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A ROOT OF $x^\lambda + ax + b = 0$ USING DEGENERATE LAMBERT W FUNCTION

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Abstract. Lambert W function is the inverse function of the function $\mathbb{L}(x) = xe^x$. And it is useful tool for finding a root of $a^x + bx + c = 0$. The degenerate exponential function $e_\lambda(t)$ has been studied by many mathematicians, and numerous related results have been published. In this paper, we define the inverse function $W_\lambda(x)$ of the function $\mathbb{L}_\lambda = xe_\lambda(x)$. Taylor expansion of $W_\lambda(x)$ are presented and a range of radius of convergence of the series is presented. We apply this $W_\lambda(x)$ to find a root of $x^\lambda + ax + b = 0$.

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1. Introduction

Let us consider the transcendental function xe^x where e is the base of the natural logarithm. In this article, we put xe^x as $\mathbb{L}(x)$. The solution of transcendental equation

$$\mathbb{L}(x) = c$$

were studied by Euler and by Lambert [5]. The inverse of the function $\mathbb{L}(x)$ is called the Lambert function and is denoted by W. If $-1/e < c < 0$, there are two real solutions, and thus two real branches of W. If we allow complex values of c, we get many solutions, and W has infinitely many complex branches. For the remainder of this paper, all parameters will be assumed to be real, and our concern will be real-valued functions of real variables. It turned out to be useful tool in combinatorics, for instance, in the enumeration of trees [4, 9]. It can be used to solve various equations involving exponentials and also occurs in the solution of delay differential equations, such as $y'(x) = ay(x-1)$ [26]. The Lambert W relation cannot be expressed in terms of elementary functions [1].

After Carlitz [2, 3], various number of degenerate versions of some special polynomials and numbers are studied. In recent years, studying degenerate versions of various special polynomials and numbers have regained interests of many mathematicians [6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. The researches have been carried out by several different methods like generating functions, combinatorial approaches, p-adic analysis and differential equations. This idea of studying degenerate versions of some special polynomials and numbers turned out to be very fruitful so as to introduce degenerate Laplace transforms and degenerate gamma functions (see [15]).

The Taylor expansion of Lambert W function is well known as follows.

$$W(x) = \sum_{n=1}^{\infty} (-n)^{n-1} \frac{x^n}{n!}.$$

This series converges if $|x| < \frac{1}{e}$.

The Lambert W function is used to solve equations in which the unknown quantity occurs both in the base and in the exponent, or both inside and outside of a logarithm. The strategy is to convert such an equation into one of the form xe^x and then to solve for x using the W function.

For example, the equation

$$a^x + bx + c = 0$$

can be solved by rewriting it as

$$\begin{aligned} a^{-x}\left(-x - \frac{c}{b}\right) &= \frac{1}{b} \\ e^{-x\ln a}\left(-x - \frac{c}{b}\right) &= \frac{1}{b} \\ \ln a\left(-x - \frac{c}{b}\right) e^{\ln a\left(-x-\frac{c}{b}\right)} &= \frac{\ln a}{b} e^{-\frac{c}{b}\ln a} \end{aligned}$$

Since $W(xe^x) = x$, apply to the last line, then we get

$$x = -\frac{1}{\ln a} W\left(\frac{\ln a}{b} e^{-\frac{c}{b}\ln a}\right) - \frac{c}{b}$$

For any nonzero $\lambda \in \mathbb{R}$ (or $\mathbb{C}$), the degenerate exponential function is defined by

$$e_\lambda(t) = (1 + \lambda t)^{\frac{1}{\lambda}} \text{ (see [2], [10],[15]).}$$

In accordance with the exponential sense, $\log(1 + \lambda t)^{\frac{1}{\lambda}}$ can be used for t to study degenerate numbers and polynomials. It is natural to think of a degenerate log function as the inverse function of the degenerate exponential function. The degenerate log function, denoted by $\log_\lambda(t)$, is defined by the generating function to be

$$\log_\lambda(t) = \frac{t^\lambda - 1}{\lambda} \text{ (see [16], [17],[18]).}$$

As λ goes to 0, $e_\lambda(t)$ converges to e^t and $\log_\lambda t$ converges to $\ln(t)$ of natural log. From these two degenerate function $e_\lambda(t)$ and $\log_\lambda(t)$, many results are published.

