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ALMOST SURE CONVERGENCE OF WEIGHTED SUMS FOR WIDELY NEGATIVE DEPENDENT RANDOM VARIABLES UNDER SUB-LINEAR EXPECTATIONS

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ABSTRACT. The sub-linear expectation space is a nonlinear expectation space having advantages of modeling the uncertainty of probability and statistics. In this paper we study the almost sure convergence for weighted sums of widely negative dependent random variables in the sub-linear expectation spaces. An almost sure convergence theorem is obtained for weighted sums of widely negative dependent random variables under sub-linear expectations. Our results extend and generalize the corresponding ones of Hu and Wu(2021) to widely negative dependent random variables under sub-linear expectation.

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1. INTRODUCTION

Limit theory is an important research topic, which is widely used in the financial sector and other fields in the study of probability theory and mathematical statistics. The classical limit theorems require strict conditions for the certainty model, whereas the certainty model hypothesis is invalid in many areas of applications because the uncertainty phenomenon cannot be explained by the certainty model. Many uncertainty phenomena can not be well modeled by using additive probabilities and additive expectations, such as most of problems in statistics, quantum mechanics, and risk management. Non-additive probabilities and non-additive expectations are useful tools for studying uncertainties in statistics, measure of risk, super-hedge pricing and modeling uncertainty in finance (c.f.[9]-[13]). Motivated by the modeling uncertainty in practice, Peng([10]-[14]) introduced a new notion of sub-linear expectations, and an alternative to the traditional probability and expectation into capacities and sub-linear expectations. At the same time, Peng gave a complete axiom system of sub-linear expectation. It makes up for the lock of application of classical probability space and its theorems in the financial field. Since sub-linear expectation provides a very flexible framework for modeling sub-linear problems, the limit theorems under sub-linear expectation have received more and more attention and research. Peng([12]-[14]) established the central limit theorem and weak law of large numbers, Chen [1], Cheng [3], Hu [4] and Hwang [6] obtained strong law of large numbers, Chen and Hu [2] obtained a law of iterated logarithm, Zhang [18] studied Donsker's invariance principle and Chung's law of the iterated logarithm, and also Zhang([19]-[22]) deeply studied sub-linear expectation space, and established a series of important inequalities such as exponential inequality, Rosenthal's inequality, the moment inequalities for the maximum partial sums, and also Self-normalized moderate deviation. Recently, Hwang [7] obtained strong convergence of sums of independent random variables under sub-linear expectations and Hu and Wu [5] studied almost sure convergence of weighted sums for END sequences in sub-linear expectation spaces, and so on.

In this paper we study the almost sure convergence for weighted sums of widely negative dependent random variables in the sub-linear expectation spaces. An almost sure convergence theorem is obtained for weighted sums of widely negative dependent random variables under sub-linear expectations. Our results extend and generalize the corresponding result of Hu and Wu [5] to widely negative dependent random variables under sub-linear expectation.

This paper is organized as follows: in Section 2, we summarize some basic notations and concepts, related properties under the sub-linear expectations and present the preliminary definitions and lemmas that are useful to obtain the main results. In Section 3, we give the main results including the proof.

2. Preliminaries

We use the framework and notations of Peng([10]-[14]). Let $(\Omega, \mathcal{F})$ be a given measurable space and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, X_2, \cdots, X_n \in \mathcal{H}$ then $\varphi(X_1, X_2, \cdots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}^n)$, where $C_{l,Lip}(\mathbb{R}^n)$ denotes the linear space of local Lipschitz functions φ satisfying

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \leq C(1 + |\mathbf{x}|^m + |\mathbf{y}|^m)|\mathbf{x} - \mathbf{y}|, \forall \mathbf{x},\ \mathbf{y} \in \mathbb{R}^n$$

for some $C > 0$, $m \in \mathbb{N}$ depending on φ. $\mathcal{H}$ is considered as a space of "random variables". In this case we denote $X \in \mathcal{H}$.

Definition 2.1. A *sub-linear expectation* $\widehat{\mathbb{E}}$ on $\mathcal{H}$ is a function $\widehat{\mathbb{E}} : \mathcal{H} \to \bar{\mathbb{R}}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$ we have

(i) Monotonicity: If $X \geq Y$ then $\widehat{\mathbb{E}}[X] \geq \widehat{\mathbb{E}}[Y]$;

(ii) Constant preserving: $\widehat{\mathbb{E}}[c] = c$;

(iii) Sub-additivity: $\widehat{\mathbb{E}}[X + Y] \leq \widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$; whenever $\widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$ is not of the form $+\infty - \infty$ or $-\infty + \infty$;

(iv) Positive homogeneity: $\widehat{\mathbb{E}}[\lambda X] = \lambda \widehat{\mathbb{E}}[X], \quad \lambda \geq 0$

Here $\bar{\mathbb{R}} = [-\infty, \infty]$. The triple $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is called a *sub-linear expectation space.*

Given a sub-linear expectation $\widehat{\mathbb{E}}$, let us denote the conjugate expectation $\widehat{\mathcal{E}}$ of $\widehat{\mathbb{E}}$ by

$$\widehat{\mathcal{E}}[X] = -\widehat{\mathbb{E}}[-X], \qquad \forall X \in \mathcal{H}.$$

