Advanced Studies in Contemporary Mathematics
32 (2022), No. 4, pp. 443 - 456
www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2022.32.4.443

SOME IDENTITIES OF DEGENERATE r-DOWLING POLYNOMIALS OF THE SECOND KIND ARISING FROM UMBRAL CALCULUS

HYE KYUNG KIM

Abstract. Recently, the author Kim and Lee [16] introduced interesting properties and identities for the degenerate r-Dowling polynomials and numbers associated with the degenerate r-Whitney numbers of the second kind. In this paper, we study methods for computing the rational coefficients of a linear combination of the degenerate r-Dowling polynomials of the second kind with degree n by using umbral calculus, for algebraic applications of [16]. We derive some interesting identities for certain special polynomials from these coefficients. Furthermore, we explore various identities of the degenerate r-Dowling polynomials arising from the falling factorials bases, the Euler polynomials bases, the Daehee polynomials bases, the degenerate r-Bell polynomials bases, the Bell polynomials bases by using umbral calculus, respectively.

2010 Mathematics Subject Classification. 05A15, 05A18, 11B83.

Keywords and phrases. degenerate r-Whitney numbers and polynomials of the second, degenerate r-Dowling numbers and polynomials of the second, falling factorials, r-Bell polynomials, umbral calculus.

1. Introduction

As a generalization of the Whitney numbers $w_m(n,k)$ and $W_m(n,k)$ of the first and second kind associated with the Dowling lattice $Q_n(G)$ for a group G with order m [1, 9], Mezö [22] introduced r-Whitney numbers of the first and second kind given by

$$m^n(x)_n=\sum_{k=0}^{n}w_{m,r}(n,k)(mx+r)^k,$$

and

$$(mx+r)^n=\sum_{k=0}^{n}W_{m,r}(n,k)m^k(x)_k, \tag{1}$$

respectively. When $r=1$, $w_m(n,k)=w_{m,1}(n,k)$ and $W_m(n,k)=W_{m,1}(n,k)$.

The r-Whitney numbers of the first and second kind are applied to various fields such as physics and engineering as well as mathematical applications [2, 5, 6, 11, 12, 15, 20, 21, 24-26]. In addition, many scholars have studied degenerate special polynomials and numbers to which the strength of psychological burdens or environmental changes can be applied [4, 13-17, 20]. Recently, the degenerate r-Dowling polynomials $D_{m,r,\lambda}(n,x)$ and numbers

This work was supported by the Basic Science Research Program, the National Research Foundation of Korea, (NRF-2021R1F1A1050151).

$D_{m,r,\lambda}(n)$ of the second kind respectively associated with the degenerate r-Whitney numbers of the second kind were studied in [16, 21]. We note that $D_{m,r,\lambda}(n,x)$ are polynomials of degree n with rational coefficients for all non-negative integer n. Thus, for each n, $\{D_{m,r,\lambda}(0,x),\ D_{m,r,\lambda}(1,x)\ \cdots,\ D_{m,r,\lambda}(n,x)\}$ forms bases for the $(n+1)$-dimensional space $\mathbb{P}_n(\mathbb{C}) = \{p(x) \in \mathbb{C}[x] |\deg p(x) \leq n\}$. Thus, we can express $p(x)$ by

$$p(x) = \alpha_{0,r} D_{m,r,\lambda}(0,x) + \alpha_{1,r} D_{m,r,\lambda}(1,x) + \cdots + \alpha_{n,r} D_{m,r,\lambda}(n,x).$$

In this paper, we study methods for computing $\alpha_{i,r}$ $(i = 0,1,2\cdots,n)$ by using umbral calculus. Applying this result, we derive some interesting identities for certain special polynomials. In addition, we derive interesting identities of the degenerate r-Dowling polynomials which derived from the falling factorials bases, the Euler polynomials bases, Daehee polynomials bases, degenerate r-Bell polynomials bases, bell polynomials bases by using umbral calculus, respectively.

First, we introduce the basic definitions and properties of the degenerate r-Dowling polynomials and umbral calculus needed in this paper.

For any $\lambda \in \mathbb{R} - \{0\}$, the degenerate exponential function $e_\lambda^x(t)$ is given by

$$(2)\qquad e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \quad \text{(see [4-17])},$$

where $(x)_{n,\lambda} = x(x-\lambda)\cdots\big(x-(n-1)\lambda)\big)$, $(n \geq 1)$ and $(x)_{0,\lambda} = 1$. When $\lambda = 1$, $(x)_0 = 1$ and $(x)_n = x(x-1)\cdots\big(x-(n-1))\big)$, $(n \geq 1)$.

The degenerate logarithm function $\log_\lambda(1+t)$, which is the compositional inverse of the degenerate exponential function $e_\lambda(t)$, is given by

$$(3)\qquad \begin{aligned} \log_\lambda(1+t) &= \sum_{n=1}^{\infty} \lambda^{n-1} (1)_{n,1/\lambda} \frac{t^n}{n!} \\ &= \frac{1}{\lambda} \sum_{n=1}^{\infty} (\lambda)_n \frac{t^n}{n!} = \frac{1}{\lambda}\big((1+t)^\lambda - 1\big), \quad \text{(see [14])}. \end{aligned}$$

For $r \geq 0$ and $m \geq 1$, from (1), it is easy to see that the generating function of the degenerate r-Whitney numbers of the second kind is

$$(4)\qquad \sum_{n=j}^{\infty} W_{m,r,\lambda}(n,j) \frac{t^n}{n!} = e_\lambda^r(t) \frac{1}{j!} \left(\frac{e_\lambda^m(t) - 1}{m} \right)^j, \quad \text{(see [16, 21])}.$$

