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A NOTE ON IDENTITIES INVOLVING SPECIAL NUMBERS AND MOMENTS OF POISSON RANDOM VARIABLE

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ABSTRACT. In this paper, we study some identities of special numbers and moments of Poisson random variable in the view of degenerate version. In particular, we give the value of moments of Poisson random variable associated with degenerate special numbers.

1. INTRODUCTION

For $\lambda \in \mathbb{R}$, the degenerate exponential function is defined by

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}, e_\lambda(t) = e_\lambda^1(t), \tag{1}$$

where

$$(x)_{0,\lambda} = 1, (x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda), \quad (n \geq 1), \quad (\text{see } [6,14]). \tag{2}$$

For $r \in \mathbb{R}$, Carlitz considered the degenerate Bernoulli numbers of order r which are given by the generating function to be

$$\left(\frac{t}{e_\lambda(t)-1}\right)^r = \sum_{n=0}^{\infty}\beta_{n,\lambda}^{(r)}\frac{t^n}{n!}, \quad (\text{see } [2,7]). \tag{3}$$

Note that $\lim_{\lambda\to 0}\beta_{n,\lambda}^{(r)} = B_n^{(r)}$, where $B_n^{(r)}$ are ordinary Bernoulli numbers of order r which are given by generating function to be

$$\left(\frac{t}{e^t-1}\right)^r = \sum_{n=0}^{\infty}B_n^{(r)}\frac{t^n}{n!}, \quad (\text{see } [1,5]).$$

It is well known that the Stirling number of the first kind is defined by

$$(x)_n = \sum_{k=o}^{n}S_1(n,k)x^k, (n \geq 0), \quad (\text{see } [3,19]). \tag{4}$$

where $(x)_0 = 1, (x)_n = x(x-1)\cdots(x-n+1), \quad (n \geq 1)$.
The Stirling number of the second kind is defined by

$$x^n = \sum_{k=0}^{n}S_2(n,k)(x)_k, \quad (n \geq 0), \quad (\text{see } [3,19]). \tag{5}$$

Recently, Kim–Kim considered the degenerate Stirling number of the first kind which is defined by

$$(x)_n = \sum_{k=0}^{n}S_{1,\lambda}(n,k)(x)_{k,\lambda}, \quad (n \geq 0), \quad (\text{see } [7]). \tag{6}$$

In the view of inversion formula of (7), the degenerate Stirling number of the second kind is defined by

$$(x)_{n,\lambda} = \sum_{k=0}^{n} S_{2,\lambda}(n,k)(x)_k, \quad (n \geq 0), \quad (\text{see [7]}). \tag{7}$$

From (7), we note that

$$\frac{1}{k!}(e_\lambda(t)-1)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!}, \quad (\text{see [7]}). \tag{8}$$

As is well known, the Bell polynomials are defined by the generating function to be

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} \phi_n(x)\frac{t^n}{n!}, \quad (\text{see [3,15,16,19]}). \tag{9}$$

when $x=1$, $\phi_n = \phi_n(1)$ are called the Bell numbers.
Recently, the degenerate Bell polynomials are defined by Kim–Kim to be

$$e^{x(e_\lambda(t)-1)} = \sum_{n=0}^{\infty} \phi_{n,\lambda}(x)\frac{t^n}{n!} \quad (\text{see [10,14,17]}). \tag{10}$$

From (8) and (10), we note that

$$\phi_{n,\lambda}(x) = \sum_{k=0}^{n} S_{2,\lambda}(n,k)x^k, (n \geq 0), \quad (\text{see [17]}). \tag{11}$$

A random variable X is a real valued function defined on a sample space. If X takes any values in a countable set, then X is called a discrete random variable. If X takes any values in a uncountable set, then X is called the continuous random variable. A random variable X taking on one of the values $0,1,2,\cdots$ is said to be the poisson random variable with parameter $\alpha(>0)$, which is denoted by $X \sim Poi(\alpha)$, if the probability mass function of X is given by

$$P(i) = P\{X=i\} = e^{-\alpha}\frac{\alpha^i}{i!}, \quad i=0,1,2,\cdot, \quad (\text{see [4,20]}). \tag{12}$$

Let $f(x)$ be the probability density function of continuous random variable X. Then, a continuous random variable X is called a uniform random variable on the interval (α,β) if the probability density function of X is given by

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha}, & \text{if } x \in (\alpha,\beta), \\ 0, & \text{if } x \notin (\alpha,\beta), \end{cases} \quad (\text{see [4,20]}). \tag{13}$$

Let $g(x)$ be the real valued function, Then the characteristic function $E[g(X)]$ is defined by

$$E[g(X)] = \begin{cases} \sum_{n=0}^{\infty} g(n)p(n), & \text{if} X \text{is discrete}, \\ \int_{-\infty}^{\infty} g(x)f(x)dx, & \text{if} X \text{is continuous}. \end{cases} \tag{14}$$

In particular, $E[X^n], (n \geq 1)$ is called the $n-th$ moments of random variable X.
The purpose of this paper is to derive some identities connection of few special numbers and moments of certain random variable by using probabilistic methods.

