Advanced Studies in Contemporary Mathematics
32 (2022), No. 4, pp. 497 - 507
www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2022.32.4.497

INF-SUP METHOD FOR ESTABLISHING INEQUALITIES

WAN-LAN WANG, QIUYAN ZHANG, AND JIN-YAN MIAO

Abstract. This paper systematically summarizes the inf-sup method for establishing inequalities. With different results and specific techniques, we not only use this method to establish the geometric-arithmetic mean inequality, but also establish the new r-th mean inverses.

2000 Mathematics Subject Classification. 26D15, 26E60.

Keywords and phrases. Infimum; supremum; new inequality; inverse.

1. Introduction

The following two inequalities have a dual relationship in the inf-sup sense:

$$\inf_{x\in E}\sum_{i=1}^{n} f_i(x) \geq \sum_{i=1}^{n}\inf_{x\in E} f_i(x) \tag{1}$$

The work was supported by the National Natural Science Foundation of China (No.12101090), Sichuan Science and Technology Program (No.2021ZYD0009) and the Scientific Research Fund of Chengdu University of Information Technology (No.KYTZ202185).

and

$$\sup_{x\in E}\sum_{i=1}^{n} f_i(x) \le \sum_{i=1}^{n}\sup_{x\in E} f_i(x), \tag{2}$$

where $f_i : E \subset \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, n$. In the establishment of inequalities, (1) must go hand in hand with (2). If E is a subset of real numbers, there exists a certain dual relationship that $\inf(E) = -\sup(-E)$, where $-E = \{-x \,|x \in E\}$. According to the specific situation, we choose to use of (1) and/or (2). The following pair of (3) and (4) have also a dual relationship:

$$\sum_{i=1}^{n}\inf_{x\in E} f_i(x) \le \inf_{x\in E} f(x) \le f(x)\ ,\ \forall x \in E \tag{3}$$

and

$$\sum_{i=1}^{n}\sup_{x\in E} f_i(x) \ge \sup_{x\in E} f(x) \ge f(x)\,,\ \ \forall x \in E, \tag{4}$$

where $f : E \subset \mathbb{R} \to \mathbb{R}$ and denote $f(x) = \sum_{i=1}^{n} f_i(x)$ by $f = \sum_{i=1}^{n} f_i$.

The earliest research on the inf-sup method can be traced back to 1993. In this year, Sándor and Szabó [1] proved many famous inequalities by (1). In 1997, Pečarić and Varoǒanec [2] proved the AG inequality in this way. Then, by using (1) and (2), Wang [3, 4, 5, 6, 7, 8] proved the Fan-type inequalities and so on, and Wang and Luo [9] proved the Hua-type inequalities. Many researchers believe that (1) is an obvious fact. They also believe that if it is used, it will be an effective method and will be more attractive. The pair of (3) and (4) is more obvious. To take full advantage of each pair of these inequalities, we should certainly encourage ourselves to innovate when using them. The examination has concerned not only the forms of the inequalities but also the methods to prove them. After further study, we can come to the conclusion that each pair of combination provided is valid, so there are various inequalities for readers to choose.

One of the key problems of this method is how to construct an appropriate function f to connect the above inequalities with the expected result. In order to find such a function, some of us may lose confidence or even give up. This is the shortcoming of this method. However, the idea of this method is very clear. As the saying goes—good writing is clear thinking. Furthermore, many people prefer to try different methods.

It should be noted that sometimes "inf" and "sup" in (1) and (2) may be replaced by "min" and "max" respectively. Once we begin to prove an inequality, if we can get more properties for "inf" and "sup", then we can create more opportunities to complete our proof. Therefore, other relevant properties should also be considered as follows (pp.183-254 in [8] or pp.487-488 and pp.581-582 in [10]):

Theorem 1.1. (i) *If f, g are both defined on the set E, then*

(5)
$$\begin{aligned}\inf\{f(x)\}+\inf\{g(x)\} &\le \inf\{f(x)+g(x)\} \le \inf\{f(x)\}+\sup\{g(x)\} \\ &\le \sup\{f(x)+g(x)\} \le \sup\{f(x)\}+\sup\{g(x)\}\end{aligned}$$

(ii) *If f, g are both defined on the set E, and nonnegative, then*

$$\begin{aligned}\inf\{f(x)\}\cdot\inf\{g(x)\} &\le \inf\{f(x)\cdot g(x)\} \le \inf\{f(x)\}\cdot\sup\{g(x)\} \\ &\le \sup\{f(x)\cdot g(x)\} \le \sup\{f(x)\}\cdot\sup\{g(x)\}.\end{aligned}$$