From Definition 2.1, it is easily shown that

$$\widehat{\mathcal{E}}[X] \leq \widehat{\mathbb{E}}[X], \quad \widehat{\mathbb{E}}[X + c] = \widehat{\mathbb{E}}[X] + c \ \text{ and } \ \widehat{\mathbb{E}}[X - Y] \geq \widehat{\mathbb{E}}[X] - \widehat{\mathbb{E}}[Y]$$

for all $X, Y \in \mathcal{H}$ with $\widehat{\mathbb{E}}[Y]$ being finite. Further, if $\widehat{\mathbb{E}}[|X|]$ is finite, then $\widehat{\mathbb{E}}[X]$ and $\widehat{\mathcal{E}}[X]$ are both finite, and if $\widehat{\mathbb{E}}[X] = \widehat{\mathcal{E}}[X]$, then $\widehat{\mathbb{E}}[X+aY] = \widehat{\mathbb{E}}[X]+a\widehat{\mathbb{E}}[Y]$ for any $a \in \mathbb{R}$.

Next, we consider the capacities corresponding to the sub-linear expectations. Let $\mathcal{G} \subset \mathcal{F}$. A function $\mathbb{V} : \mathcal{G} \to [0,1]$ is called a *capacity* if $\mathbb{V}(\emptyset) = 0, \mathbb{V}(\Omega) = 1$ and $\mathbb{V}(A \cup B) \leq \mathbb{V}(A) + \mathbb{V}(B)$ for all $A, B \in \mathcal{G}$. It is called *sub-additive* if $\mathbb{V}(A \cup B) \leq \mathbb{V}(A) + \mathbb{V}(B)$ for all $A, B \in \mathcal{G}$ with $A \cup B \in \mathcal{G}$.

Let $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ be a sub-linear space. We denote a pair $(\mathbb{V}, \mathcal{V})$ of capacities by

$$\mathbb{V}(A) := \inf\{\widehat{\mathbb{E}}[\xi] : I_A \leq \xi, \xi \in \mathcal{H}\}, \quad \mathcal{V}(A) = 1 - \mathbb{V}(A^c), \quad \forall A \in \mathcal{F},$$

where A^c is the complement set of A. Then

$$\widehat{\mathbb{E}}[f] \leq \mathbb{V}(A) \leq \widehat{\mathbb{E}}[g], \quad \widehat{\mathcal{E}}[f] \leq \mathcal{V}(A) \leq \widehat{\mathcal{E}}[g],$$

if $f \leq I_A \leq g,\ f, g \in \mathcal{H}$. It is obvious that $\mathbb{V}$ is sub-additive, i.e., $\mathbb{V}(A \cup B) \leq \mathbb{V}(A) + \mathbb{V}(B)$. But $\mathcal{V}$ and $\widehat{\mathcal{E}}$ are not. However, we have

$$\mathcal{V}(A \cup B) \leq \mathcal{V}(A) + \mathbb{V}(B) \quad \text{and} \quad \widehat{\mathcal{E}}[X+Y] \leq \widehat{\mathcal{E}}[X] + \widehat{\mathbb{E}}[Y]$$

due to the fact that

$$\mathbb{V}(A^c \cap B^c) = \mathbb{V}(A^c \setminus B) \geq \mathbb{V}(A^c) - \mathbb{V}(B) \ \text{ and } \ \widehat{\mathbb{E}}[-X-Y] \geq \widehat{\mathbb{E}}[-X] - \widehat{\mathbb{E}}[Y].$$

Further, if X is not in $\mathcal{H}$ and we define $\widehat{\mathbb{E}}$ by $\widehat{\mathbb{E}}[X] = \inf\{\widehat{\mathbb{E}}[Y] : X \leq Y, Y \in \mathcal{H}\}$, then $\mathbb{V}(A) = \widehat{\mathbb{E}}[I_A]$.

In this paper we only consider the capacity generated by a sub-linear expectation. Given a sub-linear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$, we define a capacity:

$$\mathbb{V}(A) := \widehat{\mathbb{E}}[I_A], \quad \forall A \in \mathcal{F}$$

and also define the conjugate capacity:

$$\mathcal{V}(A) := 1 - \mathbb{V}(A^c), \quad \forall A \in \mathcal{F}.$$

It is clear that $\mathbb{V}$ is a sub-additive capacity and $\mathcal{V}(A) = \widehat{\mathcal{E}}[I_A]$.

Definition 2.2. ([21]) (1) A sub-linear expectation $\widehat{\mathbb{E}} : \mathcal{H} \to R$ is called to be *countably sub-additive* if it satisfies

$$\widehat{\mathbb{E}}[X] \leq \sum_{n=1}^{\infty} \widehat{\mathbb{E}}[X_n], \text{ whenever } \ X \leq \sum_{n=1}^{\infty} X_n, X, X_n \in \mathcal{H},$$

where $X \geq 0, X_n \geq 0$ and $n \geq 1$.
(2) A function $\mathbb{V} : \mathcal{F} \to [0,1]$ is called to be *countably sub-additive* if

$$\mathbb{V}(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mathbb{V}(A_n), \ \forall A_n \in \mathcal{F}.$$

Also, we define the Choquet integrals/expectations $(C_{\mathbb{V}}, C_{\mathcal{V}})$ by

$$C_V(X) = \int_0^{\infty} V(X > x)\, dx + \int_{-\infty}^{0} (V(X > x) - 1)\, dx,$$

with V being replaced by $\mathbb{V}$ and $\mathcal{V}$ respectively.

The following definition and example can be found in Wu and Lu [16] (see Wu and Jiang [15]).

