Advanced Studies in Contemporary Mathematics
32 (2022), No. 4, pp. 479 - 486

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2022.32.4.479

SOME PROPERTIES OF GENERALIZED DEGENERATE BERNOULLI POLYNOMIALS AND NUMBERS

WONJOO KIM, LEE-CHAE JANG, AND JONGKYUM KWON

ABSTRACT. Many researchers studied degenerate Bernoulli polynomials and their applications. In this paper, we consider the generalized degenerate Bernoulli polynomials and investigate some identities of those polynomials.

1. INTRODUCTION

Carlitz studied the degenerate Bernoulli polynomials as degenerate versions of the ordinary Bernoulli polynomials. Recently, many researchers have been investigated degenerate versions of some special numbers and polynomials which has received increased attention by mathematicians with their interests not only in combinatorial and arithmetic properties.

Assume that $\lambda(\neq 0)\in\mathbb{C}$. The degenerate exponential function is defined as

$$e_\lambda^x(t)=(1+\lambda t)^{\frac{x}{\lambda}},\ (\text{see } [1-12]). \tag{1}$$

Note that $\lim_{\lambda\to 0} e_\lambda^x(t)=e^{xt}$ and $e_\lambda^1(t)=e_\lambda(t)$.

The degenerate Bernoulli polynomials are defined by

$$\frac{t}{e_\lambda(t)-1}e_\lambda^x(t)=\sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!},\ (\text{see } [1-5]). \tag{2}$$

When $x=0$, $\beta_{n,\lambda}=\beta_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers.

From (2), we easily get

$$\beta_{n,\lambda}(x)=\sum_{l=0}^{n}\binom{n}{l}(x)_{n-l,\lambda}\beta_{l,\lambda}, \tag{3}$$

where $(x)_{0,\lambda}=1$, $(x)_{n,\lambda}=x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda),\quad (n\geq 1)$.

The Stirling numbers of the first kind is defined as

$$(x)_n=\sum_{l=0}^{n}S_1(n,l)x^l,\quad (n\geq 0). \tag{4}$$

The Stirling numbers of the second kind is defined as

$$x^n=\sum_{l=0}^{n}S_2(n,l)(x)_l,\quad (n\geq 0). \tag{5}$$

Let $\log_\lambda(t)$ be the compositional inverse function of $e_\lambda(t)$.

$$\log_\lambda(1+t)=\sum_{n=1}^{\infty}\frac{\lambda^{n-1}(1)_{n,1/\lambda}}{n!}t^n,\quad (\text{see } [3]). \tag{6}$$

2010 *Mathematics Subject Classification.* 11B73; 11B83.

Key words and phrases. the generalized degenerate Bernoulli polynomials; the ordinary higher-order Bernoulli polynomials; the Frobenius-Euler polynomials.

Note that $\lim_{\lambda\to 0}\log_\lambda(1+t)=\log(1+t)$. Kim-Kim considered the degenerate Stirling numbers of the second kind $S_{2,\lambda}(n,k)$, $(n,k\geq 0)$, which are given by

$$(x)_{n,\lambda}=\sum_{k=0}^{n}S_{2,\lambda}(n,k)(x)_k,\quad (n\geq 0),(\text{see [3]}). \tag{7}$$

As the inversion formula of (7), they also considered the degenerate Stirling numbers of the first kind given by

$$(x)_n=\sum_{k=0}^{n}S_{1,\lambda}(n,k)(x)_{k,\lambda}\quad (n\geq 0),(\text{see [3]}). \tag{8}$$

From (5) and (7), we can derive the following equations:

$$\frac{1}{k!}(e^\lambda(t)-1)^k=\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!},\quad (\text{see [3,9]}). \tag{9}$$

In this paper, we define the generalized degenerate Bernoulli polynomials and numbers and investigate some properties and identities of those polynomials. Recently, Kim introduced the generalized degenerate Euler-Genocchi polynomials which are given by

$$\frac{2t^r}{e_\lambda(t)+1}e_\lambda^x(t)=\sum_{n=0}^{\infty}A_{n,\lambda}^{(r)}(x),\quad (\text{see [11]}). \tag{10}$$

The technical method of Kim-Kim (see[11]) became the core research motivation for writing our paper.