For $m \in \mathbb{N}$, the degenerate r-Dowling polynomials of the second kind are given by

$$(5)\qquad D_{m,r,\lambda}(n|x) = \sum_{j=0}^{n} W_{m,r,\lambda}(n,j) x^j, \quad (n \geq 0), \quad \text{(see [16, 21])}$$

and the generating function of degenerate r-Dowling polynomials of the second kind given by

$$(6)\qquad e_\lambda^r(t) e^{x\left(\frac{e_\lambda^m(t)-1}{m}\right)} = \sum_{n=0}^{\infty} D_{m,r,\lambda}(n|x) \frac{t^n}{n!}, \quad \text{(see [21])}.$$

When $x=1$, we get $D_{m,r,\lambda}(n)=D_{m,r,\lambda}(n|1)$ which are called the degenerate r-Dowling numbers of the second kind [16].
When $r=1$, we get $D_{m,\lambda}(n,x)=D_{m,1,\lambda}(n|x)$ which are called the degenerate Dowling polynomials of the second kind [18].

For $n\geq 0$, the Stirling numbers of the first and second kind are given by respectively

(7)
$$(x)_n=\sum_{l=0}^{n}S_1(n,l)x^l,\quad \text{and}\quad \frac{1}{k!}(\log(1+t))^k=\sum_{n=k}^{\infty}S_1(n,k)\frac{t^n}{n!},\quad (\text{see [8, 25]}).$$

and

(8)
$$x^n=\sum_{l=0}^{n}S_2(n,l)(x)_l,\quad \text{and}\quad \frac{1}{k!}(e^t-1)^k=\sum_{n=k}^{\infty}S_2(n,k)\frac{t^n}{n!},\quad (\text{see [8, 25]}).$$

The degenerate Stirling numbers of the first kind are given by

(9)
$$(x)_n=\sum_{l=0}^{n}S_{1,\lambda}(n,l)(x)_{l,\lambda}\quad \text{and}\quad \frac{1}{k!}\big(\log_\lambda(1+t)\big)^k=\sum_{n=k}^{\infty}S_{1,\lambda}(n,k)\frac{t^n}{n!}\quad (k\geq 0),\quad (\text{see [17]}).$$

The degenerate Stirling numbers of the second kind are given by

(10)
$$(x)_{n,\lambda}=\sum_{l=0}^{n}S_{2,\lambda}(n,l)(x)_l\quad \text{and}\quad \frac{1}{k!}\big(e_\lambda(t)-1\big)^k=\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!}\quad (k\geq 0),\quad (\text{see [17]}).$$

It is well known that the ordinary Bell polynomials and the generating function of them are given by

(11)
$$bel_n(x)=\sum_{k=0}^{n}S_2(n,k)x^k,\quad \text{and}\quad e^{x(e^t-1)}=\sum_{n=0}^{\infty}bel_n(x)\frac{t^n}{n!},\quad (\text{see [5, 20, 25]}),$$

respectively.
Let $s\in\mathbb{N}\bigcup\{0\}$, the s-Stirling numbers $S_{2,s}(n,j)$ of the second kind are given by

$$\frac{1}{j!}e^{st}(e^t-1)^j=\sum_{n=j}^{\infty}S_{2,s}(n+s,j+s)\frac{t^n}{n!},\quad (\text{see [3, 4, 21, 28]}).\tag{12}$$

Bell polynomials are also well known in enumerative combinatorics, whose coefficients are Stirling and s-Stirling numbers of the second kind, respectively [8, 4, 28].
Kim et al. studied the unsigned degenerate s-Stirling numbers of the second kind defined by

$$(x+s)_{n,\lambda}=\sum_{j=0}^{n}S_{2,\lambda}^{(s)}(n+s,j+s)(x)_j,\quad (n\geq 0),\quad (\text{see [21]}),\tag{13}$$

and the generating function of degenerate s-Bell polynomials given by

$$e_\lambda^s(t)e^{x(e_\lambda(t)-1)}=\sum_{n=0}^{\infty}Bel_n^{(s)}(x|\lambda)\frac{t^n}{n!},\quad \text{(see [20, 21])}. \tag{14}$$

When $x=1$, $Bel_n^{(s)}(\lambda)=Bel_n^{(s)}(1|\lambda)$ are called the degenerate s-Bell numbers.

The Bernoulli and Euler polynomials are defined by means of

(15)

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}\quad\text{and}\quad\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!},\quad\text{(see [8, 9, 25])}.$$

In the special case, $x=0$, $B_n(0)=B_n$ and $E_n(0)=E_n$ are called the n-th Bernoulli and Euler numbers.

From (15), we note that

$$B_n(x)=\sum_{l=0}^{n}\binom{n}{l}B_lx^{n-l},\quad\text{amd}\quad E_n(x)=\sum_{l=0}^{n}\binom{n}{l}E_lx^{n-l}. \tag{16}$$

and

$$E_n(x)=\sum_{l=0}^{n}\binom{n}{l}E_lx^{n-l}. \tag{17}$$

The Daehee polynomials are given by

$$\frac{\log(1+t)}{t}(1+t)^x=\sum_{n=0}^{\infty}\widetilde{D}_n(x)\frac{t^n}{n!},\quad\text{(see [15])}. \tag{18}$$

When $x=0$, $\widetilde{D}_n=\widetilde{D}_n(0)$ are called the Daehee numbers.

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all power series in the variable t over $\mathbb{C}$ with

$$\mathcal{F}=\left\{f(t)=\sum_{k=0}^{\infty}a_k\frac{t^k}{k!}\;\middle|\;a_k\in\mathbb{C}\right\}.$$

Let $\mathbb{P}=\mathbb{C}[x]$ and $\mathbb{P}_n=\{\,P(x)\in\mathbb{P}\mid degP(x)\le n\}$, $(n\ge 0)$. Then $\mathbb{P}_n$ is an $(n+1)$-dimensional vector space over $\mathbb{C}$.