Definition 2.3. A sequence of random variables $\{X_n, n \geq 1\}$ is said to *converge to* X *almost surely* V (a.s. V), showed by $X_n \to X$ a.s. V as $n \to \infty$, if $V(X_n \nrightarrow X) = 0$.

V can be replaced by $\mathbb{V}$ and $\mathcal{V}$ severally. By $\mathcal{V}(A) \leq \mathbb{V}(A)$ and $\mathcal{V}(A) + \mathbb{V}(A^c) = 1$ for any $A \in \mathcal{F}$, it is quite clear that $X_n \to X$ a.s. $\mathbb{V}$ implies $X_n \to X$ a.s. $\mathcal{V}$, but $X_n \to X$ a.s. $\mathcal{V}$ does not signify $X_n \to X$ a.s. $\mathbb{V}$. Further, $X_n \to X$ a.s. $\mathbb{V} \Leftrightarrow \mathcal{V}(X_n \to X) = 1 \Leftrightarrow \mathbb{V}(|X_n - X| \geq \epsilon, \text{i.o.}) = 0$, $\forall \epsilon > 0$, and $X_n \to X$ a.s. $\mathcal{V} \Leftrightarrow \mathcal{V}(X_n \nrightarrow X) = 0 \Leftrightarrow \mathbb{V}(X_n \to X) = 1$. In conventional probability space, it is well known $X_n \to X$ a.s. $\Leftrightarrow P(X_n \to X) = 1 \Leftrightarrow P(X_n \nrightarrow X) = 0$ from $P(A) + P(A^c) = 1$. Whereas, in the sub-linear expectation space, the formula $\mathbb{V}(A) + \mathbb{V}(A^c) = 1$ is not valid, which implies $\mathbb{V}(X_n \to X) = 1 \nLeftrightarrow \mathbb{V}(X_n \nrightarrow X) = 0$. Actually, we can have $\mathbb{V}(X_n \nrightarrow X) = 0 \Rightarrow \mathbb{V}(X_n \to X) = 1$, but $\mathbb{V}(X_n \to X) = 1 \nRightarrow \mathbb{V}(X_n \nrightarrow X) = 0$. Thus, in the sub-linear expectation space, $X_n \to X$ a.s. $\mathbb{V}$ cannot be defined with $\mathbb{V}(X_n \to X) = 1$.

Now, we will show an example (see [15],[16]) which satisfies $X_n \to X$ a.s. $\mathcal{V}$ but not $X_n \to X$ a.s. $\mathbb{V}$ as follows.

Example 2.4. Let X_n be independent G-normal random variables with $X_n \sim \mathcal{N}(0, [1/4^{2^n}, 1])$ in a sub-linear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$. $\widehat{\mathbb{E}}$ and $\mathbb{V}$ are continuous. Then $X_n \to 0$ a.s. $\mathcal{V}$; but not $X_n \to 0$ a.s. $\mathbb{V}$.

The following lemmas show that some important inequalities in classical probability theory still hold in sub-linear expectation spaces.

Lemma 2.5. (Markov's inequality) For any $X \in \mathcal{H}$, we have

$$\mathbb{V}(|X| \geq x) \leq \frac{\widehat{\mathbb{E}}[|x|^p]}{x^p}$$

for any $x > 0$ and $p > 0$.

The following lemma is introduced by Zhang [20].

Lemma 2.6. (Borel-Cantelli's lemma) Let $\{A_n, n \geq 1\}$ be a sequence of events in $\mathcal{F}$. Suppose that $\mathbb{V}$ is a countably sub-additive capacity. If $\sum_{n=1}^{\infty} \mathbb{V}(A_n) < \infty$, then $\mathbb{V}(A_n, i.o.) = 0$, where $\{A_n, i.o.\} = \cap_{n=1}^{\infty} \cup_{i=n}^{\infty} A_i$.

Now we give the definition of widely negative dependence on the sublinear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$. The concept of widely negative dependence is introduced by Lin and Feng [8] as follows.

Definition 2.7. Let $X_1, X_2, \cdots, X_{n+1}$ be real measurable random variables of $(\Omega, \mathcal{F})$.
(1) X_{n+1} is called *widely negative dependence* of $(X_1, \cdots, X_n)$ under $\widehat{\mathbb{E}}$ if for every nonnegative measure function φ_i with the same monotonicity on

$\mathbb{R}$ and $\widehat{\mathbb{E}}[\varphi_i(X_i)] < \infty, i = 1, 2, \cdots, n+1$, there exists a positive finite real function $g(n+1)$ such that

$$\widehat{\mathbb{E}}\left[\prod_{i=1}^{n+1} \varphi_i(X_i)\right] \leq g(n+1)\widehat{\mathbb{E}}\left[\prod_{i=1}^{n} \varphi_i(X_i)\right] \widehat{\mathbb{E}}\left[\varphi_{n+1}(X_{n+1})\right].$$

(2) $\{X_i\}_{i=1}^{\infty}$ is said to be *a sequence of widely negative dependent random variables* under $\widehat{\mathbb{E}}$ if for any $n \geq 1$, X_{n+1} is widely negative dependence of $(X_1, X_2, \cdots, X_n)$.
(3) $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ is said to be *an array of rowwise widely negative dependent random variables* under $\widehat{\mathbb{E}}$ if for any $n \geq 1$, $\{X_{ni}, 1 \leq i \leq k_n\}$ is a sequence of widely negative dependent random variables

Remark 2.8. For a sequence of widely negative dependent random variables $\{X_i, i \geq 1\}$, we have

$$\widehat{\mathbb{E}}\left[\prod_{i=1}^{n} \varphi_i(X_i)\right] \leq \tilde{g}(n) \prod_{i=1}^{n} \widehat{\mathbb{E}}\left[\varphi_i(X_i)\right], \text{ where } \tilde{g}(n) = \prod_{i=1}^{n} g(i)$$

for any $n \geq 1$ and every nonnegative measurable function $\varphi_i(\cdot)$ with the same monotonicity on $\mathbb{R}$ and $\widehat{\mathbb{E}}[\varphi_i(X_i)] < \infty, i = 1, 2, \cdots, n$, where $g(\cdot)$ is a positive finite real function as in Definition 2.7(1).