2. Some identities of special number arising from moments of random variable

Suppose that $X_1, X_2, \cdots, X_k$ are identically independent Poisson random variable with mean $\alpha(>0)$. Then we have

$$\begin{aligned}
E\left[e_\lambda^{X_1+X_2+\cdots+X_k}(t)\right] &= E\left[e_\lambda^{X_1}(t)\right]\cdot E\left[e_\lambda^{X_2}(t)\right]\cdots E\left[e_\lambda^{X_k}(t)\right] \\
&= e^{-\alpha}\sum_{l_1=0}^{\infty} e_\lambda^{l_1}(t)\frac{\alpha^{l_1}}{l_1!}\times e^{-\alpha}\sum_{l_2=0}^{\infty} e_\lambda^{l_2}(t)\frac{\alpha^{l_2}}{l_2!}\times\cdots\times e^{-\alpha}\sum_{l_k=0}^{\infty} e_\lambda^{l_k}(t)\frac{\alpha^{l_k}}{l_k!} \\
&= \underbrace{e^{-\alpha}e^{\alpha e_\lambda(t)}\times e^{-\alpha}e^{\alpha e_\lambda(t)}\times\cdots\times e^{-\alpha}e^{\alpha e_\lambda(t)}}_{k-\text{times}} \\
&= e^{-\alpha k}\cdot e^{\alpha k e_\lambda(t)} = e^{\alpha k(e_\lambda(t)-1)} = \sum_{n=0}^{\infty}\phi_{n,\lambda}(\alpha k)\frac{t^n}{n!}.
\end{aligned} \tag{15}$$

On the other hand, by Taylor expansion, we get

$$E\left[e_\lambda^{X_1+X_2+\cdots+X_k}(t)\right] = \sum_{n=0}^{\infty} E\left[(X_1+X_2+\cdots+X_k)_{n,\lambda}\right]\frac{t^n}{n!}. \tag{16}$$

Therefore, by (15) and (16), we obtain the following theorem.

Theorem 1. *For $n \geq 0$, we have*

$$E\left[(X_1+X_2+\cdots+X_k)_{n,\lambda}\right] = \phi_{n,\lambda}(k\alpha),$$

where $X_1, X_2, \cdots, X_k$ are identically independent Poisson random variable with mean α.

From (7), we note that

$$E\left[(X_1+X_2+\cdots+X_k)_{n,\lambda}\right] = \sum_{l=0}^{n} S_{2,\lambda}(n,l)E\left[(X_1+\cdots+X_k)_l\right] \tag{17}$$

Let $X \sim Poi(\alpha)$, Then we have

$$\begin{aligned}
\sum_{n=0}^{\infty} E\left[(X)_n\right]\frac{t^n}{n!} &= E\left[(1+t)^X\right] = e^{-\alpha}\sum_{l=0}^{\infty}(1+t)^l\frac{\alpha^l}{l!} \\
&= e^{-\alpha}\cdot e^{\alpha(1+t)} = e^{\alpha t} = \sum_{n=0}^{\infty}\alpha^n\frac{t^n}{n!}.
\end{aligned} \tag{18}$$

Comparing the coefficients on the both sides of (18), we have

$$E\left[(X)_n\right] = \alpha^n, \quad (n \geq 0). \tag{19}$$

Suppose that $X_1, X_2, \cdots, X_k$ are identically independent Poisson random variable with mean α. Then we have

$$\begin{aligned}
\sum_{n=0}^{\infty} E\left[(X_1+X_2+\cdots+X_k)_n\right]\frac{t^n}{n!} &= E\left[(1+t)^{X_1+\cdots+X_k}\right] \\
&= E\left[(1+t)^{X_1}\right]\times E\left[(1+t)^{X_2}\right]\times\cdots\times E\left[(1+t)^{X_k}\right] \\
&= \sum_{l_1=0}^{\infty}\frac{\alpha^{l_1}}{l_1!}t^{l_1}\times\sum_{l_2=0}^{\infty}\frac{\alpha^{l_2}}{l_2!}t^{l_2}\times\cdots\times\sum_{l_k=0}^{\infty}\frac{\alpha^{l_k}}{l_k!}t^{l_k} \\
&= \sum_{n=0}^{\infty}\left(\alpha^n\sum_{l_1+\cdots+l_k=n}\binom{n}{l_1,l_2,\cdots,l_k}\right)\frac{t^n}{n!}.
\end{aligned} \tag{20}$$