(iii) *Order preserving: If $f(x) \le g(x)$, $\forall x \in E$, then*

$$\inf\{f(x)\} \le \inf\{g(x)\},\ \sup\{f(x)\} \le \sup\{g(x)\}.$$

Theorem 1.2. *If $E_1 \subset E$, and f is defined on E, then*

$$\inf\{f(x)\,|x\in E_1\} \geq \inf\{f(x)\,|x\in E\}$$

and

$$\sup\{f(x)\,|x\in E_1\} \leq \sup\{f(x)\,|x\in E\}.$$

If f is continuous on $[a,b]$, then
(i) $m(x)=\inf\{f(t)|a\leq t\leq x\}$ *is decreasing on* $[a,\, b]$;
(ii) $M(x)=\sup\{f(t)|a\leq t\leq x\}$ *is increasing on* $[a,b]$.

Remark 1. Let $a=\{a_1,\dots,a_n\}$, $b=\{b_1,\dots,b_n\}$ be two real sequences. We may state some results that are similar to the above. For example,
(i) If $a,\, b\in\mathbb{R}^n$, then $\sup a{+}\inf b\leq \sup(a{+}b)\leq \sup a{+}\sup b$.
(ii) If $a,\ b,\in\mathbb{R}^n_+$, then $\sup a\inf b\leq \sup ab\leq \sup a\sup b$.

Theorem 1.3 (Ji-Chang Kuang)**.** *If f is defined on $(a,\,+\infty)$ and bounded on any finite subinterval; g is a strictly increasing function tending towards $+\infty$, then for any positive constant c,*

$$\lim_{x\to+\infty}\inf\frac{f(x{+}c){-}f(x)}{g(x{+}c){-}g(x)}\leq \lim_{x\to+\infty}\inf\frac{f(x)}{g(x)}\leq \lim_{x\to+\infty}\sup\frac{f(x)}{g(x)}$$
$$\leq \lim_{x\to+\infty}\sup\frac{f(x{+}c){-}f(x)}{g(x{+}c){-}g(x)}.$$

In this paper, we introduce the latest research progress of using some properties of infimum and supremum. For simplicity, we call this method as "Inf-Sup method". Our research is limited to the establishment of discrete inequalities. Of course, by combining some results of this chapter with other methods, some continuous versions can be also established. (1) and (2) can be used to prove some propositions about super-additivity and sub-additive properties, which are essential in quasi linearization techniques. Therefore, in a sense, the Inf-Sup method is more worthy of our study and inheritance. In a word, we will make full use of the duality principle for Inf-Sup method.

2. On the Inf-Sup Method for Proving Inequalities

In order to compare the effectiveness and characteristics of different approaches, we will give various proofs using the above inequalities.

Theorem 2.1. *If* $b=(b_1,\dots,b_n)>0$, *then* $G_n(b)\le A_n(b)$.

The inequality in theorem 2.1 is called AG inequality. In the following, using notation $\sum := \sum_{i=1}^n$ and $\prod := \prod_{i=1}^n$, we will use inf-sup method to prove Theorem 2.1.

Proof. The first proof. Choose the functions

$$g_i : E=(0,+\infty)\to\mathbb{R}\,,\ g_i(x) := b_i x^{-\alpha}+\ln x,\ i=1,\dots,n,$$

where α is a positive number. From the first derivative $g_i'(x)=x^{-\alpha-1}(x^\alpha-\alpha b_i)$ it is easy to see g_i that has minimum value at $x_{i,0}=(\alpha b_i)^{1/\alpha}$ and its value is $g_i(x_{i,0}) = \alpha^{-1}[1+\ln(\alpha b_i)]$. Thus we obtain

$$\sum_{i=1}^n \inf_{x\in E} g_i(x)=\alpha^{-1}\big[n+\ln\prod_{i=1}^n(\alpha b_i)\big].$$

Similarly, $g(x) := \sum g_i(x) = (\sum b_i)x^{-\alpha}+n\ln x$ has minimum value at $x_0 = (\alpha n^{-1}\sum_{i=1}^n b_i)^{1/\alpha}$ and its value is $\inf_{x\in E}\sum_{i=1}^n g_i(x)=\alpha^{-1}[n+n\ln(\alpha n^{-1}\sum_{i=1}^n b_i)]$.