Remark 2.9. Without loss of generality, we will assume that $g(n) \geq 1$ for any $n \geq 1$ in the sequal.

The following lemma is introduced by Lin and Feng [8].

Lemma 2.10. Suppose that $\{X_i\}_{i=1}^{\infty}$ is a sequence of widely negative dependent random variable under $\widehat{\mathbb{E}}$, and $\{\psi_i(x)\}_{i=1}^{\infty}$ is a sequence of measurable function with the same monotonicity. Then $\{\psi_i(X_i)\}_{i=1}^{\infty}$ is also a sequence of widely negative dependent random variables.

It is necessary to note that widely negative dependence under sub-linear expectations is defined through continuous functions in $C_{l,Lip}$ and the indicator function $I(|x| \leq a)$ is not continuous. Therefore, we should modify the indicator function by functions in $I(|x| \leq a)$ to ensure that the sequence of truncated random variables is also widely negative dependence.

For $0 < \mu < 1$, let $h(x) \in C_{l,Lip}(\mathbb{R})$ be an even function such that $h(x)$ is a non-increasing function for any $x > 0$ and $0 \leq h(x) \leq 1$ for all x and $h(x) = 1$ if $|x| \leq \mu$, $h(x) = 0$ if $|x| > 1$, then

$$I(|x| \leq \mu) \leq h(x) \leq I(|x| \leq 1), \quad I(|x| > 1) \leq 1 - h(x) \leq I(|x| > \mu).$$

Throughout this paper, let $\{X_n, n \geq 1\}$ be a sequence of widely negative dependent random variables in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$. C will signify a positive constant that may have different values in different places. $a_n = O(b_n)$ denotes that for a sufficiently large n, there exists $C > 0$ such that $a_n \leq Cb_n$ and $I(\cdot)$ denotes an indicator function.

3. Main Results and Proofs

Hu and Wu [5] gave the result for identical distributed extended negatively dependent random variables in the sub-linear expectation spaces. The result in this paper do not need the random variables to be identically distributed. We extend Theorem 1 in Hu and Wu [5] to widely negative dependent random variables as follows.

Theorem 3.1. Let $\{X_n; n \geq 1\}$ be a sequence of widely negative dependent random variables in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ with $\widehat{\mathbb{E}}[X_n] = \widehat{\mathcal{E}}[X_n] = 0$ and $\widehat{\mathbb{E}}[|X_n|^p] \leq C_{\mathbb{V}}(|X_n|^p)$, and let $\mathbb{V}$ be a countably sub-additive capacity. There exist a r.v. X and a constant C satisfying

$$\widehat{\mathbb{E}}[h(X_n)] \leq C\widehat{\mathbb{E}}[h(X)] \quad \textit{for all } n \geq 1,\ 0 \leq h \in C_{l,Lip}(\mathbb{R}) \tag{1}$$

and

$$C_{\mathbb{V}}(|X|^{1/\beta}) < \infty \quad \textit{for some } 0 < \beta \leq 1. \tag{2}$$

Suppose that $\tilde{g}(x)$ is a nondecreasing positive function on $[0, \infty)$ such that

$$\tilde{g}(x) = \tilde{g}(n) \textit{ when } x = n, \quad \tilde{g}(0) = 1 \quad \textit{and} \quad \frac{\tilde{g}(x)}{x^\tau} \downarrow \quad \textit{for some } 0 < \tau < 1. \tag{3}$$

Let $\{a_{nk}; 1 \leq k \leq n, n \geq 1\}$ be an array of real numbers satisfying

$$\max_{1 \leq k \leq n} |a_{nk}| = O(n^{-\beta}) \tag{4}$$

and

$$\sum_{k=1}^n |a_{nk}|^p = O(n^{-\alpha}) \quad \textit{for some } \alpha > 0, \tag{5}$$

where $p = \min\{1/\beta, 2\}$. Then we have

$$T_n = \sum_{k=1}^n a_{nk} X_k \longrightarrow 0 \quad \text{a.s.} \quad \mathbb{V}, \quad n \to \infty. \tag{6}$$

Proof. Note that $a_{nk} = a_{nk}^+ - a_{nk}^-$, where $a_{nk}^+ = \max\{0, a_{nk}\}$ and $a_{nk}^- = \max\{0, -a_{nk}\}$, and then

$$T_n = \sum_{k=1}^n a_{nk} X_k = \sum_{k=1}^n a_{nk}^+ X_k - \sum_{k=1}^n a_{nk}^- X_k.$$