In this paper, we introduce degenerate Lambert W function and its applications for finding a root of $x^\lambda + ax + b = 0$.

2. The function $\mathbb{L}_\lambda(x) = x(1 + \lambda x)^{\frac{1}{\lambda}}$

For a real number λ with $\lambda \neq 0$ and $\lambda \neq \pm 1$, from now on, we consider this condition at the rest of this article. We define

$$\mathbb{L}_\lambda(x) = x(1 + \lambda x)^{\frac{1}{\lambda}} = xe_\lambda(x).$$

As λ goes to 0, the function $\mathbb{L}_\lambda(x)$ goes to $\mathbb{L}(x) = xe^x$.

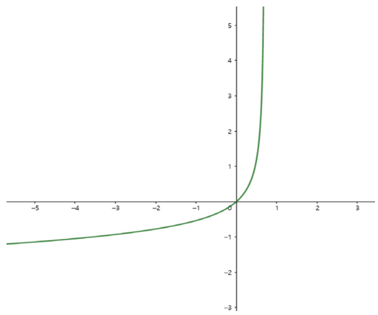

FIGURE 1. The graph of $\mathbb{L}_{-\sqrt{2}} = x(1-\sqrt{2}x)^{-\frac{1}{\sqrt{2}}}$

As if $\mathbb{L}(\ln x) = x \ln x$,

$$\mathbb{L}_{\lambda}(\log_{\lambda} x) = x \log_{\lambda}(x).$$

If $\lambda > 0$, then

$$\mathbb{L}_{\lambda}\left(-\frac{1}{\lambda}\right) = 0.$$

And $\mathbb{L}_{\lambda}(x)$ goes to positive infinity as x goes to positive infinity.

If $\lambda < 0$,

$$\lim_{x \to -\frac{1}{\lambda}-} \mathbb{L}_{\lambda}(x) = \infty.$$

Therefore, it is necessary to observe the $\mathbb{L}_{\lambda}(x)$ according to λ. To obtain the minimum or maximum value of $\mathbb{L}_{\lambda}(t)$, differentiation of $\mathbb{L}_{\lambda}(t)$ gives the following.

$$\mathbb{L}'_{\lambda}(x) = (1+(1+\lambda)x)(1+\lambda x)^{\frac{1}{\lambda}-1}.$$

Therefore we need to divide λ into three intervals: $\lambda < -1$, $-1 < \lambda < 0$ and $0 < \lambda$.

In the case of $\lambda < -1$ except for $\frac{1}{\lambda}$ or λ being odd integer, the function $\mathbb{L}_{\lambda}(x)$ is strictly increasing in $x < -\frac{1}{\lambda}$. And the function goes to positive infinity as x goes to $-\frac{1}{\lambda}$, the function goes to negative infinity as x goes to negative infinity. In this case, the range of the function $\mathbb{L}_{\lambda}(x)$ is every real number. The Figure 1. shows the graph of $\mathbb{L}_{\lambda}(x)$ if $\lambda = -\sqrt{2}$.

In the case of $-1 < \lambda < 0$ except for $\frac{1}{\lambda}$ or λ being odd integer. In this case, the function $f_{\lambda}(x)$ has minimum value $-\left(\frac{1}{1+\lambda}\right)^{\frac{1}{\lambda}+1}$ at $x = -\frac{1}{1+\lambda}$. In this case, the function $f_{\lambda}(x)$ decreases in the region $x < \lambda$ and increases $\lambda < x < -\frac{1}{\lambda}$. The function goes to positive infinity as x goes to $-\frac{1}{\lambda}$, the function goes to 0 as x goes to negative infinity. The Figure 2. shows the graph of $\mathbb{L}_{\lambda}(x)$ if $\lambda = -\frac{1}{\pi}$.