2. The generalized degenerate Bernoulli polynomials and numbers

For $r\in\mathbb{N}$, we consider the generalized degenerate Bernoulli polynomials which are given by the generating function to be

$$\frac{t^r}{e_\lambda(t)-1}e_\lambda^x(t)=\sum_{n=0}^{\infty}\mathfrak{B}_{n,\lambda}^{[r]}(x)\frac{t^n}{n!}. \tag{11}$$

Note that if $r=1$, then $\mathfrak{B}_{n,\lambda}^{[1]}(x)=\beta_{n,\lambda}(x)$ and $\mathfrak{B}_{0,\lambda}^{[r]}(x)=\mathfrak{B}_{1,\lambda}^{[r]}(x)=\cdots=\mathfrak{B}_{r-2,\lambda}^{[r]}(x)=0$, for $r\in\mathbb{N}$. When $x=0$, $\mathfrak{B}_{n,\lambda}^{[r]}=\mathfrak{B}_{n,\lambda}^{[r]}(0)$ are called the generalized degenerate Bernoulli numbers.

By (11), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{B}_{n,\lambda}^{[r]}(x)\frac{t^n}{n!}&=\frac{t^r}{e_\lambda(t)-1}e_\lambda^x(t)\\&=\left(\sum_{l=0}^{\infty}\mathfrak{B}_{l,\lambda}^{[r]}(x)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(x)_{m,\lambda}\frac{t^m}{m!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\mathfrak{B}_{l,\lambda}^{[r]}(x)_{n-l,\lambda}\right)\frac{t^n}{n!}\end{aligned} \tag{12}$$

By comparing the coefficients on both sides of (12), we obtain the following theorem.

Theorem 1. *For $r\in\mathbb{Z}$ with $r\geq 0$ and $n\in\mathbb{N}\cup\{0\}$, we have*

$$\mathfrak{B}_{n,\lambda}^{[r]}(x)=\sum_{l=0}^{n}\binom{n}{l}\mathfrak{B}_{l,\lambda}^{[r]}(x)_{n-l,\lambda}. \tag{13}$$

From (2) and (11), we observe

$$
\begin{aligned}
\sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!} &= \frac{t}{e_\lambda(t)-1}e_\lambda^x(t)\\
&= \frac{1}{t^{r-1}}\sum_{n=0}^{\infty}\mathfrak{B}_{n,\lambda}^{[r]}(x)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\mathfrak{B}_{n+r-1,\lambda}^{[r]}(x)\frac{t^n}{(n+r-1)!}\\
&= \sum_{n=0}^{\infty}\mathfrak{B}_{n+r-1,\lambda}^{[r]}(x)\frac{n!}{(n+r-1)!}\frac{(r-1)!}{(r-1)!}\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{\mathfrak{B}_{n+r-1,\lambda}^{[r]}(x)}{\binom{n+r-1}{r-1}(r-1)!}\frac{t^n}{n!}.
\end{aligned}
\tag{14}
$$

Therefore, by comparing the coefficients on both sides of (14), we obtain the following theorem.

Theorem 2. *For $r \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, we have*

$$\beta_{n,\lambda}(x) = \frac{\mathfrak{B}_{n+r-1,\lambda}^{[r]}(x)}{\binom{n+r-1}{r-1}(r-1)!}.$$

For $r \in \mathbb{N}$, Carlitz introduced the higher-order degenerate Bernoulli polynomials given by

$$\left(\frac{t}{e_\lambda(t)-1}\right)^r e_\lambda^x(t) = \sum_{n=0}^{\infty}\beta_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}, \quad \text{(see [6])}. \tag{15}$$

When $x = 0$, $\beta_{n,\lambda}^{(r)} = \beta_{n,\lambda}^{(r)}(0)$ are called the higher-order degenerate Bernoulli numbers. Note that for $r = 1$, $\beta_{n,\lambda}(x) = \beta_{n,\lambda}^{(1)}(x) = \mathfrak{B}_{n,\lambda}^{[1]}(x)$ are the degenerate Bernoulli polynomials.

From (15), we easily get $\lim_{\lambda\to 0}\beta_{n,\lambda}^{(r)}(x)\frac{t^n}{n!} = B_n^{(r)}(x)$, where $B_n^{(r)}(x)$ are the ordinary higher-order Bernoulli polynomials given by

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty}B_n^{(r)}(x)\frac{t^n}{n!}. \tag{16}$$