Inequality (1) gives

$$\alpha^{-1}\big[n+\ln\prod_{i=1}^n(\alpha b_i)\big]\le \alpha^{-1}\big[n+n\ln(\alpha n^{-1}\sum_{i=1}^n b_i)\big].$$

This yields the desired inequality. □

Proof. The second proof. We still choose the same as in the first proof:

$$g_i(x) := b_i x^{-\alpha} + \ln x,\ i = 1, \dots, n.$$

But we use the inequality in (2). For $g(x)$, we have

$$\sum_{i=1}^{n} \inf_{x\in E} g_i(x) \le g(x) \tag{6}$$

where $x \in (0, +\infty)$. For $g(x)$ in (6), we may take any $x \in (0, +\infty)$. Let us try to choose $x_1 = [\alpha A_n(b)]^{1/\alpha}$. Thus, substituting the value x_1 into (6) we obtain

$$\begin{aligned}\inf_{x\in E}\sum_{i=1}^{n} g_i(x) &= n\alpha^{-1}\big[1 + n^{-1}\ln\prod_{i=1}^{n}(\alpha b_i)\big]\\ &\le g(x_1) = n\alpha^{-1}\big\{1 + \ln\big[n^{-1}\sum_{i=1}^{n}(\alpha b_i)\big]\big\}\end{aligned}$$

AG inequality can be obtained by simplification. □

Proof. The third proof. Here is similar to the second proof, but x_1 is replaced by $x_2 = \big[\prod_{i=1}^{n}(\alpha b_i)\big]^{1/\alpha n}$. By substituting the value into (6), we obtain

$$\begin{aligned}&1 + \ln\big[\prod_{i=1}^{n}(\alpha b_i)\big]^{1/n}\\ \le&\big[n^{-1}\sum_{i=1}^{n}(\alpha b_i)\big]\big\{\big[\prod_{i=1}^{n}(\alpha b_i)\big]^{-1/n} + \ln\big[\prod_{i=1}^{n}(\alpha b_i)\big]\big\}^{1/n},\end{aligned}$$

which can be simplified into $1 \le (n^{-1}\sum_{i=1}^{n} b_i)\big/\sqrt[n]{\prod_{i=1}^{n} b_i}$, which is the AG inequality. □

Proof. The fourth proof. Now let us choose other functions as

$$h_i : E{=}(0, +\infty){\to}\mathbb{R},\ \ h_i(x) := -b_i x^{\alpha}{+}\ln x,\ i{=}1, \dots, n, \alpha{>}0.$$

By $h_i'(x) = -\alpha b_i x^{-1}[x^\alpha - (\alpha b_i)^{-1}]$, we can affirm that h_i has maximum value at $x_{i,0} = (\alpha b_i)^{-1/\alpha}$ and its value is $h_i(x_{i,0}) = \alpha^{-1}[-1-\ln(\alpha b_i)]$. Clearly, $h(x) = -(\sum_{i=1}^n b_i)x^\alpha + n\ln x$ has maximum value at $x_0 = [n(\sum_{i=1}^n(\alpha b_i)^{-1}]^{1/\alpha}$ and its value is

(7)
$$\sup_{x\in E}\sum_{i=1}^n h_i(x) = \sup_{x\in E} h(x) = n\alpha^{-1}\big[-1-\ln(n^{-1}\sum_{i=1}^n \alpha b_i)\big].$$

Let us try inequality (2). Using inequality (7), we get

$$\begin{aligned}\sup_{x\in E} h(x) &= n\alpha^{-1}\big[-1-\ln(n^{-1}\sum_{i=1}^n \alpha b_i)\big]\\ &\le \sum_{i=1}^n \sup_{x\in E} h_i(x) = \alpha^{-1}\big[-n-\ln(\prod_{i=1}^n(\alpha b_i))\big].\end{aligned} \tag{8}$$

Simplifying, we can obtain that

$$\ln\big[n(\textstyle\sum_{i=1}^n \alpha b_i)^{-1}\big] \le \ln\prod_{i=1}^n(\alpha b_i)^{-1/n},$$

which implying that $G_n(b) \le A_n(b)$. □

Proof. The fifth proof. We still choose $h_i(x) := -b_i x^\alpha + \ln x$, $i = 1,\ldots,n$ as in the fourth proof. Combining (4) with (8), we get

