum_{n=0}^{\infty}H_n^{(r)}(x|u)\frac{t^n}{n!},\ \text{(see [9, 16])}.$$

In the special case $x=0$, $H_n^{(r)}(u)=H_n^{(r)}(0|u)$ are called the *Frobenius-Euler numbers of order* r. Note that

$$H_n^{(r)}(x|u)\sim\left(\left(\frac{e^t-u}{1-u}\right)^r,t\right),$$

and

$$(e_\lambda(t)-u)^r=\sum_{b=0}^{\infty}\sum_{a=0}^{r}\binom{r}{a}(-u)^{r-a}(a)_{b,\lambda}\frac{t^b}{b!}. \tag{3.27}$$

Let $\beta_{n,\lambda}(x)=\sum_{l=0}^{n}a_{n,l}H_l^{(r)}(x|u)$. By Theorem 2.2 and (3.27), we get

$$\begin{aligned}a_{n,l}=&\frac{1}{l!}\left\langle\frac{\left(\frac{e_\lambda(t)-u}{1-u}\right)^r}{\frac{e_\lambda(t)-1}{t}}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^l\middle|x^n\right\rangle\\=&\sum_{m=l}^{n}\binom{n}{m}S_\lambda^{(1)}(m,l)\left\langle\frac{t}{e_\lambda(t)-1}\left(\frac{e_\lambda(t)-u}{1-u}\right)^r\middle|x^{n-m}\right\rangle\\=&\sum_{m=l}^{n}\sum_{a=0}^{n-m}\binom{n}{m}\binom{n-m}{a}\frac{S_\lambda^{(1)}(m,l)\beta_{a,\lambda}}{(1-u)^r}\left\langle(e_\lambda(t)-u)^r\middle|x^{n-m-a}\right\rangle\\=&\sum_{m=l}^{n}\sum_{a=0}^{n-m}\sum_{b=0}^{r}\binom{n}{m}\binom{n-m}{a}\binom{r}{b}\frac{S_\lambda^{(1)}(m,l)\beta_{a,\lambda}(-u)^{r-b}(b)_{n-m-a,\lambda}}{(1-u)^r}.\end{aligned} \tag{3.28}$$

Conversely, assume that $H_n^{(r)}(x|u)=\sum_{l=0}^{n}b_{n,l}\beta_{l,\lambda}(x)$. Then

$$\begin{aligned}b_{n,l}=&\frac{1}{l!}\left\langle\frac{\frac{\lambda(e^t-1)}{e^{\lambda t}-1}}{\left(\frac{e^t-u}{1-u}\right)^r}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^l\middle|x^n\right\rangle\\=&\left\langle\left(\frac{1-u}{e^t-u}\right)^r\frac{\lambda t}{e^{\lambda t}-1}\frac{e^t-1}{t}\middle|\left(\frac{1}{l!}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^l\right)x^n\right\rangle\\=&\sum_{m=l}^{n}\binom{n}{m}S_\lambda^{(2)}(m,l)\left\langle\left(\frac{1-u}{e^t-u}\right)^r\frac{\lambda t}{e^{\lambda t}-1}\frac{e^t-1}{t}\middle|x^{n-m}\right\rangle\\=&\sum_{m=l}^{n}\sum_{a=0}^{n-m}\binom{n}{m}\binom{n-m}{a}S_\lambda^{(2)}(m,l)H_a^{(r)}(u)\left\langle\frac{\lambda t}{e^{\lambda t}-1}\frac{e^t-1}{t}\middle|x^{n-m-a}\right\rangle\\=&\sum_{m=l}^{n}\sum_{a=0}^{n-m}\sum_{b=0}^{n-m-a}\binom{n}{m}\binom{n-m}{a}\binom{n-m-a}{b}S_\lambda^{(2)}(m,l)H_a^{(r)}(u)\lambda^bB_b\left\langle\frac{e^t-1}{t}\middle|x^{n-m-a-b}\right\rangle\\=&\sum_{m=l}^{n}\sum_{a=0}^{n-m}\sum_{b=0}^{n-m-a}\binom{n}{m}\binom{n-m}{a}\binom{n-m-a}{b}\frac{S_\lambda^{(2)}(m,l)H_a^{(r)}(u)\lambda^bB_b}{n-m-a-b+1}.\end{aligned} \tag{3.29}$$

By (3.28) and (3.29), we obtain the following theorem.