To prove the result, we need to show that

$$T_n = \sum_{k=1}^n a_{nk}^+ X_k \longrightarrow 0 \quad \text{a.s.} \quad \mathbb{V}, \quad n \to \infty \tag{7}$$

and

$$T_n = \sum_{k=1}^n a_{nk}^- X_k \longrightarrow 0 \quad \text{a.s.} \quad \mathbb{V}, \quad n \to \infty. \tag{8}$$

Without loss of generality, we may assume that $a_{nk} > 0$ for $1 \leq k \leq n,\ n \geq 1$. It suffices to prove that (7) holds, because a slight change in the proof of (7) actually shows that (8) holds. For any $\epsilon > 0$, set $N = [4/\alpha + 1]$, $A_{nk} = a_{nk}^{-1} n^{-\alpha\beta/2}$ and $B_k = \epsilon k^{\beta}/(NC)$ for $1 \le k \le n$. Let

$$\begin{aligned} X_{nk}^{(1)} &= X_k I(X_k \le A_{nk}) + A_{nk} I(X_k > A_{nk}), \\ X_{nk}^{(2)} &= X_k I(X_k > B_k), \\ X_{nk}^{(3)} &= X_k - X_{nk}^{(1)} - X_{nk}^{(2)} \\ &= (X_k - A_{nk}) I(A_{nk} < X_k \le B_k) - A_{nk} I(X_k > B_k), \\ T_n^{(i)} &= \sum_{k=1}^{n} a_{nk} X_{nk}^{(i)}, \quad i = 1, 2, 3. \end{aligned} \tag{9}$$

Let $Z_{nk} = n^{\alpha\beta/2} a_{nk} X_{nk}^{(1)}, n \ge 1$, then $\{Z_{nk}, 1 \le k \le n\}$ is a sequence of widely negative dependent random variables in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ from Lemma 2.10, and $Z_{nk} \le n^{\alpha\beta/2} a_{nk} A_{nk} = 1$ for $1 \le k \le n, n \ge 1$. Since $X_{nk}^{(1)} \le X_k, 1 \le k \le n$ and $\widehat{\mathbb{E}}[X_n] = 0, n \ge 1$, we have

$$\widehat{\mathbb{E}}[Z_{nk}] = n^{\alpha\beta/2} a_{nk} \widehat{\mathbb{E}}[X_{nk}^{(1)}] \le n^{\alpha\beta/2} a_{nk} \widehat{\mathbb{E}}[X_k] = 0. \tag{10}$$

On the other hand, since $e^z \le 1 + z + |z|^p, z \le 1, 1 \le p \le 2$, we have

$$e^{Z_{nk}} \le 1 + Z_{nk} + |Z_{nk}|^p. \tag{11}$$

Note that $1 + z \le e^z$ for $z \in \mathbb{R}$, then we get from (2), (10) and (11) that

$$\begin{aligned} \widehat{\mathbb{E}}[e^{Z_{nk}}] &\le 1 + \widehat{\mathbb{E}}[Z_{nk}] + \widehat{\mathbb{E}}[|Z_{nk}|^p] \\ &\le 1 + \widehat{\mathbb{E}}[|Z_{nk}|^p] \\ &\le \exp\left(\widehat{\mathbb{E}}[|Z_{nk}|^p]\right) \\ &\le \exp\left(\widehat{\mathbb{E}}[|X|^p]\right). \end{aligned} \tag{12}$$

Since $p < 1/\beta$ and $C_{\mathbb{V}}\left(|X|^{1/\beta}\right) < \infty$, we have

$$\widehat{\mathbb{E}}[|X|^p] \le C_{\mathbb{V}}(|X|^p) < \infty. \tag{13}$$

From Definition 2.7, (1), (3), (5), (12) and (13), we have

$$
\begin{aligned}
\widehat{\mathbb{E}}\left[\exp\left(n^{\alpha\beta/2}T_n^{(1)}\right)\right] &= \widehat{\mathbb{E}}\left[\prod_{k=1}^{n}\exp(Z_{nk})\right] \\
&\le \tilde{g}(n)\prod_{k=1}^{n}\widehat{\mathbb{E}}\left[\exp(Z_{nk})\right] \\
&\le \tilde{g}(n)\prod_{k=1}^{n}\exp\left(C\widehat{\mathbb{E}}\left[|Z_{nk}|^p\right]\right) \\
&= \tilde{g}(n)\prod_{k=1}^{n}\exp\left(Cn^{p\alpha\beta/2}a_{nk}^p\widehat{\mathbb{E}}\left[\left|X_{nk}^{(1)}\right|^p\right]\right) \\
&\le \tilde{g}(n)\exp\left(Cn^{\alpha/2}\sum_{k=1}^{n}a_{nk}^p\widehat{\mathbb{E}}\left[|X_k|^p\right]\right) \\
&\le \tilde{g}(n)\exp\left(Cn^{-\alpha/2}\widehat{\mathbb{E}}\left[|X|^p\right]\right) \\
&\le Cn^{\tau}\exp\left(Cn^{-\alpha/2}\right) \\
&\le Cn^{\tau}
\end{aligned}
\tag{14}
$$

for sufficiently large n, where C is a constant. For any $\epsilon > 0$, we have, for sufficiently large n,