In case $0 < \lambda$, the domain of $f_{\lambda}(x)$ is $\frac{1}{\lambda} < x$. In this case, the function $f_{\lambda}(x)$ decreases in the region where $-\frac{1}{\lambda} < x < \lambda$ and increases $x > \lambda$. The value of the function is 0 at $x = -\frac{1}{\lambda}$, the function goes to positive infinity

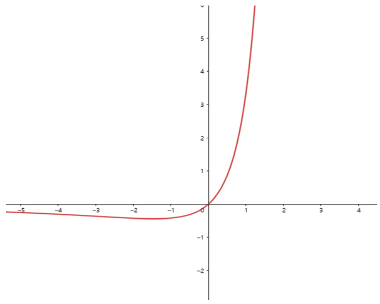

FIGURE 2. The graph of $\mathbb{L}_{-1/\pi} = x(1 - \frac{1}{\pi}x)^{-\pi}$

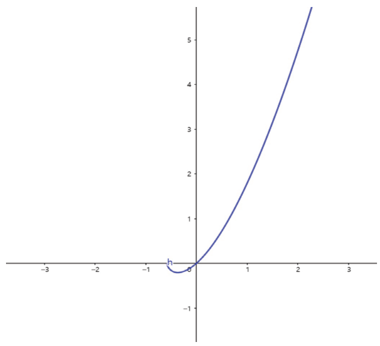

FIGURE 3. The graph of $\mathbb{L}_{\sqrt{3}} = x(1 + \sqrt{3}x)^{\frac{1}{\sqrt{3}}}$

as x goes to positive infinity. The Figure 3. shows the graph of $\mathbb{L}_\lambda(x)$ if $\lambda = \sqrt{3}$.

The following Table 1. summarizes the above results. It shows the maximum and minimum values of $\mathbb{L}(x)$ according to the domain of definition, and shows where the minimum value is obtained.

$\mathbb{L}_\lambda(x)$	$\lambda < -1$	$-1 < \lambda < 0$	$0 < \lambda$
domain	$x < -\frac{1}{\lambda}$	$x < -\frac{1}{\lambda}$	$-\frac{1}{\lambda} < x$
minimum	none	$-\left(\frac{1}{1+\lambda}\right)^{\frac{1}{\lambda}+1}$	$-\left(\frac{1}{1+\lambda}\right)^{\frac{1}{\lambda}+1}$
where $\mathbb{L}$ has minimum	none	$x = -\frac{1}{1+\lambda}$	$x = -\frac{1}{1+\lambda}$
maximum	∞	∞	∞

TABLE 1

The Lambert W function is known to be a good tool for solving various differential equations. We are going to observe how $\mathbb{L}_\lambda$ relates to the differential equation. For a natural number N, $F^{(N)}$ to denote the N-th derivative of F, that is,

$$F^{(0)} = F(t),\ F^{(N)} = \frac{d}{dt}F^{(N-1)}.$$

Then we know that

$$\begin{aligned} e_\lambda^{(N)}(t) &= \left((1+\lambda x)^{\frac{1}{\lambda}}\right)^{(N)} = \left(\frac{1}{\lambda}\right)_N \left(\frac{\lambda}{1+\lambda t}\right)^N e_\lambda(t) \\ &= \frac{(1)_{N,\lambda}}{(1+\lambda t)^N} e_\lambda(t) \end{aligned}$$

where $(x)_N$ denote, and from now on, N-th falling factorial of x, that is, $(x)_N = x(x-1)(x-2)\cdots(x-N+1)$ and $(x)_{N,\lambda}$ denote the generalized falling factorial $x(x-\lambda)(x-2\lambda)\cdots(x-(N-1)\lambda)$. This yields the following theorem.