Theorem 3.7. *For each $n \in \mathbb{N} \cup \{0\}$, we have*

$$\beta_{n,\lambda}(x) = \sum_{l=0}^{n} \left(\sum_{m=l}^{n} \sum_{a=0}^{n-m} \sum_{b=0}^{r} \binom{n}{m} \binom{n-m}{a} \binom{r}{b} \frac{S_\lambda^{(1)}(m,l) \beta_{a,\lambda}(-u)^{r-b} (b)_{n-m-a,\lambda}}{(1-u)^r} \right) H_l^{(r)}(x|u),$$

and

$$H_n^{(r)}(x|u) = \sum_{l=0}^{n} \left(\sum_{m=l}^{n} \sum_{a=0}^{n-m} \sum_{b=0}^{n-m-a} \binom{n}{m} \binom{n-m}{a} \binom{n-m-a}{b} \frac{S_\lambda^{(2)}(m,l) H_a^{(r)}(u) \lambda^b B_b}{n-m-a-b+1} \right) \beta_{l,\lambda}(x).$$

The *Mittag-Leffer polynomials* are defined by the generating function to be

$$\left(\frac{1+t}{1-t}\right)^x = \sum_{n=0}^{\infty} M_n(x) \frac{t^n}{n!}, \text{ (see [20])}.$$

When $x = 1$, $M_n = M_n(1)$ are called the *Mittag-Leffer numbers*. By the definition of Mittag-Leffer polynomials, we see that

$$M_n(x) \sim \left(1, \frac{e^t - 1}{e^t + 1}\right).$$

Let $\beta_{n,\lambda}(x) = \sum_{l=0}^{n} a_{n,l} M_l(x)$. By (1.6) and Theorem 2.2, we get

$$\begin{aligned} a_{n,l} &= \frac{1}{l!} \left\langle \frac{1}{\frac{e_\lambda(t)-1}{t}} \left(\frac{e_\lambda(t)-1}{e_\lambda(t)+1} \right)^l \middle| x^n \right\rangle \\ &= \frac{1}{2^l} \left\langle \frac{t}{e_\lambda(t)-1} \left(\frac{2}{e_\lambda(t)+1} \right)^l \middle| \left(\frac{1}{l!} \left(e_\lambda(t)-1\right)^l \right) x^n \right\rangle \\ &= \frac{1}{2^l} \sum_{k=l}^{n} \binom{n}{k} S_{2,\lambda}(k,l) \left\langle \frac{t}{e_\lambda(t)-1} \left(\frac{2}{e_\lambda(t)+1} \right)^l \middle| x^{n-k} \right\rangle \\ &= \sum_{k=l}^{n} \sum_{r=0}^{n-k} \binom{n}{k} \binom{n-k}{r} \frac{S_{2,\lambda}(k,l) \beta_{r,\lambda}}{2^l} \left\langle \left(\frac{2}{e_\lambda(t)+1} \right)^l \middle| x^{n-k-r} \right\rangle \\ &= \sum_{k=l}^{n} \sum_{r=0}^{n-k} \binom{n}{k} \binom{n-k}{r} \frac{S_{2,\lambda}(k,l) \beta_{r,\lambda} E_{n-r-k,\lambda}^{(l)}}{2^l}. \end{aligned} \tag{3.30}$$