$$\epsilon n^{\alpha\beta/2} > \ln n^2.$$

By Lemma 2.5 (Markov's inequality) and (14), we have

$$
\begin{aligned}
\sum_{n=1}^{\infty}\mathbb{V}\left(T_n^{(1)} \ge \epsilon\right) &\le \sum_{n=1}^{\infty}\exp\left\{-\epsilon n^{\alpha\beta/2}\right\}\widehat{\mathbb{E}}\left[\exp\left(n^{\alpha\beta/2}T_n^{(1)}\right)\right] \\
&\le C\sum_{n=1}^{\infty}n^{\tau}\exp\left\{-\epsilon n^{\alpha\beta/2}\right\} \\
&\le C\sum_{n=1}^{\infty}\exp\left\{-(2-\tau)\ln n\right\} \\
&= C\sum_{n=1}^{\infty}n^{-(2-\tau)} < \infty,
\end{aligned}
\tag{15}
$$

where $0 < \tau < 1$. From Lemma 2.6 (Borel-Cantelli's lemma) and $\mathbb{V}$ being the countably sub-additivity, we have for any $\epsilon > 0$

$$\mathbb{V}\left(\limsup_{n\to\infty}T_n^{(1)} \ge \epsilon\right) \le \mathbb{V}\left(T_n^{(1)} \ge \epsilon, \text{i.o.}\right) = 0,$$

and hence

$$\limsup_{n\to\infty}T_n^{(1)} \le 0 \quad \text{a.s.} \quad \mathbb{V} \tag{16}$$

Note that

$$C_{\mathbb{V}}\left(|X|^{1/\beta}\right) = \int_0^{\infty}\mathbb{V}\left(|X|^{1/\beta} > x\right)dx = \int_0^{\infty}\mathbb{V}\left(|X| > x^{\beta}\right)dx,$$

then

$$C_{\mathbb{V}}\left(|X|^{1/\beta}\right)<\infty \qquad \Longleftrightarrow \qquad \sum_{n=1}^{\infty}\mathbb{V}\left(|X|>n^{\beta}\right)<\infty.$$

Also, we have

$$\sum_{n=1}^{\infty}\mathbb{V}\left(|X|>n^{\beta}\right)<\infty \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty}\mathbb{V}\left(|X|>cn^{\beta}\right)<\infty, \quad \forall c>0 \tag{17}$$

(See [15], [16] for more details). From (1), (2) and (17), we have

$$\begin{aligned}
\sum_{k=1}^{\infty}\mathbb{V}\left(|X_k|>B_k\right) &= \sum_{k=1}^{\infty}\mathbb{V}\left(|X_k|>\frac{\epsilon k^{\beta}}{NC}\right)\\
&\le \sum_{k=1}^{\infty}\widehat{\mathbb{E}}\left[1-h\left(\frac{NC}{\epsilon k^{\beta}}X_k\right)\right]\\
&\le C\sum_{k=1}^{\infty}\widehat{\mathbb{E}}\left[1-h\left(\frac{NC}{\epsilon k^{\beta}}X\right)\right]\\
&\le C\sum_{k=1}^{\infty}\mathbb{V}\left(|X|>\mu\frac{\epsilon k^{\beta}}{NC}\right)<\infty.
\end{aligned}\tag{18}$$

By Lemma 2.6, (18) and the countably sub-additivity of $\mathbb{V}$, we have

$$\mathbb{V}\left(|X_k|>B_k, \mathrm{i.o}\right)=0,$$

and hence

$$\left(\sum_{k=1}^{\infty}X_k^2 I\left(|X_k|>B_k\right)=\infty\right)\subset\left(|X_k|>B_k,\ \mathrm{i.o.}\right), \tag{19}$$

it follows from (19) that

$$\sum_{k=1}^{\infty}X_k^2 I\left(|X_k|>B_k\right)<\infty \qquad \text{a.s.} \quad \mathbb{V}. \tag{20}$$

From Schwarz's inequality, (5) and (20), we have

$$\begin{aligned}
T_n^{(2)}=\sum_{k=1}^{n}a_{nk}X_{nk}^{(2)} &\le \left(\sum_{k=1}^{n}a_{nk}^2\right)^{1/2}\left(\sum_{k=1}^{n}X_k^2 I\left(|X_k|>B_k\right)\right)^{1/2}\\
&\le \left(\sum_{k=1}^{n}a_{nk}^p\right)^{1/2}\left(\sum_{k=1}^{\infty}X_k^2 I\left(|X_k|>B_k\right)\right)^{1/2}\\
&\le Cn^{-\alpha/2}\left(\sum_{k=1}^{\infty}X_k^2 I\left(|X_k|>B_k\right)\right)^{1/2}\\
&\longrightarrow 0 \qquad \text{a.s.} \quad \mathbb{V}, \quad n\to\infty.
\end{aligned}\tag{21}$$

Our next claim is that

$$\limsup_{n\to\infty} T_n^{(3)} \le 0 \qquad \text{a.s.} \quad \mathbb{V}. \tag{22}$$

By (4), we have

$$
\begin{aligned}
a_{nk}X_{nk}^{(3)} &\le a_{nk}(X_k - A_{nk})I\left(A_{nk} < X_k \le B_k\right) \\
&\le a_{nk}B_k \le Cn^{-\beta}\frac{\epsilon k^{\beta}}{NC} \\
&\le Ck^{-\beta}\frac{\epsilon k^{\beta}}{NC} = \frac{\epsilon}{N},
\end{aligned} \tag{23}
$$

where $0 < \epsilon < 1$. By the definition of $X_{nk}^{(3)}$ and (23), we get that if $X_k \notin (A_{nk}, B_k]$, then $a_{nk}X_{nk}^{(3)} \le 0$; if $X_k \in (A_{nk}, B_k]$, then $a_{nk}X_{nk}^{(3)} \le \epsilon/N$. So in order to make $T_n^{(3)} = \sum_{k=1}^n a_{nk}X_{nk}^{(3)} > \epsilon$, there must exist at least a positive integer N indices k such that $A_{nk} < X_k \le B_k$, which yields, for any $\epsilon > 0$,