By comparing the coefficients on the both sides of (20), we get

$$E\left[(X_1+X_2+\cdots+X_k)_n\right]=\alpha^n\sum_{l_1+l_2+\cdots+l_k=n}\binom{n}{l_1,l_2,\cdots,l_k}. \tag{21}$$

By (17) and (20), we get

$$\begin{aligned}E\left[(X_1+X_2+\cdots+X_k)_{n,\lambda}\right]&=\sum_{l=0}^{n}S_{2,\lambda}(n,l)E\left[(X_1+X_2+\cdots+X_k)_l\right]\\&=\sum_{l=0}^{n}S_{2,\lambda}(n,l)\alpha^l\sum_{l_1+l_2+\cdots+l_k=l}\binom{l}{l_1,l_2,\cdots,l_k}\end{aligned} \tag{22}$$

Therefore, by **Theorem 1** and (22), we obtain the following theorem.

Theorem 2. *For $n\geq 0$, we have*

$$\phi_{n,\lambda}(k\alpha)=\sum_{l=0}^{n}S_{2,\lambda}(n,l)\alpha^l\sum_{l_1+\cdots+l_k=l}\binom{l}{l_1,l_2,\cdots,l_k}.$$

By (6), we get

$$\begin{aligned}E\left[(X_1+X_2+\cdots+X_k)_n\right]&=\sum_{l=0}^{n}S_{1,\lambda}(n,l)E\left[(X_1+\cdots+X_k)_{l,\lambda}\right]\\&=\sum_{l=0}^{n}S_{1,\lambda}(n,l)\phi_{l,\lambda}(k\alpha).\end{aligned} \tag{23}$$

Therefore, by (20) and (23), we obtain the following theorem.

Theorem 3. *For $n\geq 0$, we have*

$$\alpha^n\sum_{l_1+\cdots+l_k=n}\binom{n}{l_1,l_2,\cdots,l_k}=\sum_{l=0}^{n}S_{1,\lambda}(n,k)\phi_{l,\lambda}(k\alpha).$$

Let U be uniformly random variable on $[0,1]$.
Then we have

$$\begin{aligned}E\left[e^{U(e_\lambda(t)-1)}\right]&=\int_{-\infty}^{\infty}e^{x(e_\lambda(t)-1)}f(x)dx\\&=\int_{-\infty}^{0}e^{x(e_\lambda(t)-1)}f(x)dx+\int_{0}^{1}e^{x(e_\lambda(t)-1)}dx+\int_{1}^{\infty}e^{x(e_\lambda(t)-1)}f(x)dx\\&=\int_{0}^{1}e^{x(e_\lambda(t)-1)}dx=\frac{1}{e_\lambda(t)-1}\left(e^{e_\lambda(t)-1}-1\right)\end{aligned} \tag{24}$$

Let $U_1, U_2, \cdots, U_k$ be uniformly independent random variables on $[0,1]$. Then, by (24), we get