From (15), we observe

$$\begin{aligned}
\sum_{n=0}^{\infty}\beta_{n,\lambda}^{(r)}(x)\frac{t^n}{n!} &= \left(\frac{t}{e_\lambda(t)-1}\right)^r e_\lambda^x(t)\\
&= \frac{1}{(e_\lambda(t)-1)^{r-1}}\frac{t^r}{e_\lambda(t)-1}e_\lambda^x(t)\\
&= (e_\lambda(t)-1)^{1-r}\frac{t^r}{e_\lambda(t)-1}e_\lambda^x(t)\\
&= \left(\sum_{i=0}^{\infty}\binom{1-r}{i}(-1)^{1-r-i}e_\lambda^i(t)\right)\left(\sum_{l=0}^{\infty}\mathfrak{B}_{l,\lambda}^{[r]}(x)\frac{t^l}{l!}\right)\\
&= \left(\sum_{i=0}^{\infty}\binom{1-r}{i}(-1)^{1-r-i}\sum_{j=0}^{\infty}(i)_{j,\lambda}\frac{t^j}{j!}\right)\left(\sum_{l=0}^{\infty}\mathfrak{B}_{l,\lambda}^{[r]}(x)\frac{t^l}{l!}\right)\\
&= \left(\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\binom{1-r-i}{i}(-1)^{1-r-i}(i)_{j,\lambda}\frac{t^j}{j!}\right)\left(\sum_{l=0}^{\infty}\mathfrak{B}_{l,\lambda}^{[r]}(x)\frac{t^l}{l!}\right)\\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{i=0}^{\infty}\binom{n}{l}\binom{1-r}{i}(-1)^{1-r-i}(i)_{n-l,\lambda}\mathfrak{B}_{l,\lambda}^{[r]}(x)\right)\frac{t^n}{n!}.
\end{aligned}$$

(17)

Therefore, by comparing the coefficients on both sides of (17), we obtain the following theorem.

Theorem 3. *For $n,m \in\in \mathbb{N}\cup\{0\}$, we have*

$$\beta_{n,\lambda}^{(r)}(x) = \sum_{l=0}^{n}\sum_{i=0}^{\infty}\binom{n}{l}\binom{1-r}{i}(-1)^{1-r-i}(i)_{n-l,\lambda}\mathfrak{B}_{l,\lambda}^{[r]}(x).$$

We also observe that

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathfrak{B}_{n,\lambda}^{[r]}(x)\frac{t^n}{n!} &= \left(\frac{t^r}{e_\lambda(t)-1}\right)e_\lambda^x(t)\\
&= (e_\lambda(t)-1)^{r-1}\left(\frac{t}{e_\lambda(t)-1}\right)^r e_\lambda^x(t)\\
&= \left(\sum_{i=0}^{\infty}\binom{r-1}{i}(-1)^{r-1-i}e_\lambda^i(t)\right)\left(\sum_{l=0}^{\infty}\beta_{l,\lambda}^{(r)}(x)\frac{t^l}{l!}\right)\\
&= \left(\sum_{i=0}^{\infty}\binom{r-1}{i}(-1)^{r-1-i}\sum_{m=0}^{\infty}(i)_{m,\lambda}\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}\beta_{l,\lambda}^{(r)}(x)\frac{t^l}{l!}\right)\\
&= \left(\sum_{m=0}^{\infty}(\sum_{i=0}^{\infty}\binom{r-1}{i}(-1)^{r-1-i}(i)_{m,\lambda}\frac{t^m}{m!})\right)\left(\sum_{l=0}^{\infty}\beta_{l,\lambda}^{(r)}(x)\frac{t^l}{l!}\right)\\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{i=0}^{\infty}\binom{n}{l}\binom{r-1}{i}(-1)^{r-1-i}(i)_{n-l,\lambda}\beta_{l,\lambda}^{(r)}(x)\right)\frac{t^n}{n!}.
\end{aligned}$$

(18)

Therefore, by comparing the coefficients on both sides of (18), we obtain the following theorem.

Theorem 4. *For $r \in \mathbb{N}$ and $n \in \mathbb{N}\cup\{0\}$, we have*

$$\mathfrak{B}_{n,\lambda}^{[r]}(x) = \sum_{l=0}^{n}\sum_{i=0}^{\infty}\binom{n}{l}\binom{r-1}{i}(-1)^{r-1-i}(i)_{n-l,\lambda}\beta_{l,\lambda}^{(r)}(x).$$