Conversely, assume that $M_n(x) = \sum_{l=0}^{n} b_{n,l} \beta_{l,\lambda}(x)$. Note that

$$\begin{aligned} \left(\frac{1+t}{1-t}\right)^x &= \left(1 + \frac{2t}{1-t}\right)^x = \sum_{m=0}^{\infty} (x)_m 2^m \frac{1}{m!} \left(\frac{t}{1-t}\right)^m \\ &= \sum_{m=0}^{\infty} (x)_m 2^m \sum_{r=m}^{\infty} L(r,m) \frac{t^r}{r!} \\ &= \sum_{a=0}^{\infty} \sum_{m=0}^{a} (x)_m 2^m L(a,m) \frac{t^a}{a!} \\ &= \sum_{a=0}^{\infty} \sum_{m=0}^{a} \sum_{r=0}^{m} 2^m L(a,m) S_1(m,r) x^r \frac{t^a}{a!}, \end{aligned}$$

and so we see that

$$M_n(x) = \sum_{m=0}^{n}\sum_{r=0}^{m} 2^m L(n,m) S_1(m,r) x^r. \tag{3.31}$$

By (1.6) and (3.31), we get

$$\begin{aligned} b_{n,l} =& \frac{1}{l!}\left\langle \frac{\lambda(e^t-1)}{e^{\lambda t}-1}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^l \middle| M_n(x)\right\rangle \\ =& \sum_{m=0}^{n}\sum_{r=0}^{m} 2^m S_1(m,r)L(n,m)\left\langle \frac{\lambda t}{e^{\lambda t}-1}\frac{e^t-1}{t}\middle| \left(\frac{1}{l!}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^l\right)x^r\right\rangle \\ =& \sum_{m=0}^{n}\sum_{r=0}^{m}\sum_{a=l}^{r}\binom{r}{a} 2^m S_1(m,r)L(n,m)\left\langle \frac{\lambda t}{e^{\lambda t}-1}\frac{e^t-1}{t}\middle| x^{r-a}\right\rangle \\ =& \sum_{m=0}^{n}\sum_{r=0}^{m}\sum_{a=l}^{r}\sum_{b=0}^{r-a}\binom{r}{a}\binom{r-a}{b} 2^m S_1(m,r)L(n,m)\lambda^b B_b\left\langle \frac{e^t-1}{t}\middle| x^{r-a-b}\right\rangle \\ =& \sum_{m=0}^{n}\sum_{r=0}^{m}\sum_{a=l}^{r}\sum_{b=0}^{r-a}\binom{r}{a}\binom{r-a}{b}\frac{2^m S_1(m,r)L(n,m)\lambda^b B_b}{r-a-b+1}. \end{aligned} \tag{3.32}$$

By (3.30) and (3.32), we obtain the following theorem.

Theorem 3.8. *For each $n \in \mathbb{N}\cup\{0\}$, we have*

$$\beta_{n,\lambda}(x) = \sum_{l=0}^{n}\left(\sum_{k=l}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r}\frac{S_{2,\lambda}(k,l)\beta_{r,\lambda}E^{(l)}_{n-r-k,\lambda}}{2^l}\right)M_l(x),$$

and

$$M_n(x) = \sum_{l=0}^{n}\left(\sum_{m=0}^{n}\sum_{r=0}^{m}\sum_{a=l}^{r}\sum_{b=0}^{r-a}\binom{r}{a}\binom{r-a}{b}\frac{2^m S_1(m,r)L(n,m)\lambda^b B_b}{r-a-b+1}\right)\beta_{l,\lambda}(x).$$

4. Conclusion

The umbral calculus which was built by G. C. Rota with a completely rigid foundation for theories based on modern ideas of linear functions, linear operators, and adjacency functions are one of the important and useful tools for studying the relationship between special functions. In addition, umbral calculus is still being actively used by many researchers.

In this paper, when the nth degenerate Bernoulli polynomials are expressed as a linear combination of special polynomials of order n or less, especially Bernoulli polynomials, Euler polynomials, Bernoulli polynomials of the second kind, Daehee polynomials, Changhee polynomials, Bell polynomials, Lah-Bell polynomials, Frobenius-Euler polynomials, Mittag-Leffer polynomials, the relationships between special polynomials are investigated by finding the coefficients.

A study to find the relationship between special functions by applying these methods to other special polynomials will be conducted in the future.

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