(9)
$$\begin{aligned}h(x) &\le n\alpha^{-1}\{-1+\ln\big[n(\sum_{i=1}^n \alpha b_i)^{-1}\big]\}\{-1+\ln\big[n(\sum_{i=1}^n \alpha b_i)^{-1}\big]\}\\ &= \sup_{x\in E} h(x) \le \sum_{i=1}^n \sup_{x\in E} h_i(x) = \alpha^{-1}\big[-n-\ln\prod_{i=1}^n(\alpha b_i)\big].\end{aligned}$$

Thus $h(x) = -(\sum b_i)x^\alpha + n\ln x \le \alpha^{-1}[-n-\ln\prod(\alpha b_i)]$. Take $x_3 = (\sum \alpha b_i/n)^{-1/\alpha}$ and substitute it into the last inequality. Simplifying gives $\ln A(\alpha b) \geq \ln G(\alpha b)$, that is, $G_n(b) \leq A_n(b)$. $\square$

Proof. The sixth proof. Set $x_4 = (\prod \alpha b_i)^{-1/n\alpha}$. The proof is essentially the same as the fifth proof. $\square$

Remark 2. As the saying goes, success and failure always go hand in hand. However, through our observation, if we can carefully arrange, (1) and (2) often work to pave the way for our coming success together. For inequality (6), other choices of x will lead to other inequalities.

The following inequalities involve the harmonic, geometric, arithmetic and quadratic means, of which (3) includes a converse of the AG inequality. Note that $M_n^{[r]}(b) := (n^{-1}\sum b_i^r)^{1/r}$ is usually referred to as the r-th mean. (7) and (4) can be used to derive some converses of the AGH inequalities as follows:

Theorem 2.2. *Let* $b = (b_1, \ldots, b_n) > 0,$ *and let* $r \geq 1$. *Then*

(i) $\ln[A_n(b)/G_n(b)] \leq A_n(b)/G_n(b) - 1$;

(ii) $\ln[A_n(b)/M_n^{[r]}(b)] \leq A_n(b)/M_n^{[r]}(b) - 1$;

(iii) $G_n(b) \leq A_n(b) \leq G_n(b) \cdot \exp[A_n(b)/G_n(b) - 1]$.

Proof. Consider the functions as shown in the fourth proof of Theorem 2.1:

$h_i : E = (0, +\infty) \to \mathbb{R}\,,\ h_i(x) := -b_i x^\alpha + \ln x,\ i = 1, \ldots, n.$

Certainly, $h(x) := \sum h_i(x) = -(\sum b_i)x^\alpha + n \ln x$. Set $x_4 = (\prod \alpha b_i)^{-1/n\alpha}$. By (7) and (4) for h we obtain

$$n\alpha^{-1}\big\{-1 - \ln\big[\big(\sum \alpha b_i\big)/n\big]\big\} = \sup h(x)$$

$$\geq h(x_4) = n\alpha^{-1}\big\{-\big(n^{-1}\sum \alpha b_i\big)/\big(\prod \alpha b_i\big)^{1/n} - \ln\big(\prod \alpha b_i\big)^{1/n}\big\}.$$

We can rewrite this as $-A_n(\alpha b)/G_n(\alpha b) - \ln G_n(\alpha b) \leq -1 - \ln A_n(\alpha b)$, which is equivalent to the desired inequality (i).

(Various choices ultimately yield different inequalities or equality, e.g. we can obtain an equality if choose $x_3 = [A_n(\alpha b)]^{1/\alpha}$.)

Choose $x_5 =: [M_n^{[r]}(\alpha b)]^{-1/\alpha} := (n^{-1}\sum \alpha^r b_i^r)^{-1/r\alpha}$. By using (4) for h, we get

$$n\alpha^{-1}\big\{-1-\ln\big[\big(\sum \alpha b_i\big)/n\big]\big\} = \sup h(x)$$
$$\geq h(x_5) = n\alpha^{-1}\big\{-\big(n^{-1}\sum \alpha b_i\big)/\big(n^{-1}\sum \alpha^r b_i^r\big)^{1/r} - \ln\big(\sum \alpha^r b_i^r\big)^{1/r}\big\}.$$

Simplifying yields $-A_n(\alpha b)/M_n^{[r]}(\alpha b) - \ln M_n^{[r]}(\alpha b) \leq -1 - \ln A_n(\alpha b)$, this is just the desired inequality in (ii).