From (11), we observe

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathfrak{B}_{n,\lambda}^{[r]}(x)\frac{t^n}{n!}=\left(\frac{t^r}{e_\lambda(t)-1}\right)e_\lambda^x(t)\\
&=(e_\lambda(t)-1)^{r-1}\left(\frac{t}{e_\lambda(t)-1}\right)^r e_\lambda^x(t)\\
&=\frac{(r-1)!}{(r-1)!}\Big((e_\lambda(t)-1)^{r-1}\Big)\left(\sum_{l=0}^{\infty}\beta_{l,\lambda}^{(r)}(x)\frac{t^l}{l!}\right)\\
&=\left((r-1)!\sum_{l=r-1}^{\infty}S_{2,\lambda}(l,r-1)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\beta_{m,\lambda}^{(r)}(x)\frac{t^m}{m!}\right)\\
&=\sum_{n=r-1}^{\infty}\left((r-1)!\sum_{l=0}^{n}\binom{n}{l}S_{2,\lambda}(l,r-1)\beta_{n-l,\lambda}^{(r)}(x)\right)\frac{t^n}{n!}.
\end{aligned}$$

(19)

Therefore, by comparing the coefficients on both sides of (19), we obtain the following theorem.

Theorem 5. *For $r\in\mathbb{N}$ and $n\in\mathbb{N}\cup\{0\}$, we have*

$$\mathfrak{B}_{n,\lambda}^{[r]}(x)=\begin{cases}\sum_{l=0}^{n}(r-1)!\binom{n}{l}S_{2,\lambda}(l,r-1)\beta_{n-l,\lambda}^{(r)}(x), & \text{if } n\geq r-1,\\ 0, & \text{if } n<r-1.\end{cases}$$

Now, we observe

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathfrak{B}_{n,\lambda}^{[r]}(x+1)\frac{t^n}{n!}=\left(\frac{t^r}{e_\lambda(t)-1}e_\lambda^x(t)\right)\Big(e_\lambda(t)-1+1\Big)\\
&=\Big(t^r e_\lambda^x(t)\Big)+\left(\frac{t^r}{e_\lambda(t)-1}e_\lambda^x(t)\right)\\
&=t^r\sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{t^l}{l!}+\sum_{n=0}^{\infty}\mathfrak{B}_{n,\lambda}^{[r]}(x)\frac{t^n}{n!}.
\end{aligned}$$

(20)

Since (20),

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\mathfrak{B}_{n,\lambda}^{[r]}(x+1)-\mathfrak{B}_{n,\lambda}^{[r]}(x)\right)\frac{t^n}{n!}=\sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{t^{l+r}}{l!}\\
&=\sum_{l=r}^{\infty}(x)_{l-r,\lambda}\frac{t^l}{(l-r)!}\\
&=\sum_{n=r}^{\infty}(x)_{n-r,\lambda}\binom{n}{r}r!\frac{t^n}{(n)!}.
\end{aligned}$$

(21)

Therefore, by comparing the coefficients on both sides of (21), we obtain the following theorem.

Theorem 6. *For $r\in\mathbb{N}$ and $n\in\mathbb{N}\cup\{0\}$, we have*

$$\mathfrak{B}_{n,\lambda}^{[r]}(x+1)-\mathfrak{B}_{n,\lambda}^{[r]}(x)=(x)_{n-r,\lambda}\binom{n}{r}r!.$$

For $u \in \mathbb{C}$, with $u \neq 1$, we consider the degenerate Frobenius-Euler polynomials which are given by the generating function to be

$$\frac{1-u}{e_\lambda(t)-u}e_\lambda^x(t)\sum_{n=0}^{\infty}\mathfrak{h}_{n,\lambda}(x\mid u)\frac{t^n}{n!}, \quad \text{(see [12])}. \tag{22}$$

When $x=0$, $\mathfrak{h}_{n,\lambda}(u)=\mathfrak{h}_{n,\lambda}(0\mid u)$ are called the degenerate Frobenius-Euler numbers.

From (22), we observe

$$\begin{aligned}
e_\lambda^x(t) &= \frac{e_\lambda(t)-u}{1-u}\frac{1-u}{e_\lambda(t)-u}e_\lambda^x(t) \\
&= \frac{1}{1-u}(e_\lambda(t)-u)\left(\frac{1-u}{e_\lambda(t)-u}e_\lambda^x(t)\right) \\
&= \frac{1}{1-u}\left(\frac{1-u}{e_\lambda(t)-u}e_\lambda^{x+1}(t)\right)-\frac{u}{1-u}\left(\frac{1-u}{e_\lambda(t)-u}e_\lambda^x(t)\right) \\
&= \frac{1}{1-u}\left(\sum_{n=0}^{\infty}\mathfrak{h}_{n,\lambda}(x+1\mid u)\frac{t^n}{n!}\right)-\frac{u}{1-u}\left(\sum_{n=0}^{\infty}\mathfrak{h}_{,\lambda}(x\mid u)\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{1-u}\mathfrak{h}_{n,\lambda}(x+1\mid u)-\frac{u}{1-u}\mathfrak{h}_{,\lambda}(x\mid u)\right)\frac{t^n}{n!}.
\end{aligned} \tag{23}$$

and

$$e_\lambda^x(t)=\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}. \tag{24}$$

Therefore, by comparing the coefficients on both sides of (23) and (24), we obtain the following theorem.