For $f(t)=\sum_{k=0}^{\infty}a_k\frac{t^k}{k!}\in\mathcal{F}$ and a fixed nonzero real number, each gives rise to the linear functional $\langle f(t)\mid\cdot\rangle$ on $\mathbb{P}$, called linear functional given by $f(t)$, which is defined by

$$\langle f(t)\mid x^n\rangle=a_n,\quad\text{for all }\ n\ge 0\quad\text{(see [24])}. \tag{19}$$

In particular $\langle t^k\mid x^n\rangle=n!\delta_{n,k}$, for all $n,\ k\ \ge 0$, where $\delta_{n,k}$ is the Kronecker's symbol.

We observe that the linear functional $\langle f(t)\mid\cdot\rangle$ agrees with the one in $\langle f(t)\mid x^n\rangle=a_k$, $(k\ge 0)$.

For each nonnegative integer k, the differential operator on $\mathbb{P}$ is given by

$$(t^k)x^n=\begin{cases}(n)_kx^{n-k}, & \text{if}\quad k\le n,\\ 0 & \text{if}\quad k\ge n,\quad\text{(see [24])}.\end{cases} \tag{20}$$

and for any power series $f(t)=\sum_{k=0}^{\infty}a_k\frac{t^k}{k!}\in\mathcal{F}$, $(f(t))x^n=\sum_{k=0}^{n}\binom{n}{k}a_kx^{n-k}$, $(n\geq 0)$.

The order $o(f(t))$ of a power series $f(t)(\neq 0)$ is the smallest integer k for which the coefficient of t^k does not vanish. The series $f(t)$ is called invertible if $o(f(t))=0$ and such series has a multiplicative inverse $1/f(t)$ of $f(t)$. $f(t)$ is called a delta series if $o(f(t))=1$ and it has a compositional inverse $\overline{f}(t)$ of $f(t)$ with $\overline{f}(f(t))=f(\overline{f}(t))=t$ [24].

Let $f(t)$ and $g(t)$ be a delta series and an invertible series, respectively. Then there exists a unique sequences $s_n(x)$ such that the orthogonality conditions holds

$$\langle g(t)\big(f(t)\big)^k \mid s_n(x)\rangle = n!\delta_{n,k},\quad (n,\ k\geq 0)\qquad \text{(see [24])}. \tag{21}$$

By the uniqueness of (21), the sequence $s_n(x)$ is called the Sheffer sequence for $(g(t),\ f(t))$, which are denoted by $s_n(x)\sim(g(t),\ f(t))$.

The sequence $s_n(x)\ \sim\ (g(t),\ f(t))$ if and only if

$$\frac{1}{g\big(\overline{f}(t)\big)}e^{x\left(\overline{f}(t)\right)} = \sum_{k=0}^{\infty}\frac{s_k(x)}{k!}t^k\quad (n,\ k\geq 0),\qquad \text{(see [24])}. \tag{22}$$

Let $s_n(x)\sim(g(t),\ f(t))$ and $r_n(x)\sim(h(t),\ l(t))$, $(n\geq 0)$. Then

(23)

$$s_n(x)=\sum_{k=0}^{n}a_{n,k}r_k(x),\quad (n\geq 0),$$

$$\text{where}\quad a_{n,k}=\frac{1}{k!}\left\langle\frac{h(\overline{f}(t))}{g(\overline{f}(t))}\big(l(\overline{f}(t))\big)^k \mid x^n\right\rangle,\quad (n,\ k\geq 0),\qquad \text{(see [24])}.$$

2. Identities associated with degenerate r-Dowling polynomials by using umbral calculus

From now on, we explore combinatorial identities between degenerate r-Dowling polynomials and special polynomials and numbers by using umbral calculus.

The compositional inverse of

$$f(t)=\log_\lambda(mt+1)^{\frac{1}{m}}=\frac{1}{\lambda}\left((mt+1)^{\frac{\lambda}{m}}-1\right)=\frac{1}{\lambda}\left(\sum_{i=1}^{\infty}\binom{\frac{\lambda}{m}}{i}(mt)^i\right) \tag{24}$$

is $\overline{f}(t)=\frac{1}{m}(e_\lambda^m(t)-1)$.

From (22) and (24), we have the Sheffer sequence

$$D_{m,r,\lambda}(n,x)\sim\left((mt+1)^{-\frac{r}{m}},\ \log_\lambda(mt+1)^{\frac{1}{m}}\right). \tag{25}$$

From (2), we note that

$$e_\lambda^m(t)=(1+\lambda t)^{\frac{m}{\lambda}}=e_{\frac{\lambda}{m}}(mt). \tag{26}$$

By (10) and (26), we have

$$\frac{1}{j!}\left(\frac{e_\lambda^m(t)-1}{m}\right)^j = \frac{1}{m^j}\frac{1}{j!}(e_{\frac{\lambda}{m}}(mt)-1)^j = \sum_{k=j}^{\infty} m^{k-j} S_{2,\frac{\lambda}{m}}(k,j)\frac{t^k}{k!}. \tag{27}$$

Theorem 2.1. *Let $p(x) \in \mathbb{P}_n(\mathbb{C})$ with $p(x) = \sum_{k=0}^{n} \alpha_{k,r} D_{m,r,\lambda}(k,x)$. Then we have*

$$\alpha_{k,r} = \frac{1}{\lambda^k k!}\sum_{j=0}^{k}\sum_{l=0}^{\deg p(x)} \binom{l}{j}\binom{\frac{1+\lambda j}{m}}{l}(-1)^{k-j} m^l p^{(l)}(0),$$

where $p^{(n)}(0) = \left.\frac{d^n p(x)}{dx}\right|_{x=0}$.