Theorem 2.1. *For any positive integer N, the differential equation*

$$F^{(N)} = Ne_\lambda^{(N-1)}(t) + te_\lambda^{(N)}(t)$$

has a solution

$$F(t) = \mathbb{L}(t) = te_\lambda(t) = t(1+\lambda t)^{\frac{1}{\lambda}}.$$

3. The degenerate Lambert $W_\lambda(z)$

We define the degenerate Lambert W function W, which is the inverse of $\mathbb{L}_\lambda(x)$, as,

$$W_\lambda(\mathbb{L}_\lambda(x)) = \mathbb{L}_\lambda(W_\lambda(x)) = x.$$

For the Taylor expansion of inverse function, Lagrange Inversion theorem says that the following theorem

$$f^{-1}(y) = \sum_{n=1}^{\infty} \left[\frac{d^{n-1}}{dx^{n-1}}\left(\frac{x}{f(x)}\right)^n\right]_{x=0} \frac{y^n}{n!}. \tag{1}$$

Note that,

$$\left(\frac{x}{\mathbb{L}_\lambda(x)}\right)^n = (1+\lambda x)^{-\frac{n}{\lambda}},$$

we get,

$$\begin{aligned} \left[\frac{d^{n-1}}{dx^{n-1}}(1+\lambda x)^{-\frac{n}{\lambda}}\right]_{x=0} &= (-1)^{n-1}\frac{n}{\lambda}(\frac{n}{\lambda}+1)\cdots(\frac{n}{\lambda}+(n-2))\lambda^{n-1} \\ &= (-1)^{n-1}(n+\lambda)(n+2\lambda)\cdots(n+(n-2)\lambda) \\ &= (-1)^{n-1} < n >_{n-1,\lambda} \end{aligned} \tag{2}$$

where $< x >_{m,\lambda}$ denotes the generalized rising factorial $< x >_{m,\lambda} = x(x+\lambda)(x+2\lambda)\cdots(x+(m-1)\lambda)$.

By (1) and (2), we get the following theorem.

Theorem 3.1. *Taylor expansion of $W_\lambda(x)$ is the following.*

$$W_\lambda(x) = \sum_{n=1}^{\infty}(-1)^{n-1} < n >_{n-1,\lambda} \frac{x^n}{n!},$$

where $< n >_{n-1,\lambda} = n(n+\lambda)(n+2\lambda)\cdots(n+(n-2)\lambda)$.

As λ goes to 0, $< n >_{n-1,\lambda}$ converges to n^{n-1}. This says that the Taylor expansion of $W_\lambda(x)$ goes to the Taylor expansion of $W(x)$, that is,

$$\lim_{\lambda\to 0} W_\lambda = W.$$

Theorem 3.2. *The radius convergence* $R(W_\lambda(x))$ *of Taylor expansion of* $W_\lambda(x)$ *is bounded as follows according to* λ.

i) $\lambda > 0$, $e^{-1}|1+\lambda|^{-1} \leq R(W_\lambda(x)) \leq e^{-1/(1+\lambda)}|1+\lambda|^{-1}$,

ii) $-1 < \lambda < 0$, $e^{-1/(1+\lambda)}|1+\lambda|^{-1} \leq R(W_\lambda(x)) \leq e^{-1}|1+\lambda|^{-1}$,

iii) $\lambda < -1$, $e^{-\frac{3\lambda-1}{2\lambda(1+\lambda)}}|1+\lambda|^{-1} \leq R(W_\lambda(x)) \leq e^{-\frac{1+\lambda}{2\lambda}}|1+\lambda|^{-1}$.

Proof. First, let us look at the case $\lambda > 0$. In this case, $n+k\lambda < n+l\lambda$ if $k < l$. So we get the following

(3)
$$\begin{aligned}\left|\frac{< n+1 >_{n,\lambda}/(n+1)!}{< n >_{n-1,\lambda}/n!}\right| &= \left|\frac{(n+1)(n+1+\lambda)(n+1+2\lambda)-\cdots(n+1+(n-1)\lambda)n!}{n(n+\lambda)(n+2\lambda)\cdots(n+(n-2)\lambda)(n+1)!}\right| \\ &= \left|\left(1+\frac{1}{n+\lambda}\right)\cdots\left(1+\frac{1}{(n+(n-2)\lambda)}\right)\left(\frac{n+1+(n-1)\lambda}{n}\right)\right| \\ &\leq \left|\left(1+\frac{1}{n+\lambda}\right)^{n-2}\left(\frac{n+1+(n-1)\lambda}{n}\right)\right| \\ &\to e|1+\lambda|\end{aligned}$$