(24)

$$
\begin{aligned}
&\left\{T_n^{(3)} \ge \epsilon\right\} \\
&= \left\{\sum_{k=1}^n a_{nk}(X_k - A_{nk})I\left(A_{nk} < X_k \le B_k\right) - \sum_{k=1}^n a_{nk}A_{nk}I\left(X_k > B_k\right) \ge \epsilon\right\} \\
&\subset \left\{\sum_{k=1}^n a_{nk}(X_k - A_{nk})I\left(A_{nk} < X_k \le B_k\right) \ge \epsilon\right\} \\
&\subset \{\text{there exists at least } N \text{ indices } k \text{ such that } A_{nk} < X_k \le B_k\} \\
&\subset \{\text{there exists at least } N \text{ indices } k \text{ such that} X_k > A_{nk}\}.
\end{aligned}
$$

Since $\{X_n, n \ge 1\}$ is a sequence of widely negative dependent random variables in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ such that there exist a random variable X and a constant C satisfying $\widehat{\mathbb{E}}[h(X_n)] \le C\widehat{\mathbb{E}}[h(X)]$ for all $n \ge 1, 0 \le h \in C_{l,Lip}(\mathbb{R})$, and $\widehat{\mathbb{E}}[|X|^p] \le C_{\mathbb{V}}(|X|^p) < \infty$, we have by (1), (5) and (24)

$$
\begin{aligned}
\mathbb{V}\left(T_n^{(3)} \ge \epsilon\right) &\le \sum_{1\le k_1<\cdots<k_N\le n} \mathbb{V}\left(X_{k_1} > A_{n_{k_1}}, \cdots, X_{k_N} > A_{n_{k_N}}\right) \\
&\le \sum_{1\le k_1<\cdots<k_N\le n} \widehat{\mathbb{E}}\left[\prod_{i=1}^N \left(1 - h\left(\frac{X_{k_i}}{A_{n_{k_i}}}\right)\right)\right] \\
&\le \tilde{g}(N) \sum_{1\le k_1<\cdots<k_N\le n} \prod_{i=1}^N \widehat{\mathbb{E}}\left[1 - h\left(a_{nk_i}n^{\alpha\beta/2}X_{k_i}\right)\right] \\
&\le \tilde{g}(N)\left(\sum_{k=1}^n \widehat{\mathbb{E}}\left[1 - h\left(a_{nk}n^{\alpha\beta/2}X_k\right)\right]\right)^N \\
&\le \tilde{g}(N)\left(C\sum_{k=1}^n \widehat{\mathbb{E}}\left[1 - h\left(a_{nk}n^{\alpha\beta/2}X\right)\right]\right)^N \\
&\le C\tilde{g}(N)\left(\sum_{k=1}^n \mathbb{V}\left(|X| > \mu a_{nk}^{-1}n^{-\alpha\beta/2}\right)\right)^N
\end{aligned} \tag{25}
$$

$$\le C\tilde{g}(n)\left(\sum_{k=1}^{n} a_{nk}^{p} n^{\alpha\beta p/2}\widehat{\mathbb{E}}\left[|X|^{p}\right]\right)^{N}$$
$$\le Cn^{\tau}n^{-\alpha N/2} = Cn^{-\left(\frac{\alpha N}{2}-\tau\right)}.$$

Set $N = [4/\alpha + 1]$, then $\alpha N \ge 4 \Leftrightarrow \frac{\alpha N}{2} \ge 2$, and hence $\frac{\alpha N}{2} - \tau \ge 2 - \tau > 1$ for $0 < \tau < 1$, it follows from (25) that

$$\sum_{n=1}^{\infty} \mathbb{V}\left(T_n^{(3)} \ge \epsilon\right) < \infty$$

for any $\epsilon > 0$. From Borel-Cantelli's lemma and $\mathbb{V}$ being the countably sub-additivity, it follows that (22) holds.
Since $T_n = \sum_{i=1}^{3} T_n^{(i)}$ in (9), it follows from (16), (21) and (22) that we get

$$\limsup_{n\to\infty} T_n \le 0 \qquad \text{a.s.} \quad \mathbb{V}. \tag{26}$$

On the other hands, since $\{X_n; n \ge 1\}$ a sequence of widely negative dependent random variables, then $\{-X_n; n \ge 1\}$ is a sequence of widely negative dependent random variables by the definition of widely negative dependent random variables. Thus if we consider $-X_n$ instead of X_n in the arguments above, then a slight change in the proof actually like as $\widehat{\mathbb{E}}\left(-X_k\right) = -\widehat{\mathcal{E}}\left(X_k\right)$ shows that we have directly the following

$$\liminf_{n\to\infty} T_n \ge 0 \qquad \text{a.s.} \quad \mathbb{V}. \tag{27}$$

Therefore, from (26) and (27), we obtain (6), which completes the proof.

Widely negative dependent random variables include extended negatively dependent random variables in sub-linear expectation space, so for extended negative dependent random variables under sub-linear expectations, the following corollary 3.2 holds and is in Hu and Wu [5].