Proof. Let $p(x) = \sum_{k=0}^{n} \alpha_{k,r} D_{m,r,\lambda}(k,x)$. Then, from (21) and (25), we observe that

$$\begin{aligned} &\langle (mt+1)^{-\frac{r}{m}}(\log_\lambda (mt+1)^{\frac{1}{m}})^k | p(x)\rangle \\ &\qquad = \sum_{k=0}^{n} \alpha_{k,r} \langle (mt+1)^{-\frac{r}{m}}(\log_\lambda (mt+1)^k) | D_{m,r,\lambda}(k,x)\rangle \\ &\qquad = \sum_{k=0}^{n} \alpha_{k,r} n! \delta_{n,k} = k!\alpha_{k,r}. \end{aligned} \tag{28}$$

By (20) and (28), we have

$$\begin{aligned} \alpha_{k,r} &= \frac{1}{k!}\langle (mt+1)^{-\frac{r}{m}}(\log_\lambda (mt+1)^{\frac{1}{m}})^k | p(x)\rangle \\ &= \frac{1}{\lambda^k k!}\langle (mt+1)^{\frac{1}{m}}((mt+1)^{\frac{\lambda}{m}}-1)^k | p(x)\rangle \\ &= \frac{1}{\lambda^k k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\langle (mt+1)^{\frac{1}{m}+\frac{\lambda}{m}j} | p(x)\rangle \\ &= \frac{1}{\lambda^k k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\sum_{l=0}^{\deg p(x)}\binom{\frac{1+\lambda j}{m}}{l} m^l \langle t^l | p(x)\rangle \\ &= \frac{1}{\lambda^k k!}\sum_{j=0}^{k}\sum_{l=0}^{\deg p(x)}\binom{l}{j}\binom{\frac{1+\lambda j}{m}}{l}(-1)^{k-j} m^l p^{(l)}(0), \end{aligned} \tag{29}$$

where $p^{(n)}(0) = \left.\frac{d^n p(x)}{dx}\right|_{x=0}$.

From (29), we attain the desired result. $\square$

Some Applications of Theorem 1. Let $p^{(n)} = \frac{d^n p(x)}{dx}$. Then we give some Applications of Theorem 1.

(a) Let $p(x)=\sum_{k=0}^{n}B_k(x)B_{n-k}(x)\in\mathbb{P}_n(\mathbb{C})$.
Then, by (16), we easily have

$$\begin{aligned}p^{(1)}(x)&=\sum_{k=1}^{n}kB_{k-1}(x)B_{n-k}(x)+\sum_{k=0}^{n-1}(n-k)B_{n-k-1}(x)B_k(x)\\&=2\sum_{k=1}^{n}kB_{k-1}(x)B_{n-k}(x),\end{aligned}$$

$$p^{(2)}(x)=2(n+1)\sum_{k=2}^{n}(k-1)B_{k-2}(x)B_{n-k}(x),$$

$$\vdots$$

$$p^{(l)}(x)=2\frac{(n+1)!}{(n-l+2)!}\sum_{k=l}^{n}(k-l+1)B_{k-l}(x)B_{n-k}(x). \tag{30}$$

Combining (30) with Theorem 1, we have

$$\begin{aligned}\sum_{k=0}^{n}B_k(x)B_{n-k}(x)=\frac{1}{\lambda^k k!}\sum_{k=1}^{n-1}\Bigg(\sum_{j=0}^{k}\sum_{l=0}^{\deg p(x)}\sum_{k=l}^{n}\binom{l}{j}\binom{\frac{1+\lambda j}{m}}{l}\\ \times\frac{(n+1)!2(k-l+1)(-1)^{k-j}m^l}{(n-l+2)!}B_{k-l}B_{n-k}\Bigg)D_{m,r,\lambda}(k,x).\end{aligned}$$

(b) Let $q(x)=\sum_{k=0}^{n}E_k(x)E_{n-k}(x)\in\mathbb{P}_n(\mathbb{C})$.
Then, by (17), we note that

$$q^{(l)}(x)=\frac{2(n+1)!}{(n+2-l)!}\sum_{k=l}^{n}(k-l+1)E_{k-l}(x)E_{n-k}(x). \tag{31}$$

Combining (31) with Theorem 1, we have

$$\begin{aligned}\sum_{k=0}^{n}E_k(x)E_{n-k}(x)=\frac{1}{\lambda^k k!}\sum_{k=1}^{n-1}\Bigg(\sum_{j=0}^{k}\sum_{l=0}^{\deg q(x)}\sum_{k=l}^{n}\binom{l}{j}\binom{\frac{1+\lambda j}{m}}{l}\\ \times\frac{(n+1)!2(k-l+1)(-1)^{k-j}m^l}{(n-l+2)!}E_{k-l}E_{n-k}\Bigg)D_{m,r,\lambda}(k,x).\end{aligned}$$

(c) Let $u(x)=\sum_{k=0}^{n}B_k(x)E_{n-k}(x)\in\mathbb{P}_n(\mathbb{C})$. Then, by (16), we easily that

$$u^{(1)}(x)=(n+1)\sum_{k=1}^{n}B_{k-1}(x)E_{n-k}(x)$$

$$u^{(2)}(x)=n(n+1)\sum_{k=2}^{n}B_{k-2}(x)E_{n-k}(x).$$

$$\vdots$$

$$u^{(l)}(x)=\frac{(n+1)!}{(n+1-l)!}\sum_{k=l}^{n}B_{k-l}(x)E_{n-k}(x). \tag{32}$$

Combining (32) with Theorem 1, we have

$$\sum_{k=0}^{n} B_k(x)E_{n-k}(x) = \frac{1}{\lambda^k k!}\sum_{k=1}^{n-1}\Bigg(\sum_{j=0}^{k}\sum_{l=0}^{\deg u(x)}\sum_{k=l}^{n}\binom{l}{j}\binom{\frac{1+\lambda j}{m}}{l} \times \frac{(n+1)!(k-l+1)(-1)^{k-j}m^l}{(n-l+1)!}B_{k-l}E_{n-k}\Bigg)D_{m,r,\lambda}(k,x).$$

Gessel [11] showed a short proof of Miki's identity for Bernoulli numbers,

$$\sum_{i=1}^{n-1}\frac{1}{i(n-i)}B_iB_{n-i} = \sum_{i=2}^{n-2}\binom{n}{i}\frac{1}{i(n-i)}B_iB_{n-i} + 2H_n\frac{B_n}{n}, \quad (n \geq 4),$$

where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ are the harmonic numbers.