$$\begin{aligned}
&E\left[e^{(U_1+U_2+\cdots+U_k)(e_\lambda(t)-1)}\right]\\
&=E\left[e^{U_1(e_\lambda(t)-1)}\right]\times E\left[e^{U_2(e_\lambda(t)-1)}\right]\times\cdots\times E\left[e^{U_k(e_\lambda(t)-1)}\right]\\
&=\frac{1}{(e_\lambda(t)-1)^k}\left(e^{e_\lambda(t)-1}-1\right)^k=\left(\frac{t}{e_\lambda(t)-1}\right)^k\frac{k!}{t^k}\frac{1}{k!}\left(e^{e_\lambda(t)-1}-1\right)^k\\
&=\frac{k!}{t^k}\left(\sum_{j=0}^{\infty}\beta_{j,\lambda}^{(k)}\frac{t^j}{j!}\right)\times\sum_{l=k}^{\infty}S_2(l,k)\frac{1}{l!}(e_\lambda(t)-1)^k\\
&=\frac{k!}{t^k}\left(\sum_{j=0}^{\infty}\beta_{j,\lambda}^{(k)}\frac{t^j}{j!}\right)\times\sum_{l=k}^{\infty}S_2(l,k)\sum_{m=l}^{\infty}S_{2,\lambda}(m,l)\frac{t^m}{m!}\\
&=\frac{k!}{t^k}\left(\sum_{j=0}^{\infty}\beta_{j,\lambda}^{(k)}\frac{t^j}{j!}\right)\times\sum_{m=k}^{\infty}\left(\sum_{l=k}^{m}S_2(l,k)S_{2,\lambda}(m,l)\right)\frac{t^m}{m!}\\
&=\left(\sum_{j=0}^{\infty}\beta_{j,\lambda}^{(k)}\frac{t^j}{j!}\right)\sum_{m=0}^{\infty}\left(\sum_{l=k}^{m+k}\frac{S_2(l,k)S_{2,\lambda}(m+k,l)}{\binom{m+k}{k}}\right)\frac{t^m}{m!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=k}^{m+k}\frac{\binom{n}{m}}{\binom{m+k}{k}}S_2(l,k)S_{2,\lambda}(m+k,l)\beta_{n-m,\lambda}^{(k)}\right)\frac{t^n}{n!}
\end{aligned}\tag{25}$$

On the other hand, by Taylor expansion, we get

$$\begin{aligned}
&E\left[e^{(U_1+U_2+\cdots+U_k)(e_\lambda(t)-1)}\right]\\
&=\sum_{l=0}^{\infty}E\left[(U_1+U_2+\cdots+U_k)^l\right]\frac{1}{l!}(e_\lambda(t)-1)^l\\
&=\sum_{l=0}^{\infty}E\left[(U_1+U_2+\cdots+U_k)^l\right]\sum_{n=l}^{\infty}S_{2,\lambda}(n,l)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}S_{2,\lambda}(n,l)E\left[(U_1+U_2+\cdots+U_k)^l\right]\right)\frac{t^n}{n!}
\end{aligned}\tag{26}$$

Therefore, by (25) and (26), we obtain the following theorem.

Theorem 4. *Let $U_1, U_2, \cdots, U_k$ be uniformly independent random variables on $[0,1]$. For $n \geq 0$, we have*

$$\begin{aligned}
&\sum_{l=0}^{n}S_{2,\lambda}(n,l)E\left[(U_1+\cdots+U_k)^l\right]\\
&=\sum_{m=0}^{n}\sum_{l=k}^{n}\frac{\binom{n}{m}}{\binom{m+k}{k}}S_2(l,k)S_{2,\lambda}(m+k,l)\beta_{n-m,\lambda}^{(k)}
\end{aligned}$$

Note that

$$E\left[e^{(U_1+U_2+\cdots+U_k)(e_\lambda(t)-1)}\right]=\frac{1}{(e_\lambda(t)-1)^k}\left(e^{e_\lambda(t)-1}-1\right)^k\tag{27}$$

Thus, by (27), we get

$$
\begin{aligned}
\left(e^{e_\lambda(t)-1}-1\right)^k &= (e_\lambda(t)-1)^k E\left[e^{(U_1+U_2+\cdots+U_k)(e_\lambda(t)-1)}\right] \\
&= \sum_{l=0}^{\infty}(e_\lambda(t)-1)^k E\left[(U_1+\cdots+U_k)^l\right]\frac{1}{l!}(e_\lambda(t)-1)^l \\
&= k!\sum_{l=k}^{\infty}E\left[(U_1+\cdots+U_k)^{l-k}\right]\frac{l!}{(i-k)!k!}\frac{1}{l!}(e_\lambda(t)-1)^l \\
&= k!\sum_{l=k}^{\infty}E\left[(U_1+\cdots+U_k)^{l-k}\right]\binom{l}{k}\sum_{n=l}^{\infty}S_{2,\lambda}(n,l)\frac{t^n}{n!} \\
&= k!\sum_{n=k}^{\infty}\left(\sum_{l=k}^{n}\binom{l}{k}E\left[(U_1+\cdots+U_k)^{l-k}\right]S_{2,\lambda}(n,l)\right)\frac{t^n}{n!}
\end{aligned} \tag{28}
$$