Theorem 7. *For $r \in \mathbb{N}$ and $n \in \mathbb{N}\cup\{0\}$, we have*

$$(x)_{n,\lambda}=\frac{1}{1-u}\mathfrak{h}_{n,\lambda}(x+1\mid u)-\frac{u}{1-u}\mathfrak{h}_{,\lambda}(x\mid u).$$

Now, we observe

$$\begin{aligned}
t^r e_\lambda^x(t) &= \frac{t^r(e_\lambda(t)-u)}{1-u}\frac{1-u}{e_\lambda(t)-u}e_\lambda^x(t) \\
&= \frac{t^r}{1-u}(e_\lambda(t)-u)\left(\sum_{m=0}^{\infty}\mathfrak{h}_{m,\lambda}(x\mid u)\frac{t^m}{m!}\right) \\
&= \frac{t^r}{1-u}\left(\sum_{j=0}^{\infty}(1)_{j,\lambda}\frac{t^j}{j!}-u\right)\left(\sum_{m=0}^{\infty}\mathfrak{h}_{m,\lambda}(x\mid u)\frac{t^m}{m!}\right) \\
&= \frac{t^r}{1-u}\left(\sum_{j=0}^{\infty}(1)_{j,\lambda}\frac{t^j}{j!}\sum_{m=0}^{\infty}\mathfrak{h}_{m,\lambda}(x\mid u)\frac{t^m}{m!}\right)-\frac{u}{1-u}t^r\left(\sum_{m=0}^{\infty}\mathfrak{h}_{m,\lambda}(x\mid u)\frac{t^m}{m!}\right) \\
&= \frac{1}{1-u}\left(\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}(1)_{n-m,\lambda}\mathfrak{h}_{m,\lambda}(x\mid u)\frac{t^{n+r}}{n!}\right)-\frac{u}{1-u}\left(\sum_{n=0}^{\infty}\mathfrak{h}_{n,\lambda}(x\mid u)\frac{t^{n+r}}{n!}\right) \\
&= \sum_{n=r}^{\infty}\left(\frac{1}{1-u}\sum_{l=0}^{n-r}\binom{n-r}{l}(1)_{n-r-m,\lambda}(n)_r\mathfrak{h}_{m,\lambda}(x\mid u)-\frac{u}{1-u}\mathfrak{h}_{n-r,\lambda}(x\mid u)(n)_r\right)\frac{t^n}{n!}.
\end{aligned} \tag{25}$$

On the other hand,

$$
\begin{aligned}
t^r e_\lambda^x(t) &= (e_\lambda(t)-1)\frac{t^r}{e_\lambda(t)-1}e_\lambda^x(t) \\
&= (e_\lambda(t)-1)\left(\sum_{m=0}^{\infty}\mathfrak{B}_{m,\lambda}^{[r]}(x)\frac{t^m}{m!}\right) \\
&= \left(\sum_{l=0}^{\infty}(1)_{l,\lambda}\frac{t^l}{l!}-1\right)\left(\sum_{m=0}^{\infty}\mathfrak{B}_{m,\lambda}^{[r]}(x)\frac{t^m}{m!}\right) \\
&= \left(\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}(1)_{l,\lambda}\mathfrak{B}_{n-l,\lambda}^{[r]}(x)\frac{t^n}{n!}\right)-\left(\sum_{n=0}^{\infty}\mathfrak{B}_{n,\lambda}^{[r]}(x)\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}(1)_{l,\lambda}\mathfrak{B}_{n-l,\lambda}^{[r]}(x)-\mathfrak{B}_{n,\lambda}^{[r]}(x)\right)\frac{t^n}{n!}.
\end{aligned}
\tag{26}
$$

Therefore, by comparing the coefficients on both sides of (25) and (26), we obtain the following theorem.