Theorem 2.2. *For $n \geq 0$, we have*

$$D_{m,r,\lambda}(n,x) = \sum_{d=0}^{n}\Bigg(\sum_{l=d}^{n}\sum_{s=0}^{l}\binom{l}{s}\frac{1}{l!m^l}(-1)^{l-s}(r+ms)_{n,\lambda}S_2(l,d)\Bigg)(x)_d$$

Proof. From (22) and (25), we consider the following two Sheffer sequences:

(33)

$$D_{m,r,\lambda}(m,x) \sim \left((mt+1)^{-\frac{r}{m}},\ \log_\lambda(mt+1)^{\frac{1}{m}}\right) \quad \text{and} \quad (x)_n \sim (1,\ e^t - 1),$$

since $e^{x\log(1+t)} = (1+t)^x = \sum_{n=0}^{\infty}(x)_n\frac{t^n}{n!}$.

From (23) and (33), we have

$$D_{m,r,\lambda}(m,x) = \sum_{d=0}^{n} a_{n,d}(x)_d, \tag{34}$$

where, by (2) and (8)

$$\begin{aligned} a_{n,d} &= \frac{1}{d!}\left\langle e_\lambda^r(t)\left(e^{\frac{e_\lambda^m(t)-1}{m}} - 1\right)^d \Big| x^n\right\rangle \\ &= \sum_{l=d}^{n} S_2(l,d)\frac{1}{l!m^l}\left\langle e_\lambda^r(t)(e_\lambda^m(t)-1)^l | x^n\right\rangle \\ &= \sum_{l=d}^{n} S_2(l,d)\frac{1}{l!m^l}\sum_{s=0}^{l}\binom{l}{s}(-1)^{l-s}\left\langle e_\lambda^{r+ms}(t)|x^n\right\rangle \\ &= \sum_{l=d}^{n}\sum_{s=0}^{l}\binom{l}{s}\frac{1}{l!m^l}(-1)^{l-s}(r+ms)_{n,\lambda}S_2(l,d). \end{aligned} \tag{35}$$

Combining (34) with (35), we get the desired the identity. □

Theorem 2.3. *For $n \geq 0$, we have*

$$D_{m,r,\lambda}(n,x) = \frac{1}{2}\sum_{d=0}^{n}\Bigg(\sum_{l=0}^{d}\binom{d}{l}\frac{(-1)^{d-l}}{d!m^d}\Bigg(\sum_{\alpha=0}^{n}\binom{n}{\alpha}(ml)_{\alpha,\lambda}D_{m,r,\lambda}(n-\alpha) + (ml+r)_{n,\lambda}\Bigg)E_d(x)$$

and

$$D_{m,r,\lambda}(n,x)=\frac{1}{2}\sum_{d=0}^{n}\sum_{l=d}^{n}\binom{n}{l}\left[1+\sum_{j=0}^{n-l}S_{2,\frac{\lambda}{m}}(n-l,j)m^{n-l-j}\right]W_{m,r,\lambda}(l,d)E_d(x),$$

where $E_n(x)$ are the Euler polynomials.

Proof. From (15), (22) and (25), we consider the following two Sheffer sequences:

(36)

$$D_{m,r,\lambda}(n,x)\sim\left((mt+1)^{-\frac{r}{m}},\ \log_\lambda(mt+1)^{\frac{1}{m}}\right)\quad\text{and}\quad E_n(x)\sim\left(\frac{e^t+1}{2},\ t\right).$$

From (23) and (36), we have

$$D_{m,r,\lambda}(n,x)=\sum_{d=0}^{n}a_{n,d}E_d(x), \tag{37}$$

where, by (2) and (6),

(38)

$$\begin{aligned}
a_{n,d}&=\frac{1}{d!}\left\langle e_\lambda^r(t)\left(\frac{e^{\frac{e_\lambda^m(t)-1}{m}}+1}{2}\right)\left(\frac{e_\lambda^m(t)-1}{m}\right)^d\middle|x^n\right\rangle\\
&=\frac{1}{d!}\frac{1}{2m^d}\left\langle e_\lambda^r(t)\left(e^{\frac{e_\lambda^m(t)-1}{m}}+1\right)(e_\lambda^m(t)-1)^d\middle|x^n\right\rangle\\
&=\frac{1}{d!}\frac{1}{2m^d}\sum_{l=0}^{d}\binom{d}{l}(-1)^{d-l}\left\langle e_\lambda^r(t)\left(e^{\frac{e_\lambda^m(t)-1}{m}}+1\right)e_\lambda^{ml}(t)\middle|x^n\right\rangle\\
&=\frac{1}{d!}\frac{1}{2m^d}\sum_{l=0}^{d}\binom{d}{l}(-1)^{d-l}\left[\left\langle e_\lambda^r(t)e^{\frac{e_\lambda^m(t)-1}{m}}e_\lambda^{ml}(t)\middle|x^n\right\rangle+\left\langle e_\lambda^{ml+r}(t)\middle|x^n\right\rangle\right]\\
&=\frac{1}{d!}\frac{1}{2m^d}\sum_{l=0}^{d}\binom{d}{l}(-1)^{d-l}\left[\sum_{\alpha=0}^{n}(ml)_{\alpha,\lambda}\binom{n}{\alpha}\left\langle e_\lambda^r(t)e^{\frac{e_\lambda^m(t)-1}{m}}\middle|x^{n-\alpha}\right\rangle+(ml+r)_{n,\lambda}\right]\\
&=\frac{1}{d!}\frac{1}{2m^d}\sum_{l=0}^{d}\binom{d}{l}(-1)^{d-l}\left[\sum_{\alpha=0}^{n}\binom{n}{\alpha}(ml)_{\alpha,\lambda}D_{m,r,\lambda}(n-\alpha)+(ml+r)_{n,\lambda}\right].
\end{aligned}$$