From (28), we have

$$
\begin{aligned}
&\sum_{n=k}^{\infty}\left(\sum_{l=k}^{n}E\left[(U_1+\cdots+U_k)^{n-k}\right]\binom{l}{k}S_{2,\lambda}(n,l)\right)\frac{t^n}{n!} \\
&= \frac{1}{k!}\left(e^{e_\lambda(t)-1}-1\right)^k = \sum_{l=k}^{\infty}S_2(l,k)\frac{1}{l!}(e_\lambda(t)-1)^l \\
&= \sum_{l=k}^{\infty}S_2(l,k)\sum_{n=l}^{\infty}S_{2,\lambda}(n,l)\frac{t^n}{n!} = \sum_{n=k}^{\infty}\left(\sum_{l=k}^{n}S_{2,\lambda}(n,l)S_2(l,k)\right)\frac{t^n}{n!}
\end{aligned} \tag{29}
$$

Therefore, by comparing the coefficients on the both sides of (29), we obtain the following theorem.

Theorem 5. *Let $u_1,u_2,\cdots,u_k$ be uniformly independent random variable on* $[0,1]$. *For $n,k\geq 0$ with $n\geq k$, we have*

$$
\sum_{l=k}^{n}S_{2,\lambda}(n,l)S_2(l,k) = \sum_{l=k}^{n}E\left[(U_1+U_2+\cdots+U_k)^{n-k}\right]\binom{l}{k}S_{2,\lambda}(n,l)
$$

Note that

$$
\begin{aligned}
E\left[(U_1+\cdots+U_k)^{n-k}\right] &= \sum_{l_1+l_2+\cdots+l_k=n-k}\binom{n}{l_1,\cdots,l_k}E[U_1^{l_1}]\cdots E[U_k^{l_k}] \\
&= \sum_{l_1+\cdots+l_k=n-k}\binom{n}{l_1,\cdots,l_k}\frac{1}{l_1+1}\cdot\frac{1}{l_2+1}\cdots\frac{1}{l_k+1} = \sum_{l_1+\cdots+l_k=n}\binom{n-k}{l_1,l_2,\cdots,l_k}
\end{aligned}
$$

Corollary 6. *For $n,k\geq 0$, we have*

$$
\begin{aligned}
&\sum_{l=k}^{n}\binom{l}{k}\sum_{l_1+\cdots+l_k=n}\binom{n-k}{l_1,l_2,\cdots,l_k}S_{2,\lambda}(n,l) \\
&= \sum_{l=k}^{n}S_{2,\lambda}(n,l)S_2(l,k)
\end{aligned}
$$

It is known that the cauchy number $C_n,(n\geq 0)$, is defined by

$$
\int_0^1 (x)_n\,dx = C_n,(n\geq 0) \tag{30}
$$

Thus, by (30), we get

$$\sum_{n=o}^{\infty} C_n \frac{t^n}{n!} = \int_0^1 (1+t)^x dx = \frac{t}{\log(1+t)} \tag{31}$$

Let U be uniform random variable on $[0,1]$. Then we have

$$E\left[(1+t)^U\right] = \int_0^1 (1+t)^x dx = \frac{t}{\log(1+t)} = \sum_{n=o}^{\infty} C_n \frac{t^n}{n!} \tag{32}$$

Let $\log_\lambda t$ be the compositional inverse function of $e_\lambda(t)$ such that $e_\lambda(\log_\lambda(t)) = \log_\lambda(e_\lambda(t)) = t$. Then we have

$$\log_\lambda(1+t) = \sum_{n=1}^{\infty} \frac{(1)_{n,\frac{1}{\lambda}} \lambda^{n-1}}{n!} t^n, \quad (\text{see [7]}). \tag{33}$$

From (6), we have

$$\frac{1}{k!}(\log_\lambda(1+t))^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k) \frac{t^n}{n!}, \quad (\text{see [7]}). \tag{34}$$

By (34), we get

$$\begin{aligned} E[(1+t)^U] &= E\left[e_\lambda^U(\log_\lambda(1+t))\right] \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}(\log_\lambda(1+t))^k E[(U)_{k,\lambda}] \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} S_{1,\lambda}(n,k) E[(U)_{k,\lambda}] \right) \frac{t^n}{n!} \end{aligned} \tag{35}$$

Therefore, by (32) and (35), we obtain the following theorem.

Theorem 7. *For $n \geq 0$, we have*

$$C_n = \sum_{k=0}^{n} S_{1,\lambda}(n,k) E[(U)_{k,\lambda}],$$

Where U is uniform random variable on the interval $[0,1]$.

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