Corollary 3.2. Let $\{X_n; n \ge 1\}$ be a sequence of identically distributed END random variables in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ with $\widehat{\mathbb{E}}[X_1] = \widehat{\mathcal{E}}[X_1] = 0$, $\widehat{\mathbb{E}}[|X_1|^p] \le C_{\mathbb{V}}(|X_1|^p)$, and

$$C_{\mathbb{V}}(|X_1|^{1/\beta}) < \infty \quad \textit{for some } 0 < \beta \le 1,$$

where $p = \min\{1/\beta, 2\}$, and let $\mathbb{V}$ be a countably sub-additive capacity. Let $\{a_{nk}; 1 \le k \le n, n \ge 1\}$ be an array of real numbers satisfying

$$\max_{1\le k\le n} |a_{nk}| = O(n^{-\beta})$$

and

$$\sum_{k=1}^{n} |a_{nk}|^p = O(n^{-\alpha}) \quad \textit{for some } \alpha > 0,$$

where $p = \min\{1/\beta, 2\}, 0 < \beta \le 1$. Then we have

$$T_n = \sum_{k=1}^{n} a_{nk}X_k \longrightarrow 0 \quad \text{a.s.} \quad \mathbb{V}, \quad n \to \infty.$$

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References

[1] Z.J. Chen, Strong lasw of large numbers for sub-linear expectation, *Sci. China Math.*, **59**(5) (2016), 945–954.

[2] Z.J. Chen & F. Hu, A law of the iterated logarithm under sublinear expectations, *J. Financial Engineering*, **1**(2), Article number: 1450015, (2014). Available at https://doi.org/10.1142/S2345768614500159.

[3] H. Cheng, A strong law of large numbers for sub-linear under a general moment condition, *Stat. & Probab. Lett.*, **116** (2016), 248–258.

[4] C. Hu, Strong laws of large numbers for sublinear expectation under controlled 1st moment condition, *Chines Ann Math. Ser. B*, **39**(5) (2019), 791–804.

[5] R. Hu & Q. Wu, Almost sure convergence of weighted sums for END sequences in sub-linear expectation spaces, *J. Jilin Univ.*, **59**(3) (2021), 531–536.

[6] K.S. Hwang, Note on strong law of large number under sub-linear expectation, *East Asian Math. J.*, **36**(1) (2020), 25–34.

[7] K.S. Hwang, A note on strong convergence of sums of independent random variables under sub-linear expectation, *Advanced Stud. Contem. Math.*, **30**(3) (2020), 435–445.

[8] Y. Lin & X. Feng, Complete convergence and strong law of large numbers for arrays of random variables under sublinear expectations, *Communications in Statistics-Theory & Methods*, **49**(23) (2020), 5866-5882.

[9] M. Marinacci, Limit laws for non-additive probabilities and their frequentist interpretation, *J. Econom, Theory*, **84** (1999), 145–195.

[10] S.G. Peng, G-expectation, G-Brownian motion and related stochastic calculus of Ito type. In: F.E. Benth, et al., *Stoch. Anal. Appli.*, (2006), Proceedings of the 2005 Abel Symposium. Springer, Berlin-Heidelberg, 541–567.

[11] S.G. Peng, Nonlinear expectations and stochastic calculus under uncertainty-with robust central limit theorem and G-Brownian motion, (2010), arXiv:1002.4546v1 [math.PR].

[12] S.G. Peng, A new central limit theorem under sublinear expectations, *J. Math.*, **53**(8) (2008), 1989–1994.

[13] S.G. Peng, Survey on normal distributions, central limit theorem, Brownian motion and the related stochastic calculus under sublinear expectations, *Sci. China Ser. A: Math.*, **52**(7) (2009), 1391–1411.

[14] S.G. Peng, Law of large numbers and central limit theorem under nonlinear expectations, *Probab., Uncertain. & Quanti. risk*, **4**(4) (2019), 1–8.

[15] Q. Wu & Y. Jiang, Strong law of large numbers and Chover's law of the iterated logarithm under sub-linear expectations, *J. Math. Anal. Appl*, **460**(1) (2018), 252–270.

[16] Q. Wu & J. Lu, Another form of Chover's law of the iterated logarithm under sublinear expectations, *Revista de la Real Academia de Ciencias Exactas, Fsicas y Naturales, Serie A. Matematicas*, **114**(1), Article number: 22 (2020).

[17] Z. Zhan & Q. Wu, Strong laws of large numbers for weighted sums of extended negatively dependent random variables under sub-linear expectations, *Communications in Statistics-Theory & Methods*, **51**(5) (2022), 1197–1216.

[18] L.X. Zhang, Donsker's invariance principle under the sub-linear expectation with an application to Chung's law of the iterated logarithm, *Commun. Math. Stat.*, **3** (2015), 287–214.

[19] L.X. Zhang, Exponential inequalities under the sub-linear expectations with applications to laws the iterated logarithm, *Sci. China Math.*, **59**(12) (2016), 2503–2526.

[20] L.X. Zhang, Rosenthal's inequalities for independent and negatively dependent random variables under sub-linear expectations with applications, *Sci. China Math.*, **59**(4) (2016), 751–768.

[21] L.X. Zhang, Self-normalized moderate deviation and laws of the iterated logarithm under g-expectation, *Commun. Math. & Stati.*, **4** (2016), 229–263.

[22] L.X. Zhang, Strong limit theorems for extended independent and extended negatively dependent random variables under non-linear expectations, *Acta Mathematica Scientia*, **42** (2022), 467–490.

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