Combining (37) with (38), we obtain the first identity.
In another way, we observe that by (4) and (27)

$$\begin{aligned}
a_{n,d}&=\frac{1}{d!}\left\langle e_\lambda^r(t)\left(\frac{e^{\frac{e_\lambda^m(t)-1}{m}}+1}{2}\right)\left(\frac{e_\lambda^m(t)-1}{m}\right)^d\middle|x^n\right\rangle\\
&=\frac{1}{2}\sum_{l=d}^{n}W_{m,r,\lambda}(l,d)\binom{n}{l}\left\langle e^{\frac{e_\lambda^m(t)-1}{m}}+1\middle|x^{n-l}\right\rangle\\
&=\frac{1}{2}\sum_{l=d}^{n}W_{m,r,\lambda}(l,d)\binom{n}{l}\left[1+\sum_{j=0}^{n-l}S_{2,\frac{\lambda}{m}}(n-l,j)m^{n-l-j}\right].
\end{aligned} \tag{39}$$

combining with (37) and (39), we get the second identity. □

Theorem 2.4. *For $n \geq 0$, we have*

$$D_{m,r,\lambda}(n,x) = \sum_{d=0}^{n}\left(\sum_{l=d}^{n}\sum_{j=0}^{n-l}\binom{n}{l}\frac{m^{n-l-j}}{j+1}S_{2,\frac{\lambda}{m}}(n-l,j)W_{m,r,\lambda}(l,d)\right)B_d(x).$$

where $B_n(x)$ are the ordinary Bernoulli polynomials.

Proof. From (15), (22) and (25), we consider the following two Sheffer sequence as follows:

(40)

$$D_{m,r,\lambda}(n,x) \sim \left((mt+1)^{-\frac{r}{m}},\ \log_\lambda(mt+1)^{\frac{1}{m}}\right) \quad \text{and} \quad B_n(x) \sim \left(\frac{e^t-1}{t},\ t\right).$$

From (23) and (40), we have

$$D_{m,r,\lambda}(n,x) = \sum_{d=0}^{n} a_{n,d}B_d(x), \tag{41}$$

where, by (4) and (27) we get

$$\begin{aligned} a_{n,d} &= \frac{1}{d!}\left\langle \left(e^{\frac{e_\lambda^m(t)-1}{m}}-1\right)\left(\frac{e_\lambda^m(t)-1}{m}\right)^{-1} e_\lambda^r(t)\left(\frac{e_\lambda^m(t)-1}{m}\right)^d \middle| x^n\right\rangle \\ &= \sum_{l=d}^{n} W_{m,r,\lambda}(l,d)\binom{n}{l}\left\langle \sum_{j=1}^{\infty}\frac{1}{j!}\left(\frac{e_\lambda^m(t)-1}{m}\right)^{j-1}\middle| x^{n-l}\right\rangle \\ &= \sum_{l=d}^{n} W_{m,r,\lambda}(l,d)\binom{n}{l}\left\langle \sum_{j=0}^{\infty}\frac{1}{(j+1)!}\left(\frac{e_\lambda^m(t)-1}{m}\right)^{j}\middle| x^{n-l}\right\rangle \\ &= \sum_{l=d}^{n} W_{m,r,\lambda}(l,d)\binom{n}{l}\sum_{j=0}^{n-l}\frac{m^{n-l-j}}{j+1}S_{2,\frac{\lambda}{m}}(n-l,j). \end{aligned} \tag{42}$$

Combining (41) with (42), we get the desired identity. □

Theorem 2.5. *For $n \geq 0$, we have*

$$D_{m,r,\lambda}(n,x) = \sum_{d=0}^{n}\left(\sum_{l=d+1}^{n}\frac{d+1}{l}S_1(l,d+1)W_{m,r,\lambda}(n,l-1)\right)\widetilde{D}_d(x).$$

where $\widetilde{D}_n(x)$ are the Daehee polynomials.

Proof. From (18), (22) and (25), we consider the following two Sheffer sequence:

(43)

$$D_{m,r,\lambda}(n,x) \sim \left((mt+1)^{-\frac{r}{m}},\ \ \log_\lambda(mt+1)^{\frac{1}{m}}\right) \quad \text{and} \quad \widetilde{D}_n(x) \sim \left(\frac{e^t-1}{t},\ \ e^t-1\right).$$