Now we prove (iii): Combining (4) with (9), we obtain

$$h(x) \leq n\alpha^{-1}\big\{-1+\ln\big[n/(\sum \alpha b_i)\big]\big\} \leq \alpha^{-1}\big[-n-\ln\prod(\alpha b_i)\big],$$

where $h(x) = (-\sum b_i)x^\alpha + n\ln x$. As the above, choosing $x_4 = (\prod \alpha b_i)^{-1/n\alpha}$, we have

$$\begin{aligned} n\alpha^{-1}\big\{-n^{-1}\sum(\alpha b_i)\big/\big(\prod \alpha b_i\big)^{1/n} - \ln\big(\prod \alpha b_i\big)^{1/n}\big\} \\ \leq n\alpha^{-1}\big\{-1+\ln\big[n\big/\big(\sum \alpha b_i\big)\big]\big\} \\ \leq n\alpha^{-1}\big\{-1-n^{-1}\ln\prod \alpha b_i\big\}. \end{aligned}$$

or

$$1+\ln G_n(\alpha b) \leq 1+\ln A_n(\alpha b) \leq A_n(\alpha b)/G_n(\alpha b)+\ln G_n(\alpha b).$$

The desired form in (iii) can be obtained . □

Remark 3. We can also prove Theorem 2.2 with familiar inequalities, such as $\ln x \leq x-1$. Although there are some converses of the AG inequality, the second inequality in (iii) may be one of some concise results. When we use (ii) and (iii), there are several expressions to choose from, since the combination number is $2\binom{3}{3} + 2\binom{2}{3} = 8$.

Corollary 2.3. *If* $b = (b_1, \ldots, b_n) > 0$, *then*

$$G_n(b) \leq A_n(b) \leq G_n(b) \cdot \exp\big[A_n(b)\big/H_n(b) - 1\big].$$

Proof. This is a direct consequence of the conclusion (iii) in Theorem 2.2. Alternately, choosing $x = [n\big/\sum (\alpha b_i)^{-1}]^{-1/\alpha}$, we can also obtain the corollary in a similar way. □

3. Acknowledgement

The authors are grateful to the reviewers for their keen observations and helpful comments which led to the present improved version of the paper as it stands.

References

[1] J. Sándor and V. E. S. Szabó, *On an inequality for the sum of infimums of functions*, J. Math. Anal. Appl., **204**, 3(1996), 646-654.

[2] J. E. Pečarić and S. Varošanec, *A new proof of the arithmetic mean-the geometric mean inequality*, J. Math. Anal. Appl., **215**, 2(1997), 577-578.

[3] W.-l. Wang, *Some inequalities involving means and their converses*, J. Math. Anal. Appl., **238**, 2(1999), 567-579.

[4] W.-l. Wang, H. J. Ma and L. Chen, *The Sándor-Szabó method for establishing inequalities*, J. Chengdu Univ. (Nat. Sci. Ed), **19**, 3(2000), 20-26. (in Chinese)

[5] W.-l. Wang, *The counterpart of Fan's inequality and its related results*, J. Ineq. P. Appl. Math., **9**, 4(2008), 1-8.

[6] W.-l. Wang and J.-y. Miao, *Comments and proofs for two well-known inequalities*, J. Chengdu Univ.(Nat. Sci. Ed.), **34**, 4(2015), 354-356. (in Chinese)

[7] W.-l. Wang and P. F. Wang, *A class of inequalities for the symmetric functions*, Acta Math. Sinica, **27**, 4(1984), 485-497. (In Chinese)

[8] J.P. D'Angelo and D. B. West, *Mathematical Thinking: Problem-Solving and Proofs(2nd Ed)*, Prentice-Hall, Inc. 2000.

[9] W.-l. Wang and Z. Luo, *Some Generalizations for Inequalities of Hua-Wang Type*, J. Math. Res. Exp., **22**, 2(2002), 575-582.

[10] J.-C. Kuang, *Applied Inequalities(5th Ed)*, Shandong Science and Technology Press, 2021 (in Chinese).

School of Mathematics, Chengdu University, Chengdu, 610106, P.R. China
E-mail address: wanlanwang@163.com

College of Applied Mathematics, Chengdu University of Information Technology, Chengdu, 610225, P.R. China
E-mail address: zqy1607@cuit.edu.cn (Corresponding author)

Sichuan Province Airport Group Company Limited, Chengdu, 610202, P.R. China
E-mail address: 954599851@qq.com