Theorem 8. *For $r \in \mathbb{N}$ and $n \in \mathbb{N}\cup\{0\}$, we have*

$$
\begin{aligned}
&\sum_{l=0}^{n}\binom{n}{l}(1)_{l,\lambda}\mathfrak{B}_{n-l,\lambda}^{[r]}(x)-\mathfrak{B}_{n,\lambda}^{[r]}(x) \\
&= \begin{cases} \frac{1}{1-u}\sum_{l=0}^{n-r}\binom{n-r}{l}(1)_{n-r-m,\lambda}(n)_r\mathfrak{h}_{m,\lambda}(x\mid u)-\frac{u}{1-u}\mathfrak{h}_{n-r,\lambda}(x\mid u)(n)_r, & \text{if } n \geq r, \\ 0, & \text{if } n < r. \end{cases}
\end{aligned}
$$

3. Conclusion

In this paper, we introduced the generalized degenerate Bernoulli polynomials in (11) and obtained their distribution result Theorem 1. We also obtained some relation identities between the generalized degenerate Bernoulli polynomials and the degenerate Bernoulli polynomials in Theorem 2, and between the generalized degenerate Bernoulli polynomials and the higher-order degenerate Bernoulli polynomials in Theorem 3 5. In particular, we obtained some identities between the degenerate Frobenius-Euler numbers and the generalized Bernoulli polynomials in Theorem 8. In the future, by using the idea for number r of (11), we can study by applying this idea to some special polynomials, for example, the Frobenius-Euler polynomials, Appell-type Changhee polynomials, and the Apostol-Bernoulli polynomials, etc.

References

[1] Carlitz, L. *A degenerate Staudt-Clausen theorem.* Arch. Math. (Basel) **7** (1956), 28-33.

[2] Carlitz, L. *Degenerate Stirling, Bernoulli and Eulerian numbers.* Utilitas Math. **15** (1979), 51-88.

[3] Kim, D. S.; Kim, T. *A note on a new type of degenerate Bernoulli numbers,* Russ. J. Math. Phys. **27** (2020), no. 2, 227-235.

[4] Kim, D. S.; Kim, T. *Some identities of degenerate special polynomials,* Open Math. **13** (2015), Art. 37.

[5] Kim, D. S.; Kim, T.; J. Seo. *Higher-order Bernoulli, Frobenious-Euler and Euler polynomials,* J. Comput. Anal. Appl. **17** (2014),no.1, 147-155.

[6] Kim, T.; Kim, D. S. *Degenerate polyexponential functions and degenerate Bell polynomials,* J. Math. Anal. Appl. **487** (2020), no. 2,124017, DOI 10.1016/j.jmaa.2020.124017.

[7] Kim, T.; Kim, D. S., Kwon, J.; Lee, H. *Degenerate polyexponential functions and type 2 degenerate poly-Bernoulli numbers and polynomials,* Adv. Diff. Equ, (2020), no.1, 1-12. DOI 10.1186/s13662-020-02636-7.

[8] Kim, T.; Kim, D. S. *Identities for degenerate Bernoulli polynomials and Korobov polynomials of the first kind,* Sci. Chana Math. **62** (2019), 62, 999-1028.

[9] Kim, T. *A note on degenerate Stirling polynomials of the second kind,* Proc. Jangjeon Math. Soc.**20** (2017), 62, 319-331.

[10] Kim, D. S.; Kim, T. *Representations by degenerate Bernoulli polynomials arising from Volkenborn integral,* Math. Methods Appl. Sci. **45** (2022), no. 11, 6615-6634.

[11] Kim, T.; Kim, D. S.; Kim, H. K. *On generalized degenerate Euler-Genocchi polynomials,*arXiv:2208.07549v1.

[12] Kim, T.; Kwon, H.; Seo, J. J. *Some identities of degenerate Frobenius-Euler polynomials and numbers,* Proc. Jangjeon Math. Soc. **19** (2016), no. 1, 157-163.

DEPARTMENT OF APPLIED MATHEMATICS, KYUNG HEE UNIVERSITY, SEOUL 02447, REPUBLIC OF KOREA
Email address: wjookim@khu.ac.kr

GRADUATE SCHOOL OF EDUCATION, KONKUK UNIVERSITY, SEOUL 05029, REPUBLIC OF KOREA
Email address: Lcjang@konkuk.ac.kr

DEPARTMENT OF MATHEMATICS EDUCATION, GYEONGSANG NATIONAL UNIVERSITY, JINJU 52828, REPUBLIC OF KOREA
Email address: mathkjk26@gnu.ac.kr