From (23) and (43), we have

$$D_{m,r,\lambda}(n,x) = \sum_{d=0}^{n} a_{n,d}\widetilde{D}_d(x), \tag{44}$$

where, from (4) and (18), we have

(45)
$$\begin{aligned} a_{n,d} &= \frac{1}{d!}\left\langle \left(e^{\frac{e_\lambda^m(t)-1}{m}}-1\right)\left(\frac{e_\lambda^m(t)-1}{m}\right)^{-1} e_\lambda^r(t)\left(e^{\frac{e_\lambda^m(t)-1}{m}}-1\right)^d \middle| x^n\right\rangle \\ &= (d+1)\sum_{l=d+1}^{n} S_1(l,d+1)\frac{1}{l!}\left\langle \left(\frac{e_\lambda^m(t)-1}{m}\right)^{l-1} e_\lambda^r(t)\middle| x^n\right\rangle \\ &= (d+1)\sum_{l=d+1}^{n} S_1(l,d+1)\frac{1}{l!}\left\langle (l-1)!\sum_{s=l-1}^{\infty} W_{m,r,\lambda}(s,l-1)\frac{t^s}{s!}\middle| x^n\right\rangle \\ &= \frac{d+1}{l}\sum_{l=d+1}^{n} S_1(l,d+1)W_{m,r,\lambda}(n,l-1). \end{aligned}$$

Combining (44) with (45), we get the desired identity. □

Theorem 2.6. *For $n \geq 0$, we have*

$$\begin{aligned} D_{m,r,\lambda}(n,x) = \sum_{d=0}^{n}\Bigg(\sum_{l=d}^{n}\sum_{j=l}^{n}\sum_{i=0}^{n-j}\binom{n}{j}\frac{(-1)^i\langle s\rangle_i m^{k-2j}}{i!m^i} S_{1,\lambda}(l,d) \\ \times S_{2,\frac{\lambda}{m}}(n-j,j)W_{m,r,\lambda}(j,l)\Bigg) Bel_d^{(r)}(x|\lambda), \end{aligned}$$

where $Bel_n^{(s)}(x|\lambda)$ are the degenerate s-Bell polynomials.

Proof. By (14) and (22), we note that we get

$$Bel_n^{(s)}(x|\lambda) \sim ((1+t)^{-s},\ \ \log_\lambda(1+t)). \tag{46}$$

From (4), (25) and (46), we have

$$D_{m,r,\lambda}(n,x) = \sum_{d=0}^{n} a_{n,d} Bel_d^{(s)}(x|\lambda), \tag{47}$$

where, from (4), (9) and (27), we get

(48)

$$\begin{aligned}
a_{n,d} &= \frac{1}{d!}\left\langle \left(1+\frac{e_\lambda^m(t)-1}{m}\right)^{-s} e_\lambda^r(t)\left(\log_\lambda\left(1+\frac{e_\lambda^m(t)-1}{m}\right)\right)^d \Big| x^n\right\rangle \\
&= \left\langle \left(1+\frac{e_\lambda^m(t)-1}{m}\right)^{-s} e_\lambda^r(t)\sum_{l=d}^{\infty} S_{1,\lambda}(l,d)\frac{\left(\frac{e_\lambda^m(t)-1}{m}\right)^l}{l!}\Big| x^n\right\rangle \\
&= \left\langle \left(1+\frac{e_\lambda^m(t)-1}{m}\right)^{-s} e_\lambda^r(t)\sum_{l=d}^{\infty} S_{1,\lambda}(l,d)\sum_{j=l}^{\infty} W_{m,r,\lambda}(j,l)\frac{t^j}{j!}\Big| x^n\right\rangle \\
&= \sum_{j=d}^{n}\sum_{l=d}^{j} S_{1,\lambda}(l,d)W_{m,r,\lambda}(j,l)\binom{n}{j}\left\langle \left(1+\frac{e_\lambda^m(t)-1}{m}\right)^{-s} \Big| x^{n-j}\right\rangle \\
&= \sum_{l=d}^{n} S_{1,\lambda}(l,d)\sum_{j=l}^{n}\binom{n}{j}W_{m,r,\lambda}(j,l)\left\langle \sum_{i=0}^{\infty}\langle s\rangle_i\frac{(-1)^i}{i!}\left(\frac{e_\lambda^m(t)-1}{m}\right)^i \Big| x^{n-j}\right\rangle \\
&= \sum_{l=d}^{n}\sum_{j=l}^{n}\sum_{i=0}^{n-j}\binom{n}{j}\frac{(-1)^i\langle s\rangle_i}{i!m^i}S_{1,\lambda}(l,d)W_{m,r,\lambda}(j,l)\left\langle \frac{1}{i!}\left(\frac{e_\lambda^m(t)-1}{m}\right)^i \Big| x^{n-j}\right\rangle \\
&= \sum_{l=d}^{n}\sum_{j=l}^{n}\sum_{i=0}^{n-j}\binom{n}{j}\frac{(-1)^i\langle s\rangle_i}{i!m^i}S_{1,\lambda}(l,d)W_{m,r,\lambda}(j,l)m^{k-2j}S_{2,\frac{\lambda}{m}}(n-j,j).
\end{aligned}$$

Combining (47) with (48), we have the identity. □

Theorem 2.7. *For $n \geq 0$, we have*

$$D_{m,r,\lambda}(n,x) = \sum_{d=0}^{n}\left(\sum_{l=d}^{n} S_1(l,d)W_{m,r,\lambda}(n,l)\right) bel_d(x).$$

where $bel_n(x)$ are the Bell polynomials.