as n goes to infinity. And we know that

(4)
$$\begin{aligned}\left|\frac{< n+1 >_{n,\lambda}/(n+1)!}{< n >_{n-1,\lambda}/n!}\right| &= \left|\frac{(n+1)(n+1+\lambda)(n+1+2\lambda)-\cdots(n+1+(n-1)\lambda)n!}{n(n+\lambda)(n+2\lambda)\cdots(n+(n-2)\lambda)(n+1)!}\right| \\ &\geq \left|\left(1+\frac{1}{n+(n-2)\lambda}\right)^{n-2}\left(\frac{n+1+(n-1)\lambda}{n}\right)\right| \\ &\to e^{\frac{1}{1+\lambda}}|1+\lambda|.\end{aligned}$$

The equation (3) and (4) says that $e^{-1}|1+\lambda|^{-1} \leq R(W_\lambda(x)) \leq e^{-1/(1+\lambda)}|1+\lambda|^{-1}$ if $\lambda > 0$.

Next, let us look at the case where $-1 < \lambda < 0$. In this case, $n+l\lambda < n+k\lambda$ if $k < l$. So we get the following.

(5)
$$\begin{aligned}\left|\frac{< n+1 >_{n,\lambda})/(n+1)!}{< n >_{n-1,\lambda}/n!}\right| &= \left|\frac{(n+1)(n+1+\lambda)(n+1+2\lambda)\cdots(n+1+(n-1)\lambda)n!}{n(n+\lambda)(n+2\lambda)\cdots(n+(n-2)\lambda)(n+1)!}\right| \\ &= \left|\left(1+\frac{1}{n+\lambda}\right)\cdots\left(1+\frac{1}{(n+(n-2)\lambda)}\right)\left(\frac{n+1+(n-1)\lambda}{n}\right)\right| \\ &\leq \left|\left(1+\frac{1}{(n+(n-2)\lambda)}\right)^{n-2}\left(\frac{n+1+(n-1)\lambda}{n}\right)\right| \\ &\to e^{1/(1+\lambda)}|1+\lambda|\end{aligned}$$

Similarly the equation (4), we get

(6)
$$\left|\frac{<n+1>_{n,\lambda})/(n+1)!}{<n>_{n-1,\lambda}/n!}\right| = \left|\frac{(n+1)(n+1+\lambda)(n+1+2\lambda)\cdots(n+1+(n-1)\lambda)n!}{n(n+\lambda)(n+2\lambda)\cdots(n+(n-2)\lambda)(n+1)!}\right|$$
$$\geq \left|\left(1+\frac{1}{(n+\lambda)}\right)^{n-2}\left(\frac{n+1+(n-1)\lambda}{n}\right)\right|$$
$$\to e|1+\lambda|$$

From the equations (5) and (6), we get $e^{-1/(1+\lambda)}|1+\lambda|^{-1} \leq R(W_{\lambda}(x)) \leq e^{-1}|1+\lambda|^{-1}$ if $-1<\lambda<0$.

In the case $\lambda<-1$, there exist the smallest positive integer m such that $n+m\lambda<0$. The integer m can be expressed as $m=-\frac{n}{\lambda}$.