Proof. From (11), (22) and (25) , we consider the following two Sheffer sequences:

(49)

$$D_{m,r,\lambda}(n,x) \sim \left((mt+1)^{-\frac{r}{m}},\ \ \log_\lambda(mt+1)^{\frac{1}{m}}\right) \quad \text{and} \quad bel_n(x) \sim (1,\ \ \log(1+t)).$$

From (23) and (49), we have

$$D_{m,r,\lambda}(n,x) = \sum_{d=0}^{n} a_{n,d}\, bel_d(x), \tag{50}$$

where, from (4) and (7), we have

$$\begin{aligned} a_{n,d} &= \frac{1}{d!}\left\langle e_\lambda^r(t)\left(\log\left(1+\frac{e_\lambda^m(t)-1}{m}\right)\right)^d\middle| x^n\right\rangle \\ &= \left\langle e_\lambda^r(t)\sum_{l=d}^{\infty}S_1(l,d)\frac{1}{l!}\left(\frac{e_\lambda^m(t)-1}{m}\right)^l\middle| x^n\right\rangle \quad (51)\\ &= \sum_{l=d}^{n}S_1(l,d)\left\langle \sum_{j=l}^{\infty}W_{m,r,\lambda}(j,l)\frac{t^j}{j!}\middle| x^n\right\rangle = \sum_{l=d}^{n}S_1(l,d)W_{m,r,\lambda}(n,l). \end{aligned}$$

Combining (50) with (51), we get the identity. □

References

[1] M. Benoumhani, *On Whitney numbers of Dowling lattices,* Discrete Math. 159 (1996) 13-33.
[2] M. Benoumhani, *On some numbers related to Whitney numbers of Dowling lattices,* Adv. Appl. Math. 19 (1997) 106-116.
[3] A.Z. Broder, *The r-Stirling numbers,* Discrete Math. 49 (1984) 241-259.
[4] N.P. Cakic, G.V. Milovanovic, *On generalized Stirling numbers and polynomials.* Math. Balk. 18 (2004), 241-248 .
[5] L. Carlitz, *Degenerate Stirling, Bernoulli and Eulerian numbers,* Utilitas Math. 15 (1979), 51-88.
[6] J.A D. Cillar, R.B. Corcino, *A q-analogue of Qi formula for r-Dowling numbers,* Commun. Korean Math. Soc. 35 (2020), No. 1, 21-41.
[7] G.-S. Cheon, J.-H. Jung, *r-Whitney numbers of Dowling lattices,* Discrete Math. 312 (2012), no. 15, 2337-2348. https://doi.org/10.1016/j.disc.2012.04.001
[8] L. Comtet, Advanced Combinatorics: The art of finite and infinite expansions, Reidel, Dordrecht (1974)
[9] G. V. Dunne, C. Schubert, *Bernoulli number identities from quantum field theory and topological string theory,* Commun. Number Theory Phys. 7 (2013), no. 2, 225-249.
[10] T.A. Dowling, *A class of gemometric lattices bases on finite groups,* J. Combin. Theory Ser. B, 14 (1973), 61-86.
[11] I. M. Gessel, *On Mikis identities for Bernoulli numbers,* J. Number Theory 110 (2005), no. 1, 75-82.
[12] E. Gyimesi, *A comprehensive study of r-Dowling polynomials,* Aequationes Math. 92 (2018), no. 3, 515-527. https://doi.org/10.1007/s00010-017-0538-z
[13] E. Gyimesi, G. Nyul, *New combinatorial interpretations of r-Whitney and r-Whitney-Lah numbers,* Discrete Appl. Math. 255 (2019), 222-233. https://doi.org/10.1016/j.dam.2018.08.020
[14] D.S. Kim, T. Kim, *A note on a new type of degenerate Bernoulli numbers,* Russ. J. Math. Phys. 27 (2020), no. 2, 227-235.
[15] D.S. Kim, T. Kim, *Daehee numbers and polynomials,* Appl. Math. Sci. 7 (2013), 5969-5976. https://doi.org/10.12988/ams.2013.39535
[16] H.K. Kim, D.S. Lee, *Some properties of degenerate r-Dowling polynomials and numbers of the second Kind,* CMES, (2022), pp 18. DOI: 10.32604/cmes.2022.022103
[17] T. Kim, *A note on degenerate Stirling polynomials of the second kind,* Proc. Jangjeon Math. Soc. 20(3) (2017), 319-331.
[18] T. Kim, D.S. Kim, *Degenerate Whitney numbers of the first and second kinds of Dowling lattics,* Russ. J. Math. Phys., 29(3) (2022), arXiv:2013.08904v1.
[19] T. Kim, D.S. Kim, D.V. Dolgy, S.H. Rim, *Some identities on the Euler numbers arising from Euler basis polynomials,* Ars. Combin. 109 (2013), 433-446.
[20] T. Kim, D.S. Kim, G-W. Jang, *Extended stirling polynomials of the second kind and extended Bell polynomials,* Proc. Jangjeon Math. Soc. 20(3), 365-376 (2017)

[21] T. Kim, D.S. Kim, *Degenerate r-Whitney numbers and degenerate r-Dowling polynomials via boson operator,* Advances in Applied Mathematics, 140 (2022), 102394. https://doi.org/10.1016/j.aam.2022.102394.
[22] I. Mezö, *A new formula for the Bernoulli polynomials,* Results Math., **58** (2010) 329-35.
[23] H. Miki, *A relation between Bernoulli numbers,* J. Number Theory, 10 (1978), no. 3, 297-302.
[24] S. Roman, *The umbral calculus,* Pure and Applied Mathematics, 1984.
[25] G.-C. Rota, *On the foundations of combinatorial theory I, theory of Mobius functions,* Z. Wahrscheinlichkeitstheorie, 2 (1964) 340-368.
[26] M. Shattuck, *Some combinatorial identities for the r-Dowling polynomials,* Notes on Number Theory and Discrete Math. 25 (2019), No.2, 145-154, DOI:107546/nntdm.2019252145-154.
[27] K. Shiratani, S. Yokoyama, *An application of p-adic convolutions,* Mem. Fac. Sci. Kyushu Univ. Ser. A 36 (1982), no. 1, 73-83.
[28] Y. Simsek, *Generating functions for generalized Stirling type numbers, Array type polynomials, Eulerian type polynomials and their applications,* Fixed Point Theory and Applications (2013), 2013:87.

Department of Mathematics Education, Daegu Catholic University, Gyeongsan 38430, Republic of Korea
E-mail address: hkkim@cu.ac.kr