(7)
$$\left|\frac{<n+1>_{n,\lambda})/(n+1)!}{<n>_{n-1,\lambda}/n!}\right| = \left|\frac{(n+1)(n+1+\lambda)(n+1+2\lambda)\cdots(n+1+(n-1)\lambda)n!}{n(n+\lambda)(n+2\lambda)\cdots(n+(n-2)\lambda)(n+1)!}\right|$$
$$= \left|\left(\frac{n+\lambda+1}{n+\lambda}\right)\cdots\left(\frac{(n+(n-2)\lambda)+1}{(n+(n-2)\lambda)}\right)\left(\frac{n+1+(n-1)\lambda}{n}\right)\right|$$
$$= \left|\left(\frac{n+\lambda+1}{n+\lambda}\right)\right|\cdots\left|\left(\frac{n+(m-1)\lambda+1}{n+(m-1)\lambda}\right)\right|$$
$$\times \left|\left(\frac{n+m\lambda+1}{n+m\lambda}\right)\right|\cdots\left|\left(\frac{n+(n-2)\lambda)+1}{n+(n-2)\lambda}\right)\right|\left|\left(\frac{n+1+(n-1)\lambda}{n}\right)\right|$$
$$\leq \left|\left(1+\frac{1}{n+(m-1)\lambda}\right)\right|^{m-1}\left|\left(1+\frac{1}{n+(n-2)\lambda}\right)\right|^{n-m-1}\left|\left(\frac{n+1+(n-1)\lambda}{n}\right)\right|$$
$$\to e^{\frac{3\lambda-1}{2\lambda(1+\lambda)}}|1+\lambda|.$$

Similarly the equations (4) and (6) we get

(8)
$$\left|\frac{<n+1>_{n,\lambda})/(n+1)!}{<n>_{n-1,\lambda}/n!}\right| = \left|\frac{(n+1)(n+1+\lambda)(n+1+2\lambda)\cdots(n+1+(n-1)\lambda)n!}{n(n+\lambda)(n+2\lambda)\cdots(n+(n-2)\lambda)(n+1)!}\right|$$
$$\geq \left|\left(1+\frac{1}{n+\lambda}\right)\right|^{m-1}\left|\left(1+\frac{1}{n+m\lambda}\right)\right|^{n-m-1}\left|\left(\frac{n+1+(n-1)\lambda}{n}\right)\right|$$
$$\to e^{\frac{1+\lambda}{2\lambda}}|1+\lambda|.$$

From the equations (7) and (8), we get $e^{-\frac{3\lambda-1}{2\lambda(1+\lambda)}}|1+\lambda|^{-1} \leq R(W_{\lambda}(x)) \leq e^{-\frac{1+\lambda}{2\lambda}}|1+\lambda|^{-1}$ if $\lambda<-1$. $\square$

When $\lambda>-1$, the radius convergence of $W_{\lambda}(x)$ converges to $1/e$ as λ goes to 0. This is the same as $W_{\lambda}(x)$ converges to $W(x)$ as λ goes to 0.

The next theorem shows how to solve an equation $x^{\lambda}+ax+b=0$ using W_{λ}.

Theorem 3.3. *For real a, λ and negative real b with $-\left(\frac{1}{1-\lambda}\right)^{1-\frac{1}{\lambda}} < \frac{a(-b)^{\frac{1}{\lambda}-1}}{\lambda}$ if $\lambda < 1$. Then equation $x^{\lambda} + ax + b = 0$ has a solution*

$$x = -\frac{b\lambda}{a}W_{\lambda}(\frac{a(-b)^{\frac{1}{\lambda}-1}}{\lambda}).$$

Proof. The equation

$$x^{\lambda} + ax + b = 0$$

can be solved by rewriting it as

$$\begin{aligned} x^{\lambda} &= (-ax - b) \\ x\left(\frac{a}{b}x + 1\right)^{-1/\lambda} &= (-b)^{\frac{1}{\lambda}} \end{aligned}$$

Replace $\frac{a}{b}x$ with $-\lambda y$, then $x = -\frac{b\lambda}{a}y$ and

$$\begin{aligned} -\frac{b\lambda}{a}y(1-\lambda y)^{-1/\lambda} &= (-b)^{\frac{1}{\lambda}} \\ y(1-ry)^{-1/\lambda} &= \frac{a(-b)^{\frac{1}{\lambda}-1}}{\lambda}. \end{aligned}$$

Since $W_{\lambda}(x(1+\lambda x)^{\frac{1}{\lambda}}) = x$, apply to the last line and Table 1, then we get

$$y = W_{\lambda}(\frac{a(-b)^{\frac{1}{\lambda}-1}}{\lambda}).$$

Therefore

$$x = -\frac{b\lambda}{a}W_{\lambda}(\frac{a(-b)^{\frac{1}{\lambda}-1}}{\lambda}).$$

□

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